

Pacific Journal of Mathematics

PLURISUBHARMONIC DEFINING FUNCTIONS

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Let Ω be a bounded pseudoconvex open set in n -dimensional complex Euclidean space C^n with a smooth (\mathcal{C}^∞)-boundary. It has been known for some time that it is not always possible to choose a defining function ρ which is plurisubharmonic in a neighborhood of $\bar{\Omega}$. We study here the question whether for every point $p \in \partial\Omega$, there exists an open neighborhood on which ρ can be chosen to be plurisubharmonic. Our main conclusion is that this is not always the case.

1. Notation and results. In what follows, Ω will always be a bounded open set in C^n with \mathcal{C}^∞ -boundary. This means that there exists a real-valued \mathcal{C}^∞ -function $\rho: C^n \rightarrow R$ such that $\Omega = \{\rho < 0\}$ and $d\rho \neq 0$ on $\partial\Omega$. Let $z = (z_1, z_2, \dots, z_n)$, $z_j = x_j + iy_j$, denote complex coordinates in C^n , and define

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

DEFINITION 1. The set Ω is pseudoconvex if for every $p \in \partial\Omega$, we have

$$(1) \quad \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p) t_i \bar{t}_j \geq 0$$

whenever

$$t = (t_1, \dots, t_n) \in C^n - (0) \quad \text{and} \quad \sum_{i=1}^n \frac{\partial \rho}{\partial z_i}(p) t_i = 0.$$

If we have strict inequality in (1) for all $p \in \partial\Omega$, then Ω is said to be strongly pseudoconvex.

DEFINITION 2. A real-valued \mathcal{C}^2 -function, u , defined on an open set V in C^n is plurisubharmonic if

$$\sum_{i,j=1}^n \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(p) t_i \bar{t}_j \geq 0$$

whenever $p \in V$ and $t = (t_1, \dots, t_n) \in C^n - (0)$.

If we have strict inequality for all $p \in V$, then u is strictly plurisubharmonic.

The following results are known:

THEOREM 3 [2]. *If Ω is strongly pseudoconvex, then ρ may be chosen to be strictly plurisubharmonic in some neighborhood of $\bar{\Omega}$.*

The next example shows that the theorem fails in general if we drop the hypothesis of *strong* pseudoconvexity.

EXAMPLE 4 [1]. There exists a bounded pseudoconvex domain Ω in C^2 , with \mathcal{C}^∞ -boundary, such that no (\mathcal{C}^2) defining function ρ exists with

$$\sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p) t_i \bar{t}_j \geq 0$$

whenever

$$p \in \partial\Omega \quad \text{and} \quad t = (t_1, \dots, t_n) \in C^n.$$

There exists an example, similar to the one above, which has a real analytic boundary.

EXAMPLE 5. Let

$$\begin{aligned} \Omega &= \Omega_K = \{(z_1, z_2) \in (C - (0)) \times C; \sigma \\ &= |z_2 + e^{i \ln z_1 \bar{z}_1}|^2 - 1 + K(\ln z_1 \bar{z}_1)^4 < 0\}. \end{aligned}$$

Then, if, $K > 1$ is sufficiently large, Ω is a bounded pseudoconvex domain in C^2 with smooth real analytic boundary, such that no \mathcal{C}^2 defining function, ρ , exists such that

$$\sum_{i,j=1}^2 \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p) t_i \bar{t}_j \geq 0$$

whenever $p \in \partial\Omega$ and $(t_1, t_2) \in C^2$.

The details will be given in the next section.

EXAMPLE 6. There exists a bounded pseudoconvex domain Ω in C^3 , with \mathcal{C}^∞ -boundary, and a point $p \in \partial\Omega$ such that whenever ρ is a \mathcal{C}^2 defining function for Ω ,

$$\sum_{i,j=1}^3 \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(q) t_i \bar{t}_j < 0$$

for some (t_1, \dots, t_n) and $q \in \partial\Omega$ arbitrarily close to p .

This example shows that one does not have plurisubharmonic

defining functions for pseudoconvex domains, even locally, in general.

2. Examples.

EXAMPLE 5. Clearly, Ω is bounded in $(C - (0)) \times C$. If $\partial\sigma/\partial z_2 = 0$, then $z_2 = -e^{i \ln z_1 \bar{z}_1}$. Hence, if $d\sigma = 0$, then $0 = z_1 \partial\sigma/\partial z_1 = 4K(\ln z_1 \bar{z}_1)^3$. This implies that $|z_1| = 1$ and $z_2 = -1$. At such points, $\sigma(z_1, z_2) = -1$, so $d\sigma \neq 0$ on $\partial\Omega$.

To show that Ω is pseudoconvex, we compute the Leviform

$$\begin{aligned} \mathcal{L} &= \frac{\partial^2 \sigma}{\partial z_1 \partial \bar{z}_1} \left| \frac{\partial \sigma}{\partial z_2} \right|^2 - \frac{\partial^2 \sigma}{\partial z_1 \partial \bar{z}_2} \frac{\partial \sigma}{\partial z_2} \frac{\partial \sigma}{\partial \bar{z}_1} - \frac{\partial^2 \sigma}{\partial \bar{z}_1 \partial z_2} \cdot \frac{\partial \sigma}{\partial z_1} \cdot \frac{\partial \sigma}{\partial \bar{z}_2} \\ &\quad + \frac{\partial^2 \sigma}{\partial z_2 \partial \bar{z}_2} \cdot \left| \frac{\partial \sigma}{\partial z_1} \right|^2 \end{aligned}$$

to obtain

$$\begin{aligned} \mathcal{L} &= \frac{z_2 \bar{z}_2 + K(\ln z_1 \bar{z}_1)^4 + 12K(\ln z_1 \bar{z}_1)^2}{z_1 \bar{z}_1} \cdot |z_2 + e^{i \ln z_1 \bar{z}_1}|^2 \\ &\quad + 4K \frac{(\ln z_1 \bar{z}_1)^3}{z_1 \bar{z}_1} (i \bar{z}_2 e^{i \ln z_1 \bar{z}_1} - i z_2 e^{-i \ln z_1 \bar{z}_1}) + 16K^2 \frac{(\ln z_1 \bar{z}_1)^6}{z_1 \bar{z}_1} \end{aligned}$$

on $\partial\Omega$.

If $|z_2 + e^{i \ln z_1 \bar{z}_1}| \geq 1/2$, we have

$$\mathcal{L} \geq 3K(\ln z_1 \bar{z}_1)^2 / z_1 \bar{z}_1 - 16K |\ln z_1 \bar{z}_1|^3 / z_1 \bar{z}_1,$$

since $|z_2| \leq 2$ on $\partial\Omega$. If K is sufficiently large, then $|\ln z_1 \bar{z}_1| < 3/16$ on $\partial\Omega$ and hence $\mathcal{L} \geq 0$.

Consider next a boundary point where $|z_2 + e^{i \ln z_1 \bar{z}_1}| < 1/2$. Then $K(\ln z_1 \bar{z}_1)^4 \geq 3/4$, since $\sigma(z_1, z_2) = 0$. Hence

$$\begin{aligned} \mathcal{L} &\geq -16K |\ln z_1 \bar{z}_1|^3 / z_1 \bar{z}_1 + 16K^2 (\ln z_1 \bar{z}_1)^6 / z_1 \bar{z}_1 \\ &= 16K |\ln z_1 \bar{z}_1|^3 / z_1 \bar{z}_1 (-1 + K(\ln z_1 \bar{z}_1)^4 / |\ln z_1 \bar{z}_1|) \end{aligned}$$

which is nonnegative if K is sufficiently large.

Assume next that ρ is a \mathcal{C}^2 defining function for Ω such that

$$\sum_{i,j=1}^2 \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} (p) t_i \bar{t}_j \geq 0$$

whenever $p \in \partial\Omega$ and $(t_1, t_2) \in \mathbb{C}^2$. In particular, $\rho = h\sigma$ for some \mathcal{C}^1 function $h > 0$. We observe that $\partial^2 \rho / \partial z_1 \partial \bar{z}_1(z_1, z_2) = 0$ whenever $|z_1| = 1$ and $z_2 = 0$. (All such points are in $\partial\Omega$.) Therefore, $\partial^2 \rho / \partial \bar{z}_1 \partial z_2(z_1, z_2) = 0$ at these points also. Hence

$$\left(\frac{\partial h}{\partial \bar{z}_1} \frac{\partial \sigma}{\partial z_2} + h \frac{\partial^2 \sigma}{\partial \bar{z}_1 \partial z_2}\right)(e^{i\theta}, 0) \equiv 0$$

and so

$$\frac{\partial}{\partial \bar{z}_1}(he^{i \ln z_1 \bar{z}_1})(e^{i\theta}, 0) \equiv 0 .$$

Multiplying with $e^{i \operatorname{Log} z_1}$ we get that

$$\frac{\partial}{\partial \bar{z}_1}(he^{-2 \operatorname{Arg} z_1})(e^{i\theta}, 0) \equiv 0$$

which implies that $h(e^{i\theta}, 0) = ce^{2\theta}$ for some constant $c > 0$. This is of course impossible.

In the next example, we localize the above idea suitably.

EXAMPLE 6. Let us use coordinates (w, z_1, z_2) in \mathbf{C}^3 with $w = \eta + i\zeta$ and $z_j = x_j + iy_j, j = 1, 2$. We pick a \mathcal{C}^∞ , convex function $\chi_1(t): \mathbf{R} \rightarrow \mathbf{R}$ such that $\chi_1(t) = 0$ when $t \leq 1$ and $\chi_1(t) > 0$ when $t > 0$. Define $\sigma_1: \mathbf{C}^3 \rightarrow \mathbf{R}$ by

$$\sigma_1 = \eta + \eta^2 + K\zeta^2 + K(y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)\zeta^2 + \chi_1(x_1^2 + x_2^2) ,$$

and let $\Omega_1 = \{\sigma_1 < 0\}$. Here $K \gg 1$ is a constant which will be chosen later.

LEMMA 7. *The set Ω_1 is bounded and pseudoconvex with \mathcal{C}^∞ -boundary for all K sufficiently large.*

Proof. Computation shows that $d\sigma_1 = 0$ only at points $(-1/2, x_1, x_2)$ with $x_1^2 + x_2^2 \leq 1$. Since $\sigma_1 = -1/4$ at these points, it follows that $d\sigma_1 \neq 0$ on $\partial\Omega_1$. Further computation shows that σ_1 is plurisubharmonic in a neighborhood of $\bar{\Omega}_1$ if K is sufficiently large.

In the following K , sufficiently large, is fixed.

The next step is to make an infinite number of perturbations of the boundary of Ω_1 . Let $p_j = (0, 1/2^j, 0), j = 1, 2, \dots$ and let $\mathbf{B}(p_j, r) = \{(w, z_1, z_2); (|w|^2 + |z_1 - 1/2^j|^2 + |z_2|^2)^{1/2} < r\}$ be the ball centered at p_j of radius r . Choose functions $\chi^{(j)} \in \mathcal{C}_0^\infty(\mathbf{B}(p_j, 1/2^{j+2}))$ with $\chi^{(j)} \equiv 1$ on $\mathbf{B}(p_j, 1/2^{j+3})$ and $\chi^{(j)} \geq 0, j = 1, 2, \dots$. Observe that $\operatorname{supp} \chi^{(i)} \cap \operatorname{supp} \chi^{(j)} = \emptyset$ whenever $i \neq j$. We may arrange that $|d\chi^{(j)}| \leq C_j \chi^{(j)}$ and $|\partial\chi^{(j)}/\partial y_k| \leq C_j |y_k|$ for suitable C_1, C_2, \dots , and $k = 1, 2$. Let $\varepsilon = \{\varepsilon_j\}_{j=1}^\infty$ denote a rapidly decreasing sequence, $\varepsilon_1 > \varepsilon_2 > \dots > 0$ and define

$$\sigma_2 = \sigma_1 + \sum_{j=1}^\infty \varepsilon_j \chi^{(j)} \cdot (y_1^2 + y_2^2) \cdot x_2^2 .$$

Clearly σ_2 is a \mathcal{C}^∞ -function, and if $\Omega_2 = \{\sigma_2 < 0\}$, then $d\sigma_2 \neq 0$ on $\partial\Omega_2$ and Ω_2 is a bounded domain which is pseudoconvex at every point in $\partial\Omega_2 - \bigcup_j B(p_j, 1/2^{j+2})$.

LEMMA 8. *The set Ω_2 is pseudoconvex if ε decreases sufficiently fast.*

Proof. Fix a $j \geq 1$. It suffices to show that $\sigma_1 + \varepsilon_j \chi^{(j)} \cdot (y_1^2 + y_2^2)x_2^2$ is plurisubharmonic in $B(p_j, 1/2^{j+2})$ for all small enough $\varepsilon_j > 0$. This is checked by a direct computation.

We fix a sequence $\{\varepsilon_j\}$ decreasing sufficiently fast.

To complete the construction of the example, we will perturb σ_2 inside each $B(p_j, 1/2^{j+3})$. More precisely, let $\chi_{(j)} \in \mathcal{C}_0^\infty(B(p_j, 1/2^{j+3}))$ with

$$\int_{\mathbb{R}} \left(\frac{\partial \chi_{(j)}}{\partial x_1} + \chi_{(j)} \right) (0, x_1, 0) dx_1 \neq 0$$

for each j , $\chi_{(j)} \geq 0$. We may assume that $|\partial \chi_{(j)} / \partial \eta|, |\partial \chi_{(j)} / \partial \zeta|, |\partial \chi_{(j)} / \partial y_k|, |\partial \chi_{(j)} / \partial x_2| \leq C_j (|\eta| + |\zeta| + |x_2| + |y_1| + |y_2|)$, $k = 1, 2$, C_j some constant.

If $\delta = \{\delta_j\}_{j=j_0}^\infty, \delta_{j_0} > \delta_{j_0+1} > \dots > 0$ is any sufficiently rapidly decreasing sequence,

$$\sigma = \sigma_2 + \sum_{j=j_0}^\infty \delta_j \chi_{(j)} \cdot (\eta + \zeta y_1)$$

is a \mathcal{C}^∞ -function and $d\sigma \neq 0$ on $\partial\Omega, \Omega = \{\sigma < 0\}$. Moreover, Ω is a bounded domain which is pseudoconvex on $\partial\Omega - \bigcup B(p_j, 1/2^{j+3})$.

LEMMA 9. *The set Ω is pseudoconvex if δ decreases sufficiently fast, and j_0 is sufficiently large.*

Proof. Fix a $j \gg 1$. It suffices to show that Ω is pseudoconvex at those boundary points which are in $B(p_j, 1/2^{j+3})$ for all δ_j sufficiently small. In $B(p_j, 1/2^{j+3}), \sigma = \eta + \eta^2 + K\zeta^2 + K(y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)\zeta^2 + \varepsilon_j(y_1^2 + y_2^2) \cdot x_2^2 + \delta_j \chi_{(j)} \cdot (\eta + \zeta y_1)$. Differentiating, we obtain:

$$\begin{aligned} \frac{\partial \sigma}{\partial w} &= \frac{1}{2} + \eta - iK\zeta - i\zeta(y_1^2 + y_2^2) + \delta_j \frac{\partial \chi_{(j)}}{\partial w} \cdot (\eta + \zeta y_1) \\ &\quad + \frac{1}{2} \delta_j \chi_{(j)} - \frac{i}{2} \delta_j \chi_{(j)} y_1, \end{aligned}$$

$$\begin{aligned} \frac{\partial \sigma}{\partial z_1} &= -2iK(y_1^3 + y_1 y_2^2) - i y_1 \zeta^2 - i \varepsilon_j y_1 x_2^2 \\ &\quad + \delta_j \frac{\partial \chi_{(j)}}{\partial z_1} \cdot (\eta + \zeta y_1) - \frac{i}{2} \delta_j \chi_{(j)} \cdot \zeta, \end{aligned}$$

$$\begin{aligned}
\frac{\partial \sigma}{\partial z_2} &= -2iK(y_1^2 y_2 + y_2^3) - iy_2 \zeta^2 - i\varepsilon_j y_2 x_2^2 + \varepsilon_j (y_1^2 + y_2^2) x_2 \\
&\quad + \delta_j \frac{\partial \chi_{(j)}}{\partial z_2} \cdot (\eta + \zeta y_1), \\
\frac{\partial^2 \sigma}{\partial w \partial \bar{w}} &= \frac{1}{2} + \frac{K}{2} + \frac{1}{2} (y_1^2 + y_2^2) + \delta_j \frac{\partial^2 \chi_{(j)}}{\partial w \partial \bar{w}} \cdot (\eta + \zeta y_1) \\
&\quad + \frac{1}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial w} + \frac{i}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial w} \cdot y_1 + \frac{1}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial \bar{w}} \\
&\quad - \frac{i}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial \bar{w}} \cdot y_1, \\
\frac{\partial^2 \sigma}{\partial w \partial \bar{z}_1} &= \zeta y_1 + \delta_j \frac{\partial^2 \chi_{(j)}}{\partial w \partial \bar{z}_1} \cdot (\eta + \zeta y_1) + \frac{i}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial w} \cdot \zeta \\
&\quad + \frac{1}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial \bar{z}_1} - \frac{i}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial \bar{z}_1} y_1 + \frac{1}{4} \delta_j \chi_{(j)}, \\
\frac{\partial^2 \sigma}{\partial w \partial \bar{z}_2} &= \zeta y_2 + \delta_j \frac{\partial^2 \chi_{(j)}}{\partial w \partial \bar{z}_2} \cdot (\eta + \zeta y_1) + \frac{1}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial \bar{z}_2} - \frac{i}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial \bar{z}_2} \cdot y_1, \\
\frac{\partial^2 \sigma}{\partial z_1 \partial \bar{z}_1} &= 3Ky_1^2 + Ky_2^2 + \frac{1}{2} \zeta^2 + \frac{1}{2} \varepsilon_j x_2^2 + \delta_j \frac{\partial^2 \chi_{(j)}}{\partial z_1 \partial \bar{z}_1} \cdot (\eta + \zeta y_1) \\
&\quad + \frac{i}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial z_1} \cdot \zeta - \frac{i}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial \bar{z}_1} \cdot \zeta, \\
\frac{\partial^2 \sigma}{\partial z_1 \partial \bar{z}_2} &= 2Ky_1 y_2 - i\varepsilon_j y_1 x_2 + \delta_j \frac{\partial^2 \chi_{(j)}}{\partial z_1 \partial \bar{z}_2} \cdot (\eta + \zeta y_1) - \frac{i}{2} \delta_j \frac{\partial \chi_{(j)}}{\partial \bar{z}_2} \cdot \zeta
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 \sigma}{\partial z_2 \partial \bar{z}_2} &= Ky_1^2 + 3Ky_2^2 + \frac{1}{2} \zeta^2 + \frac{\varepsilon_j x_2^2}{2} - i\varepsilon_j x_2 y_2 + i\varepsilon_j y_2 x_2 \\
&\quad + \frac{1}{2} \varepsilon_j (y_1^2 + y_2^2) + \delta_j \frac{\partial^2 \chi_{(j)}}{\partial z_2 \partial \bar{z}_2} \cdot (\eta + \zeta y_1).
\end{aligned}$$

Observe that $\eta = 0(\zeta^2 + y_1^2 + y_2^2)$ on $\partial\Omega \cap \mathbf{B}(p_j, 1/2^{j+3})$. Hence there is a $D_j \gg 1$ such that for all sufficiently small $\delta_j > 0$, $\partial^2 \sigma / \partial w \partial \bar{w} \geq K/2$,

$$\begin{aligned}
\left| \frac{\partial^2 \sigma}{\partial w \partial \bar{z}_1} - \zeta y_1 - \frac{1}{4} \delta_j \frac{\partial \chi_{(j)}}{\partial x_1} - \frac{1}{4} \delta_j \chi_{(j)} \right| &\leq D_j \delta_j \| (w, iy_1, z_2) \|, \\
\left| \frac{\partial^2 \sigma}{\partial w \partial \bar{z}_2} - \zeta y_2 \right| &\leq D_j \delta_j \| (w, iy_1, z_2) \|, \\
\frac{\partial^2 \sigma}{\partial z_1 \partial \bar{z}_1} &\geq (3K - 1)y_1^2 + (K - 1)y_2^2 + \frac{1}{4} \zeta^2 + \frac{1}{4} \varepsilon_j x_2^2, \\
\left| \frac{\partial^2 \sigma}{\partial z_1 \partial \bar{z}_2} - 2Ky_1 y_2 + i\varepsilon_j y_1 x_2 \right| &\leq D_j \delta_j \| (w, iy_1, z_2) \|^2
\end{aligned}$$

and

$$\frac{\partial^2 \sigma}{\partial z_2 \partial \bar{z}_2} \geq Ky_1^2 + 3Ky_2^2 + \frac{1}{4}\zeta^2 + \frac{\varepsilon_j}{4}x_2^2.$$

We compute the Leviform,

$$\begin{aligned} \mathcal{L}_\sigma &= \sigma_{w\bar{w}}t_0\bar{t}_0 + 2 \operatorname{Re} \sigma_{w\bar{z}_1}t_0\bar{t}_1 + 2 \operatorname{Re} \sigma_{w\bar{z}_2}t_0\bar{t}_2 \\ &\quad + \sigma_{z_1\bar{z}_1}t_1\bar{t}_1 + 2 \operatorname{Re} \sigma_{z_1\bar{z}_2}t_1\bar{t}_2 + \sigma_{z_2\bar{z}_2}t_2\bar{t}_2 \end{aligned}$$

for vectors (t_0, t_1, t_2) such that

$$t_0 = (-1/\sigma_w) \cdot (\sigma_{z_1}t_1 + \sigma_{z_2}t_2).$$

Using the above estimates, we obtain

$$\begin{aligned} \mathcal{L}_\sigma &\geq \left((3K - 2)y_1^2 + (K - 2)y_2^2 + \frac{1}{8}\zeta^2 + \frac{1}{8}\varepsilon_j x_2^2 \right) t_1\bar{t}_1 \\ &\quad + \left((K - 2)y_1^2 + (3K - 2)y_2^2 + \frac{1}{8}\zeta^2 + \frac{\varepsilon_j}{8}x_2^2 \right) t_2\bar{t}_2 \\ &\quad + 2 \operatorname{Re} (2Ky_1y_2 - i\varepsilon_j y_1x_2) t_1\bar{t}_2 \\ &\quad + 2 \operatorname{Re} \left(\frac{1}{4} \delta_j \frac{\partial \chi_{(j)}}{\partial x_1} + \frac{1}{4} \delta_j \chi_{(j)} \right) \cdot \left[\left(\frac{-1}{\frac{1}{2} + \frac{1}{2} \delta_j \chi_{(j)}} \right) \cdot \frac{-i}{2} \right. \\ &\quad \left. \times \delta_j \chi_{(j)} \zeta t_1 \right] \bar{t}_1 \end{aligned}$$

which clearly is nonnegative.

Assume that there exists a \mathcal{C}^2 -function $\rho: \mathbb{C}^3 \rightarrow \mathbb{R}$, such that $\Omega = \{\rho < 0\}$ and $d\rho \neq 0$ on $\partial\Omega$, with a nonnegative complex Hessian on some neighborhood U of 0 in $\partial\Omega$.

Let $\gamma_i, i = 1, 2, 3, 4$, be straight lines in the (x_1, x_2) -plane,

$$\gamma_1 \text{ goes from } \left(\frac{1}{2^j} - \frac{1}{2^{j+2}}, 0 \right) \text{ to } \left(\frac{1}{2^j} + \frac{1}{2^{j+2}}, 0 \right),$$

$$\gamma_2 \text{ goes from } \left(\frac{1}{2^j} + \frac{1}{2^{j+2}}, 0 \right) \text{ to } \left(\frac{1}{2^j} + \frac{1}{2^{j+2}}, \frac{1}{2^{j+2}} \right),$$

$$\gamma_3 \text{ goes from } \left(\frac{1}{2^j} + \frac{1}{2^{j+2}}, \frac{1}{2^{j+2}} \right) \text{ to } \left(\frac{1}{2^j} - \frac{1}{2^{j+2}}, \frac{1}{2^{j+2}} \right) \text{ and}$$

$$\gamma_4 \text{ goes from } \left(\frac{1}{2^j} - \frac{1}{2^{j+2}}, \frac{1}{2^{j+2}} \right) \text{ to } \left(\frac{1}{2^j} - \frac{1}{2^{j+2}}, 0 \right).$$

We fix j so large that each $\gamma_i \subset U$. The function $\rho = \sigma h$ for some \mathcal{C}^1 -function $h > 0$.

We will show that $\int_{r_1} d(\ln h) \neq 0$ for all small enough $\delta_j > 0$, while

$$\int_{r_i} d(\ln h) = 0, \quad i = 2, 3, 4.$$

First consider the curves γ_2 and γ_4 . There $\rho = (\eta + \eta^2 + K\zeta^2 + K(y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)\zeta^2)h$ from which it follows that $\partial^2\rho/\partial z_2\partial\bar{z}_2 \equiv 0$ on $\gamma_2 \cup \gamma_4$. Hence $\partial^2\rho/\partial w\partial\bar{z}_2 \equiv 0$ on $\gamma_2 \cup \gamma_4$ as well. This reduces to the equation $\partial h/\partial\bar{z}_2 = 0$ from which it follows that $\int_{r_i} d(\ln h) = 0, i = 2, 4$. Similarly $\int_{r_3} d(\ln h) = 0$.

Finally, consider the curve γ_1 . Here $\sigma = \eta + \eta^2 + K\zeta^2 + K(y_1^2 + y_2^2)^2 + (y_1^2 + y_2^2)\zeta^2 + \varepsilon_j\chi^{(j)} \cdot (y_1^2 + y_2^2) \cdot x_2^2 + \delta_j\chi_{(j)} \cdot (\eta + \zeta y_1)$. Clearly $\partial^2\rho/\partial z_1\partial\bar{z}_1 \equiv 0$ on γ_1 and hence $\partial^2\rho/\partial w\partial\bar{z}_1 \equiv 0$ there also. This reduces to the equation

$$\partial^2\sigma/\partial w\partial\bar{z}_1 \cdot h + \partial\sigma/\partial w \cdot \partial h/\partial\bar{z}_1 \equiv 0 \quad \text{on } \gamma_1.$$

Hence

$$\frac{\partial}{\partial x_1}(\ln h) = (-\delta_j)(\partial\chi_{(j)}/\partial x_1 + \chi_{(j)})/(1 + \delta_j\chi_{(j)}).$$

Since we choose $\chi_{(j)}$ such that

$$\int_{\mathbb{R}} \left(\frac{\partial\chi_{(j)}}{\partial x_1} + \chi_{(j)} \right) (0, x_1, 0) dx_1 \neq 0,$$

it follows that $\int d(\ln h) \neq 0$ for all small enough $\delta_j > 0$.

So $\int_{r_1+\dots+r_4} d(\ln h) \neq 0$, which contradicts the assumption that h was well defined.

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Received May 12, 1978.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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Manufactured and first issued in Japan

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