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In this paper the structure of finitely generated splitting rings for the Goldie theory is studied. First, right non-singular finitely generated splitting rings with essential socle which either are right finite dimensional or are right orders in a semiprimary ring are characterized. This characterization is in terms of an explicit triangular matrix structure for R. Then right nonsingular finitely generated splitting rings with zero socle are shown to be right finite dimensional if and only if they are right orders in a semiprimary ring. An explicit triangular structure is given for this class of rings as well. For certain classes of right non-singular right finite dimensional finitely generated splitting rings with zero socle, the structure theorem can be simplified somewhat. Then right nonsingular right finite dimensional finitely generated splitting rings are characterized as a certain essential product of a ring with essential socle and one with zero socle. Right nonsingular finitely generated splitting rings which are right orders in a semiprimary ring are shown to be a direct product of a ring with essential socle and a ring with zero socle. Finally, some comments are made showing how some of these results can be applied to bounded splitting rings and splitting rings.

1. **Preliminaries.** In this paper all rings R are associative with identity and all R-modules are unital. Unless indicated otherwise, all modules are right modules. A left or right R-module M will be denoted by ${}_{R}M$ or M_{R} respectively. The socle of an R-module M will be denoted by soc (M); the socle of R will always mean the socle of R_{R} unless indicated otherwise.

If M is a right R-module and $X \subseteq M$, then the right annihilator of X is denoted by $r_{R}(X)$ or $r(X)$ if there is no ambiguity. Similarly for a left R-module M and $X \subseteq M$, $l_{R}(X)$ is the left annihilator of X.

A submodule K of an R-module M is an *essential submodule* of M if $K \cap L \neq 0$ for all nonzero submodules L of M. A right (left) ideal I of R is essential in R if I is essential in $R_{R}({}_{R}R)$. Let $\mathscr{S}(R)$ denote the family of all essential right ideals of R. For any right R-module M, $Z(M) = \{x \in M : r(x) \in \mathscr{S}(R)\}$ is called the *singular submodule* of M. The singular submodule of a left R-module is defined similarly. An R-module M is *singular* if $Z(M) = M$; M is *nonsingular* if $Z(M) = 0$. A ring R is right (left)-nonsingular if

$R_R(_R R)$ is a nonsingular module.

A submodule A of a module B is closed in B provided that B/A is nonsingular, and we let $L^*(B)$ denote the collection of all closed submodules of B. For any submodule A of B, there is a smallest closed submodule C in B which contains A, called the *closure* of A in B. The closure of a two-sided ideal of R is a two-sided ideal of R. By a "two-sided ideal of $L^*(R)$" we will mean a two-sided ideal of R belonging to $L^*(R)$. We use S^0 to stand for the localization functor associated with a right nonsingular ring: $S^0 R$ is the maximal right quotient ring and $S^0 R$ is a regular self-injective ring.

The R-module M is of *finite (Goldie) dimension* over R if every direct sum of nonzero submodules of M contains only a finite number of summands. We say that R has right finite (Goldie) dimension if R_R has finite (Goldie) dimension. If R is a right nonsingular right finite dimensional ring, then $S^0 R$ is semisimple Artinian and $S^0 R$ is left flat by [26].

Any hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ corresponds to an idempotent kernel functor σ. (See [12], [16] or [30] for the appropriate definitions and details.) If R is right nonsingular, then the class of all singular right R-modules forms the torsion class of a torsion theory which is usually known as the Goldie theory. A right nonsingular ring R is said to have the *splitting property* (SP), if $Z(M)$ is a direct summand of M for every right R-module M. A ring R is said to have the *finitely generated splitting property* (FGSP), if $Z(M)$ is a direct summand of M for every finitely generated right R-module M.

If R is a ring, the Jacobson radical of R will be denoted by $J(R)$. The prime radical of a ring R is denoted by $N(R)$. A ring is *semiprimary* if $R/J(R)$ is semisimple Artinian and $J(R)$ is nilpotent. In this case, $N(R) = J(R)$.

A nonzero element of a ring R is a left (right) regular element if it has zero left (right) annihilator. A regular element is a left and right regular element. A ring Q is said to be a *classical right quotient ring* of R (R is called a right order in Q) if (i) $R \subseteq Q$, (ii) every element of Q has the form ac^{-1} when a, $c \in R$ with c a regular element in R, and (iii) every regular element of R has an inverse in Q. Classical left quotient rings are defined similarly.

An R-module M is *almost finitely generated* (AFG) if $M/\mathrm{soc}\,(M)$ is finitely generated. An R-module M is *almost finitely related* (AFR) if there exists an exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is a finitely generated free R-module and K is AFG. We refer the reader to [11] or [16] for further details.

Formal triangular matrix rings will play a central role in this paper. For a more detailed discussion of such rings see [15] or [17]. The following proposition is due to Goodearl [17, Proposition 3 and Proposition 4] or [15, Propositions 4.2 and 4.4].

PROPOSITION 1.1. *Let* $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ *where B is a $C-A$ bimodule and $_C B$ is faithful. Then the following statements hold:*

(a) *A right ideal I of R belongs to $\mathscr{S}(R)$ if and only if it contains a right ideal of the form* $\begin{bmatrix} J & 0 \\ K & 0 \end{bmatrix}$ *where $J \in \mathscr{S}(A)$ and K_A is essential in B_A.*

(b) *R_R is nonsingular if and only if A_A and B_A are both non-singular.*

(c) $\mathrm{soc}\,(R_R) = \begin{bmatrix} \mathrm{soc}\,(A_A) & 0 \\ \mathrm{soc}\,(B_A) & 0 \end{bmatrix}.$

(d) *If R_R is nonsingular, then* $S^0 R = \begin{bmatrix} S^0 A & X \\ S^0 B & T \end{bmatrix}$ *where $X = \mathrm{Hom}_A(S^0 B, S^0 A)$ and C is a unital subring of $T = \mathrm{End}_A(S^0 B)$.*

2. **Rings with essential socle and FGSP.** In this section a characterization is given of right nonsingular rings R with essential right socle having FGSP which either are right finite dimensional or possess a semiprimary classical right quotient ring. We will use the methods of [8] to give a matrix representation of such rings which will be quite explicit if R is right finite dimensional and has a semiprimary classical right quotient ring.

The following result due to Goodearl [16, Theorem 4.9] specialized to the case when R has essential right socle is the starting point of our investigation.

THEOREM 2.1. *Let R be a right nonsingular ring with essential right socle. Then R has FGSP if and only if*

(a) *$\{r \in R : xr \in A\}$ is AFG for any $x \in S^0 R$ and any finitely generated submodule A of $(S^0 R)_R$.*

(b) *If I is any left ideal of R containing $\mathrm{soc}\,(R_R)$, then I is flat and $\mathrm{Tor}_1{}^R(S^0 R, R/I) = 0$.*

If R is right nonsingular with essential right socle which either is right finite dimensional or possesses a semiprimary classical right quotient ring, then $\mathrm{soc}\,(R_R) = Re$ for $e^2 = e$ by Gordon [20, Theorem 1.2]. Let $f = 1 - e$. Then R is isomorphic to $\begin{bmatrix} eRe & 0 \\ fRe & fRf \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ where A a semisimple Artinian, $_C B_A$ is a $C - A$-bimodule faithful as a C-module. By Proposition 1.1 $S^0 R$ is isomorphic to $\begin{bmatrix} A & X \\ B & T \end{bmatrix}$ where $X = \mathrm{Hom}_A(B, A)$ and $T = \mathrm{End}_A(B) = f(S^0 R)f$. We will use

this notation throughout this section.

We proceed with a sequence of lemmas, the first of which is [8, Lemma 3.4] included here for completeness.

LEMMA 2.2. $\mathrm{Tor}_1{}^R(S^0R, R/I) = 0$ *for all left ideals* I *of* R *contained in* $C = Rf$ *if and only if* T *is a right* C-*flat overring of* C.

LEMMA 2.3. *Let* S *be a semihereditary ring and suppose that* S *is embedded in a left finite dimensional ring* U *such that* U *is right* S-*flat. Then* S *possesses a left Artinian classical left quotient ring.*

Proof. By Gordon and Small [21], it is sufficient to show that S is left finite dimensional. Since U_S is S-flat, $U(L \cap K) = UL \cap UK$ for left ideals K and L of S by [3, Lemma 1.10]. From this it follows that if $I_1 \oplus \cdots \oplus I_n$ is a direct sum of left ideals of S, then $UI_1 \oplus \cdots \oplus UI_n$ is a direct sum of left ideals of U. Since U is left finite dimensional, then S is left finite dimensional.

REMARK. Semihereditary rings with semiprimary classical left quotient rings are characterized by [10, Theorem 3.7].

Using the notation that has been prescribed, we have the following lemma.

LEMMA 2.4. *If* T *is right* C-*flat and* C *possesses a classical left quotient ring* Q', *then* $Q' \subseteq T$.

Proof. First we show that a regular element d of C is a regular element of T. Since T is semisimple Artinian, it suffices to show that the left annihilator of d in T is zero. For let $td = 0$ for $t \in T$. Since T is right C-flat, by [2, Problem 5, p. 122] there exists $\{t_j\} \subseteq T$, $\{\lambda_j\} \subseteq C$ such that $t = \sum t_j \lambda_j$ and $\lambda_j d = 0$ for all j. Since d is a regular element of C, $\lambda_j = 0$ for all j and $t = 0$.

Thus every regular element of C is invertible in T since T is semisimple Artinian. Hence $C \subseteq Q' \subseteq T$.

Now we state the first main result of this section.

THEOREM 2.5. *Let* R *be a right nonsingular right finite dimensional ring which has an essential right socle and FGSP. Then* R *is isomorphic to* $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ *where*

(i) A *is a semisimple Artinian ring;*

(ii)   *B is a C − A bimodule finitely generated as an A module;*

(iii)   *C is a semihereditary ring possessing a left Artinian classical left quotient ring $Q' \subseteq T = \mathrm{End}_A(B)$ such that T is a right flat Q'-module.*

Conversely, any such ring is a right nonsingular right finite dimensional ring which has an essential right socle and FGSP.

Proof. Let R satisfy the hypothesis of the theorem. By the remarks following Theorem 2.1 we may assume that R is isomorphic to $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} eRe & 0 \\ fRe & fRf \end{bmatrix}$ where A is semisimple Artinian, B is a $C − A$ bimodule and $C \subseteq T = \mathrm{End}_A(B)$. Since R is right finite dimensional, B is finitely generated as a right A-module.

Now we show that C is semihereditary and T is right C-flat. Let I be a left ideal of C. Then I is a left ideal of R contained in Rf; so by Theorem 2.1, $Re \oplus I$ is a flat left ideal of R. Hence I is R-flat and by [9; Proposition 2.1] I is C-flat. Since R_R is finite dimensional, $T = \mathrm{End}_A(B)$ is semisimple Artinian. As C is a unital subring of T, by [23, Corollary 3.2] any finitely generated C-flat module is C-projective. It then follows that C is left semihereditary; hence by Small [28, Theorem 3] C is semihereditary. By Theorem 2.1 (b) we have that $\mathrm{Tor}_1^R(S^0R, R/(Re \oplus I)) = 0$ for all left ideals I of C. Since $R = Re \oplus Rf$, $R/I \cong Re \oplus Rf/I$; but $Rf/I \cong R/(Re \oplus I)$. It follows that $\mathrm{Tor}_1^R(S^0R, R/I) = 0$ since Re is left R-projective. By Lemma 2.2, T is right C-flat.

By Lemma 2.3, C has a left Artinian classical left quotient ring Q' and by Lemma 2.4 $Q' \subseteq T$. Since Q' is a classical left quotient ring of C, T is right Q'-flat by [30, Proposition 3.11, p. 232].

Conversely, let R be as in the conclusion of the theorem. It follows from the triangular structure of R that R is a right nonsingular right finite dimensional ring with essential right socle. In order to show that R has FGSP we will show that R satisfies conditions (a) and (b) of Theorem 2.1. To see (b) let I be a left ideal of R containing $\mathrm{soc}\,(R_R) = Re$. Then $I = Re \oplus If$. Thus If is a left ideal of C; so If is C-flat since C is semihereditary. Therefore If is R-flat by [9, Proposition 2.1] and I is R-flat. Since Q' is a classical left quotient ring of C and $T_{Q'}$ is Q'-flat, it follows that T_C is C-flat. By Lemma 2.2, $\mathrm{Tor}_1^R(S^0R, R/If) = 0$ and it follows that $\mathrm{Tor}_1^R(S^0R, R/I) = 0$. Thus (b) holds.

To establish (a), by [16, Proposition 4.8] it is equivalent to show that all finitely generated submodules M of $(S^0R)_R$ are AFR. Furthermore, by the discussion of [16, p. 65] it is sufficient to show that M/Me is C-finitely related. Now

$$S^0R = \begin{bmatrix} e(S^0R)e & e(S^0R)f \\ f(S^0R)e & f(S^0R)f \end{bmatrix}$$

and $e(S^0R)f$ is isomorphic to a direct sum of direct summands of $T = f(S^0R)f$ as a right T-module. Since T is right C-flat, then $e(S^0R)f$ is right C-flat; hence $(S^0R)f$ is a right C-flat module. M/Me is a finitely generated submodule of $(S^0R)f$ and hence is C-flat as C is semihereditary. Therefore M/Me is C-projective since finitely generated flat C-modules are C-projective by [23, Corollary 3.2]. Thus M/Me is finitely related as a C-module; so M is AFR.

REMARK. If $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ satisfies the conditions of the theorem, then B is a left Q'-module as $Q' \subseteqq T = \mathrm{End}_A(B)$. Hence $\begin{bmatrix} A & 0 \\ B & Q' \end{bmatrix}$ is a semiprimary ring with $T_{Q'}$ right Q'-flat. Thus by [8, Theorem 3.5], $\begin{bmatrix} A & 0 \\ B & Q' \end{bmatrix}$ is a splitting ring.

We now turn our attention to right nonsingular rings with essential right socle which possess a semiprimary classical right quotient ring. As we have observed earlier, $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} eRe & 0 \\ fRe & fRf \end{bmatrix}$ where A is a semisimple Artinian ring and $_CB_A$ is a bimodule faithful as a C-module. In addition, finitely generated flat $R(C)$-modules are $R(C)$-projective by [23, Corollary 3.2]. A result similar to Theorem 2.5 holds in this case; however, we first need the following lemma which will be useful later in this paper.

LEMMA 2.6. *Let R be a right nonsingular ring isomorphic to*

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

where A is a right finite dimensional semiprime ring, B is a $C - A$ bimodule faithful as a left C-module and C has a semiprimary classical right quotient ring Q' such that $Q' \subseteqq T = \mathrm{End}_A(S^0B)$. Then R has a semiprimary classical right quotient ring Q where

$$Q = \begin{bmatrix} S^0A & 0 \\ S^0B & Q' \end{bmatrix}.$$

Proof. The hypothesis implies that Q is a semiprimary ring. It remains to show that Q is a classical right quotient ring of R. Let $d = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ be a regular element of R where $a \in A$, $b \in B$, and $c \in C$. Then a and c are regular elements of A and C respectively; so they have inverses x and y in S^0A and Q' respectively as S^0A

and Q' are classical right quotient rings of A and C. Since S^0B is a left Q'-module and c is invertible in Q', $cS^0B = S^0B$. Hence there exists $w \in S^0B$ such that $cw + bx = 0$. Then

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} x & 0 \\ w & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and it follows that d is invertible in Q.

Let $x = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in Q$ where $a \in S^0A$, $b \in S^0B$, and $c \in Q'$. Since $B(S^0A) = S^0B$ and since A and C have classical right quotient rings S^0A and Q' respectively, $a = a_1 d_1^{-1}$, $b = b_1 d_2^{-1}$, and $c = c_1 d_3^{-1}$ where $a_1 \in A$, $b_1 \in B$, $c_1 \in C$, d_1 and d_2 are regular elements in A while d_3 is a regular element of C. Let d be a common left multiple of d_1^{-1} and d_2^{-1}; i.e., $d_1^{-1}d$ and $d_2^{-1}d \in A$. Then

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & d_3 \end{bmatrix} \in R \;.$$

Since $\bar{d} = \begin{bmatrix} d & 0 \\ 0 & d_3 \end{bmatrix}$ is a regular element in R, $\lambda = r\bar{d}^{-1}$ for $\bar{d}$ a regular element of R and some $r \in R$. Hence Q is the classical right quotient ring of R.

We have the following analogue to Theorem 2.5.

PROPOSITION 2.7. *Let R be a right nonsingular ring with essential right socle which possesses a semiprimary classical right quotient ring Q and FGSP. Then R is isomorphic to $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ where*

(i)   *A is a semisimple Artinian ring;*

(ii)   *B is a $C - A$ bimodule;*

(iii)   *C is a semihereditary ring possessing a semiprimary classical right quotient ring $Q' \subseteq T = \mathrm{End}_A(B)$ and T is a flat right C-module.*

Conversely any such ring R is a right nonsingular ring with essential right socle which possesses a semiprimary classical right quotient ring and FGSP.

Proof. Let R satisfy the hypothesis of the theorem. Then the techniques used in proving Theorem 2.5 can be used to show R satisfies all of (i)-(iii) except that C possesses a semiprimary classical right quotient ring $Q' \subseteq T$. But this follows from the fact that $C = fRf$, $Q \subseteq S^0R$ and fQf is a semiprimary classical right quotient ring of C by [18, Theorem A].

For the converse, again the techniques of Theorem 2.5 apply to show that R is a right nonsingular ring with essential right socle and FGSP. By Lemma 2.6, R possesses a semiprimary classical right quotient ring.

Using the fact that a ring which possesses a classical left quotient ring and a classical right quotient ring possesses a classical two-sided quotient ring, Theorem 2.5 and Proposition 2.7 combine to give the following.

THEOREM 2.8. *Let R be a right nonsingular right finite dimensional ring with essential right socle and FGSP which possesses a semiprimary classical right quotient ring. Then R is isomorphic to $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ where*
(i) *A is a semisimple Artinian ring;*
(ii) *B is a $C - A$ bimodule finitely generated as an A-module;*
(iii) *C is a semihereditary ring possessing a left Artinian two-sided classical quotient ring $Q' \subsetneqq T = \operatorname{End}_A(B)$ such that T is a right Q'-flat.*
Conversely any such ring R is a right nonsingular right finite dimensional ring with essential right socle and FGSP which possesses a semiprimary classical right quotient ring.

Suppose that R satisfies either Theorem 2.5 or Theorem 2.8 and has homogeneous right socle. Then S^0R is a finite dimensional matrix ring over a division ring and it follows that T is also. Now Q' is a hereditary ring since C is semihereditary; also T is right Q'-flat. Hence by [10, Theorem 2.1] Q' has a very explicit structure in terms of matrix rings over division rings which in turn induces an explicit structure on C by [10, Theorem 3.7 and Theorem 3.11]. Thus a quite explicit representation can be obtained if R has homogeneous socle. Now we will give an explicit subdirect representation of R in terms of rings with homogeneous socle.

If R is the subdirect product of R_1 and R_2, we define E_1 and E_2 by the relations $E_1 \times 0 = R \cap (R_1 \times 0)$ and $0 \times E_2 = R \cap (0 \times R_2)$. E_1 and E_2 are two-sided ideals of R_1 and R_2 respectively and R is said to be a (right) *split product* of R_1 and R_2 if E_1 and E_2 are direct summands of R_1 and R_2 respectively as right ideals. For further details we refer the reader to [13].

Let R be a right finite dimensional right nonsingular ring with essential right socle. Then there exists orthogonal idempotents $\{e_1, \cdots, e_t\}$ such that $Re_1 \oplus \cdots \oplus Re_t = \operatorname{soc}(R_R)$ and Re_i is homogeneous for $i = 1, \cdots, t$. By Gordon [19, Theorem 2.2] R is a unique subdirect product of rings $R_1, \cdots, R_t$ such that R_i has essential

homogeneous right socle isomorphic to Re_i for $i = 1, \cdots, t$. The minimal right ideals of R_i are R-isomorphic to the minimal right ideals of R by [19, p. 312]; hence each R_i is a right nonsingular right R-module. Now $S^0R \cong S^0R_1 \times \cdots \times S^0R_t$ (ring direct product). For any $X \subset \{1, \cdots, t\}$ let π denote the projection map of S^0R onto $\bigtimes_{i \in X} S^0R_i$. Let $R_X = \pi_X(R)$, $e = e_1 + \cdots + e_t$ and $f = 1 - e$. Let $e_X = \pi_X(e)$ and $f_X = \pi_X(f)$. Letting T and C be as in Theorem 2.5, set $T_X = f_X T f_X$ and $C_X = f_X R f_X$. For a partition X and Y of $\{1, \cdots, t\}$, R is a subdirect product of R_X and R_Y while C is a subdirect product of C_X and C_Y. Let E_X and E_Y be the two-sided ideals of R_X and R_Y respectively by defining $E_X \times 0 = (R_X \times 0) \cap R$ and $0 \times E_Y = (0 \times R_Y) \cap R$. Let $E'_X = f_X E_X f_X$ and $E'_Y = f_Y E_Y f_Y$. Using this notation, we have the following theorem.

THEOREM 2.9. *Let R be a right nonsingular right finite dimensional ring with essential right socle. Then the following statements are equivalent*:

(i) *R has FGSP*;

(ii) *For any partition X and Y of $\{1, \cdots, t\}$*

(a) *R_X and R_Y are right nonsingular right finite dimensional rings with essential right socle and FGSP*;

(b) *C is a right split product of C_X and C_Y.*

If, in addition, R possesses a semiprimary classical right quotient ring, then R has FGSP if and only if, in addition, R_X and R_Y both possess semiprimary classical right quotient rings.

Proof. (i) $\Rightarrow$ (ii): It follows from the construction of R_X and R_Y that R_X and R_Y are right nonsingular right finite dimensional rings with essential socle. It follows from what we have observed that the right socles of R_X and R_Y are R-nonsingular; so R_X and R_Y are R-nonsingular. Since R_X and R_Y are homomorphic images of R, R_X, and R_Y have FGSP by [16, Proposition 1.11].

Since $S^0R = S^0R_X \times S^0R_Y$, it follows that $T = T_X \times T_Y$. Since T is right C-flat and C is semihereditary by Theorem 2.5, then C is a split product of C_X and C_Y respectively as right ideals by [9, Proposition 3.8].

(ii) $\Rightarrow$ (i): To show that R has FGSP, it suffices to show that C is semihereditary and T_C is C-flat by Theorem 2.5. By (ii) (a) and Theorem 2.5, C_X and C_Y are semihereditary and T_X and T_Y are respectively right C_X and C_Y-flat. Since C is a split product of C_X and C_Y, E'_X and E'_Y are direct summands of C_X and C_Y respectively as right ideals. By [9, Proposition 2.1] T_X is right C-flat since $C_X = C/(0 \times E'_Y)$ and $0 \times E'_Y$ is a direct summand of C as a right

ideal. Similarly T_Y is right C-flat. Hence $T = T_X \times T_Y$ is C-flat. By [9, Theorem 3.10] C is semihereditary and R has FGSP.

In order to show the last statement let, in addition, R have a semiprimary classical right quotient ring Q and FGSP. By Lemma 2.6 to show that R_X and R_Y have semiprimary classical right quotient rings it suffices to show that C_X and C_Y have semiprimary classical right quotient rings $Q_X \subsetneq T_X$ and $Q_Y \subsetneq T_Y$. By [18, Theorem A] C has a semiprimary classical right quotient ring $Q' \subsetneq T$. Since C is a split product of C_X and C_Y by (ii), then C_X and C_Y have semiprimary classical right quotient rings Q'_X and Q'_Y where Q' is the split product of Q'_X and Q'_Y by [9, Theorem 3.17]. Hence it follows that $Q'_X \subsetneq T_X$ and $Q'_Y \subsetneq T_Y$ as desired.

Now assume that R_X, R_Y, C_X, C_Y, satisfy the conditions of the theorem and R_X, R_Y have semiprimary classical right quotient rings Q_X and Q_Y. By Theorem 2.8, C_X and C_Y have semiprimary classical right quotient rings $Q'_X \subsetneq T_X$ and $Q'_Y \subsetneq T_Y$ respectively. C is a split product of C_X and C_Y; so by [9, Theorem 3.17] C has a semiprimary classical right quotient ring Q' which is a split product of Q'_X and Q'_Y. Since $T = T_X \times T_Y$, $Q' \subsetneq T$. Hence R has a semiprimary classical right quotient ring by Lemma 2.6.

We conclude this section with some examples. The first is an example which satisfies Theorem 2.7 but not Theorem 2.5. Let R be the ring

$$\begin{bmatrix} Q & 0 \\ R & Z \end{bmatrix}.$$

Where Q is the rational numbers, R is the real numbers and Z is the ring of integers. Then R has semiprimary classical right quotient ring $\begin{bmatrix} Q & 0 \\ R & Q \end{bmatrix}$ but is clearly not right finite dimensional. Note that R has homogeneous right socle.

The second example is an example of a ring which satisfies the conditions of Theorem 2.5 but does not have a semiprimary classical right quotient ring. Let F be a field and x an indeterminate over F. As usual $F[x]$ is the ring of polynomials over F and $F(x)$ the quotient field of $F[x]$. By [18, p. 42–43] the ring

$$R = \begin{bmatrix} F(x) & 0 & 0 \\ F(x) & F & 0 \\ F(x) & F[x] & F[x] \end{bmatrix}$$

possesses a semiprimary classical left quotient ring but not a semi-primary classical right quotient ring. The ring $C = \begin{bmatrix} F & 0 \\ F[x] & F[x] \end{bmatrix}$

has a classical left quotient ring $Q' = \begin{bmatrix} F & 0 \\ F(x) & F(x) \end{bmatrix}$ and $T = \begin{bmatrix} F(x) & F(x) \\ F(x) & F(x) \end{bmatrix}$ is right Q'-flat. Hence R satisfies the conditions of Theorem 2.5. Again R has homogeneous socle.

3. **Rings with zero socle and FGSP.** In this section the structure of right nonsingular rings with zero socle and FGSP is examined. It is shown that if R is a right nonsingular ring with zero socle and FGSP, then R is right finite dimensional if and only if R possesses a semiprimary classical right quotient ring. This is contrary to the situation in the previous section. Then the structure of right nonsingular right finite dimensional rings having zero socle and FGSP is given as a certain triangular matrix ring and reduces the problem of characterizing rings with zero socle and FGSP to studying semiprime rings. Finally these results are applied to certain classes of rings to obtain more explicit results. Many of the ideas of this section were motivated by the ideas of Goodearl's paper [17].

We begin by studying the structure of right nonsingular rings R which have FGSP, zero socle and a semiprimary classical right quotient ring. We need the follwing technical lemma which is a slight modification of [17, Lemma C].

LEMMA 3.1. *Let R have zero socle and FGSP, and let M be a two-sided ideal in $L^*(R)$ whose left annihilator H is an essential right ideal of R. Then M is a direct summand of R as a right ideal.*

Proof. Since $M/(M \cap H) \cong (M + H)/H$ is right singular and R/M is right nonsingular, by FGSP the exact sequence

$$0 \longrightarrow M/(M \cap H) \longrightarrow R/(M \cap H) \longrightarrow R/M \longrightarrow 0$$

splits. Then there exists an $m \in M$ such that $mR + M \cap H = M$ and $m^2 - m \in M \cap H$. Then it follows that $m^3 - m \in M \cap H$; so that $m^4 - m^2 = 0$. Hence $m^2 = e$ is an idempotent of R. Now $eR + M \cap H = M + M \cap H = M$. Therefore $eM = M^2$ multiplying the previous equation on the right by M. Hence $eR = M^2$ and M^2 is a direct summand of R_R.

If $M^2 \neq M$, by [17, Lemma B] M_R must have a proper essential submodule K which contains M^2. It follows that $M/K = Z(R/K)$ and M/K is a direct summand of R/K. Thus there is a right ideal J of R such that $M + J = R$ and $M \cap J = K$. Since $M = M^2 + JM$, we have that $M \subseteq J$. But this implies that $M = K$, a contradiction.

Let R be a right nonsingular ring with zero socle having FGSP and suppose that R possesses a semiprimary classical right quotient ring Q. Then Q is a right nonsingular ring and $\operatorname{soc}(Q_Q) = Qe$ for an idempotent e of Q. If $f = 1 - e$, then Q is isomorphic to

$$\begin{bmatrix} eQe & 0 \\ fQe & fQf \end{bmatrix}.$$

Let $M = R \cap fQ$. fQ is a two-sided ideal of Q; so M is a two-sided ideal of R. $Qe \subsetneqq l_Q(fQ)$; so $R \cap Qe \subsetneqq l_R(M)$. Hence $H = l_R(M)$ is essential as a right ideal of R. Furthermore $R/M \cong R/(R \cap fQ) \cong (R + fQ)/fQ$; so R/M is a right nonsingular R-module. Hence by Lemma 3.1, $M = f'R$ for f' an idempotent element of R. Since M_R is essential in fQ, it follows that $fQ = f'Q$. Let $e' = 1 - f'$. Now $Q = eQ \oplus fQ$; so $eQ = e'Q$. Since $e'Qf' \subsetneqq e'Qf'Q = eQfQ = 0$, $Q = \begin{bmatrix} e'Qe' & 0 \\ f'Qe' & f'Qf' \end{bmatrix}$. Since $eQ = e'Q$, then $e'Qe' = eQe$. Since $fQe = (fQ)(eQe) = (f'Q)(e'Qe') = f'Qe'$, then $Qe = eQe + fQe = e'Qe' + f'Qe' = Qe'$. It follows that we can take the idempotents e and f of Q to be idempotent elements of R as well and that

$$R = \begin{bmatrix} eRe & 0 \\ fRe & fRf \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}.$$

Since fQe is a faithful fQf-module, it follows that B is a faithful left C-module. Hence by Proposition 1.1 we have

$$S^0R = \begin{bmatrix} S^0A & X \\ S^0B & T \end{bmatrix}$$

where $T = \operatorname{End}_A(S^0B)$ and C is a unital subring of T. Since Q is the classical right quotient ring of R, we may assume that $R \subsetneqq Q \subsetneqq S^0R$ and

$$Q \cong \begin{bmatrix} S^0A & 0 \\ S^0B & Q' \end{bmatrix}$$

where $Q' = fQf \subsetneqq T$. We will assume that R, Q, f, e and S^0R are as described and that R has FGSP for Lemmas 3.2 through 3.4.

Applying the proof of [17, Lemma D] to the idempotent f and noting that the proof of [17, Lemma D] only uses FGSP, we have the following.

LEMMA 3.2. $S^0C = T$.

LEMMA 3.3. *C is semihereditary and T is the two-sided maximal quotient ring of C.*

Proof. C has a semiprimary classical right quotient ring by [18, Theorem A] and hence finitely generated flat left and right C-modules are C-projective. Since Re is essential in R_R, it follows from [16, Proposition 4.2] that C is semihereditary. It also follows from Lemma 2.2 that T is right C-flat. Since T is the maximal right quotient ring of C by Lemma 3.2, every finitely generated nonsingular right C-module can be embedded in a finite direct sum of copies of T. Hence any finitely generated nonsingular right C-module is C-projective. Therefore by [4, Theorem 2.3] $Z(T \otimes_C T) = 0$; so $T \otimes_C T \cong T$. Thus by [5, Proposition 2.2], T is a left quotient ring of C.

Let U be the maximal left quotient ring of C chosen so that $C \subseteq T \subseteq U$. Then U is a left flat C-module by [3, Theorem 2.1]; so again by the argument used in the previous paragraph, every finitely generated nonsingular left C-module is C-projective. Thus by [4, Theorem 2.3] $Z(U \otimes_C U) = 0$ and by [5, Proposition 2.2] U is a right quotient ring of C. Therefore it follows that $U \subseteq T$ or $U = T$.

LEMMA 3.4. *T is semisimple Artinian and C has a two-sided Artinian classical two-sided quotient ring Q'.*

Proof. Since S^0R is a direct product of full linear rings being the maximal quotient ring of the semiprimary ring Q, T is the direct product of full linear rings as well. By Lemma 3.3, T is two-sided self-injective and T is semisimple Artinian by Osofsky [24]. Hence the classical right quotient ring Q' of C exists and is a two-sided classical quotient ring as well as two-sided Artinian by Lemma 2.3.

REMARK. Theorem 3.14 of [10] gives a complete characterization of C and Q'.

We now show that for right nonsingular rings R with zero socle and FGSP, R_R is finite dimensional if and only if R possesses a semiprimary classical right quotient ring.

PROPOSITION 3.5. *Let R be a right nonsingular ring with zero socle and FGSP. Then R_R is finite dimensional if and only if R possesses a semiprimary classical right quotient ring.*

Proof. Suppose that R has a semiprimary classical right quotient ring Q. Then we know that $R = \begin{bmatrix} eRe & 0 \\ fRe & fRf \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ where

$Qe = \mathrm{soc}\,(Q_Q)$. Since Q is semiprimary, $eQe = S^0A$ is semisimple Artinian. Now T is semisimple Artinian by Lemma 3.4; so it follows that S^0R is semisimple Artinian. Consequently R_R is finite dimensional by [26, Theorem 1.6].

Conversely let R_R be finite dimensional. Let $\mathcal{A}$ be the set of two-sided ideals in $L^*(R)$ whose left annihilator is essential as a right ideal. Since R_R is finite dimensional, $L^*(R)$ has ACC; hence $\mathcal{A}$ has a maximal element, say M. R/M is a semiprime ring by the proof of Theorem 2 in [17].

By Lemma 3.1, $M = fR$ for an idempotent f. Set $e = 1 - f$. Then Re is the left annihilator of M so R is isomorphic to $\begin{bmatrix} eRe & 0 \\ fRe & fRf \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ where A is semiprime and $_CB$ is faithful. Thus by Proposition 1.1, C is a unital subring of $T = \mathrm{End}_A(S^0B)$. The proofs of Lemmas 3.3 and 3.4 still apply to show that T is the two-sided maximal quotient ring of the semihereditary ring C and T is semisimple Artinian. Hence by Lemma 2.3, C has a two-sided Artinian two-sided classical quotient ring Q'. Therefore R possesses a semiprimary classical right quotient ring by Lemma 2.5.

REMARK. We have actually shown that the classical right quotient ring of R is right Artinian.

If $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ is a formal triangular matrix ring, for convenience we label the following two-sided ideals of R:

$$R_{12} = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \text{ and } R_{23} = \begin{bmatrix} 0 & 0 \\ B & C \end{bmatrix}.$$

The following theorem is the main result of this section.

THEOREM 3.6. *Let R be a right nonsingular right finite dimensional ring with zero socle and FGSP. Then R is isomorphic to*

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

where

(i) *A is a right nonsingular right finite dimensional semiprime ring with zero socle and FGSP;*

(ii) *B is a $C - A$ bimodule such that B_A is nonsingular and finite dimensional;*

(iii) *C is a semihereditary ring possessing a two-sided Artinian*

two-sided classical quotient ring Q' such that $Q' \subseteq T = \operatorname{End}_A(S^0B)$ and T is the two-sided maximal quotient ring of C;

(iv) $\operatorname{Ext}^1_A(Z(N/NR_{23}), W) = 0$ *where N is any finitely generated nonsingular right R-module and W is any singular A-module.*

Conversely, any such matrix ring is a right nonsingular right finite dimensional ring with zero socle and PGSP.

Proof. Let R have FGSP. Since $A = R/R_{23}$, A_R is right nonsingular. Hence the singular submodule of any right A-module is the same whether considered as an A-module or an R-module. Hence it follows that A has FGSP. The remainder of the conditions of (i)–(ii) follow from Proposition 1.1, Lemma 3.1 and the fact that R_R is finite dimensional. Condition (iii) is a consequence of Lemmas 3.3 and 3.4.

To see condition (iv), let N be any finitely generated right nonsingular R-module and W a singular A-module. Then W is R-singular; so by FGSP the exact sequence

$$0 \longrightarrow W \longrightarrow X \longrightarrow N \longrightarrow 0$$

splits. Since tensoring preserves direct sums, we have the following split exact sequence of A-modules,

$$0 \longrightarrow W \longrightarrow X/XR_{23} \longrightarrow N/NR_{23} \longrightarrow 0 .$$

Since A has FGSP, $N/NR_{23} = Z(N/NR_{23}) \oplus U$. Hence we have $0 \cong \operatorname{Ext}^1_A(N/NR_{23}, W) \cong \operatorname{Ext}^1_A(Z(N/NR_{23}), W)$.

The proof of the converse will require a sequence of lemmas. For Lemmas 3.7 through 3.10, we will assume that $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ which satisfies (i)–(iv) of Theorem 3.6.

LEMMA 3.7. *$_CB$ is flat and $_RR_{23}$ is flat.*

Proof. First we will show that $_CB$ is nonsingular. S^0B is a left T-module and since T is semisimple Artinian, S^0B is T-nonsingular. T is the maximal left quotient ring of C; so S^0B is C-nonsingular as well. Hence $_CB$ is nonsingular.

Since T is the two-sided maximal quotient ring of C, any finitely generated nonsingular left C-module can be embedded into a finite direct sum of copies of T. Since C is semihereditary, by [3, Theorem 2.1] $_CT$ is flat. Therefore any finitely generated nonsingular left C-module is flat. By a direct limit argument, any nonsingular left C-module is flat; hence $_CB$ is flat.

Since $_CB = {}_RB$, B is a flat R-module by [9, Proposition 2.1]. C

is a direct summand of $_RR$; so $_RC$ is flat. As left R-modules, $R_{23} = B \oplus C$; so R_{23} is left R-flat.

The next lemma is motivated by [17, Lemma S].

LEMMA 3.8.   *Let* $K \in L^*(B^n)$ *and* $J = \{x \in C^n : xB \subseteq K\}$. *Then* $JB = K$ *where* n *is a positive integer.*

Proof. Let $L = \{x \in T^n : x(S^0B) \subseteq S^0K\}$. Any element of J or L can be identified with an endomorphism of S^0B^n. Using this identification, we will show that $J \subseteq L$. Let $x \in J$. Since x maps B^n into K, x induces a homomorphism $\bar{x} : S^0B^n/B^n \to S^0B^n/K$. Now $S^0B^n = S^0K \oplus M$; so $S^0B^n/K \cong S^0K/K \oplus M$. Since S^0B^n/B^n is a singular module, $\bar{x}(S^0B^n/B^n) \subseteq S^0K/K$. Hence $x(S^0B^n) \subseteq S^0K$ and $x \in L$.

Thus the natural mapping $\eta : C^n/J \to T^n/L$ is a well defined homomorphism. J and L are right C-modules, and we claim that η is a monomorphism of right C-modules. For let $\eta(x + J) = 0$. Then $x(S^0B) \subseteq S^0K$. But $xB \subseteq B^n$; so $xB \subseteq B^n \cap S^0K$. Now $K \subseteq B^n \cap S^0K$ and $B^n \cap S^0K$ is an essential extension of K. But K is closed in B^n; so $K = B^n \cap S^0K$ and $xB \subseteq K$. Thus $x \in J$ and η is a monomorphism.

C is semihereditary and T is the two-sided maximal quotient ring of C; hence by [4, Theorem 2.3], $Z(T \otimes_C T) = 0$. Therefore by the proof of [5, Theorem 2.3], the mapping $T \otimes_C T \to T$ is an isomorphism. In addition, T is semisimple and T_C is flat. All of this along with the fact that $_CB$ is flat by Lemma 3.7 yield the following sequence of monomorphisms.

$$(C^n/J) \otimes_C B \longrightarrow (T^n/L) \otimes_C B \longrightarrow ((T^n/L) \otimes_T T) \otimes_C B$$
$$\longrightarrow (T^n/L) \otimes_T (T \otimes_C B)$$
$$\longrightarrow (T^n/L) \otimes_T (T \otimes_C S^0B)$$
$$\longrightarrow (T^n/L) \otimes_T (T \otimes_C (T \otimes_T S^0B))$$
$$\longrightarrow (T^n/L) \otimes_T ((T \otimes_C T) \otimes_T S^0B)$$
$$\longrightarrow (T^n/L) \otimes_T (T \otimes_T S^0B) \longrightarrow (T^n/L) \otimes_T S^0B .$$

Now $(C^n/J) \otimes_C B$ is isomorphic to B^n/JB while $(T^n/L) \otimes_T S^0B$ is isomorphic to $S^0B^n/L(S^0B)$. By the monomorphism of $(C^n/J) \otimes_C B$ into $(T^n/L) \otimes_T S^0B$ established in the preceeding paragraph, we have that there is a natural monomorphism α of B^n/JB into $S^0B^n/L(S^0B)$. Now the image of α is $(B^n + L(S^0B))/L(S^0B) \cong B^n/(B^n \cap L(S^0B))$. $JB \subseteq B^n \cap L(S^0B)$; so it follows that $JB = B^n \cap L(S^0B)$.

Finally we show that $JB = K$. Since S^0K is a direct summand of S^0B^n, there is an idempotent $n \times n$ matrix p over T such that $p(S^0B^n) = S^0K$. For $x \in K$, $x = u_1b_1 + \cdots + u_nb_n$ where $u_i \in T^n$, $b_i \in$

S^0B. Since $pu_i(S^0B) \subseteq p(S^0B^n) = S^0K$, $pu_i \in L$. Now $x = px$; so $x \in L(S^0B) \cap B^n = JB$. Thus $K \subseteq JB$; so $K = JB$.

LEMMA 3.9. *If N is a finitely generated nonsingular right R-module, then N/NR_{12} is a projective right R/R_{12}-module.*

Proof. Using the techniques of [17, Lemma R], N/NR_{12} can be embedded in a finite direct product of copies of $(S^0R)/(S^0R)R_{12}$. Now

$$(S^0R)/(S^0R)R_{12} = \begin{bmatrix} S^0A & X \\ S^0B & T \end{bmatrix} / \begin{bmatrix} S^0A & 0 \\ S^0B & 0 \end{bmatrix};$$

so it suffices to show that X_C and T_C are right R-flat. We have already observed that T_C is flat using condition (iii). X_C is isomorphic to direct sum of direct summands of T_C; so X_C is flat as C is semihereditary. Hence N/NR_{12} is C-flat as C is semihereditary and hence N/NR_{12} is R/R_{12}-projective.

The following lemma is essentially [17, Lemma T]. We include it here for completeness.

LEMMA 3.10. *If N is a finitely generated nonsingular right R-module, then $\mathrm{Tor}_1^R(N, R/R_{23}) = 0$.*

Proof. We will show that the mapping $N \otimes_R R_{23} \to N$ is a monomorphism. Since R_{23} is left R-flat by Lemma 3.7, the mapping $f: NR_{23} \otimes_R R_{23} \to N \otimes_R R_{23}$ is monic. Since R_{23} is idempotent, f is an isomorphism and it suffices to show that $NR_{23} \otimes R_{23} \to NR_{23}$ is monic.

Since N is finitely generated right nonsingular, $NR_{23} \cong R_{23}^n/H$ for some $H \in L^*(R_{23}^n)$ and for some positive integer n. Then it follows that $H = \begin{bmatrix} 0 & 0 \\ K & J \end{bmatrix}$ for some $K \in L^*(B^n)$ and $J \subseteq C^n$. Since H is a submodule of R_{23}^n, $JB \subseteq K$. Given $x \in C^n$ such that $xB \subseteq K$, we see that $\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} R_{12} \subseteq H$. Since R_{12} is essential in R_R, $\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \in H$ and $x \in J$. Therefore $J = \{x \in C^n : xB \subseteq K\}$ and by Lemma 3.8, $JB = K$.

Now $HR_{23} = \begin{bmatrix} 0 & 0 \\ JB & J \end{bmatrix} = H$; hence the mapping $H/HR_{23} \to R_{23}^n/(R_{23}^n)R_{23}$ is monic. Since R_{23}^R is right projective, $\mathrm{Tor}_1^R(R_{23}^n/H, R/R_{23}) = 0$; i.e., $\mathrm{Tor}_1^R(NR_{23}, R/R_{23}) = 0$. Thus $NR_{23} \otimes R_{23} \to NR_{23}$ is monic as required.

We now can return to the proof of the converse of Theorem 3.6.

Proof. (Converse of Theorem 3.6.) By the triangular structure

of R, R is a right nonsingular right finite dimensional ring with zero socle.

In order to show that R has FGSP, we must show that $\text{Ext}^1_R(N, W) = 0$ whenever N is a finitely generated right nonsingular R-module and W is a right singular R-module by [6, Proposition 1.11]. Since it suffices to show that $\text{Ext}^1_R(N, W/WR_{12}) = 0$ and $\text{Ext}^1_R(N, WR_{12}) = 0$, we may assume that either $WR_{12} = 0$ or $WR_{23} = 0$.

Case I. Let $WR_{12} = 0$. Consider any short exact sequence

$$E: 0 \longrightarrow W \longrightarrow V \longrightarrow N \longrightarrow 0$$

of right R-modules. Since R/R_{12} is left R-projective, we obtain a second short exact sequence

$$E^*: 0 \longrightarrow W \longrightarrow V/VR_{12} \longrightarrow N/NR_{12} \longrightarrow 0 .$$

By Lemma 3.9, N/NR_{12} is a projective right R/R_{12}-module; hence E^* splits and it follows that E splits.

Case II. Let $WR_{23} = 0$. Consider any short exact sequence

$$E: 0 \longrightarrow W \longrightarrow V \longrightarrow N \longrightarrow 0$$

of right R-modules. By Lemma 3.10, $\text{Tor}^R_1(N, R/R_{23}) = 0$; so we have a second exact sequence

$$E^*: 0 \longrightarrow W \longrightarrow V/VR_{23} \longrightarrow N/NR_{23} \longrightarrow 0 .$$

Since R_{23} is a two-sided ideal of $L^*(R)$, any singular submodule of any R/R_{23}-module is the same whether considered as an R/R_{23}-module or as an R-module. Hence W is a singular R/R_{23}-module. $R/R_{23} = A$ has FGSP by hypothesis; so $N/NR_{23} = Z(N/NR_{23}) \oplus U$ where U is a finitely generated nonsingular A-module. Hence by (iv) and the fact that A has FGSP, we have

$$\text{Ext}^1_R(N/NR_{23}, W) = \text{Ext}^1_A(N/NR_{23}, W)$$
$$\cong \text{Ext}^1_A(Z(N/NR_{23}), W) \oplus \text{Ext}^1_A(U, W) = 0 .$$

Therefore E^* splits and it follows that E splits.

Certain conditions on R guarantee the fact that condition (iv) of Theorem 3.6 is always satisfied. One of these conditions is that B_A is injective. For in this case, by the proof of [17, Lemma Q], NR_{23} is a direct summand of N for any finitely generated right nonsingular R-module N. Hence N/NR_{23} is A-nonsingular; so condition (iv) always holds if A has FGSP. The following proposition gives a necessary and sufficient condition for B_A to be injective.

We will assume in this proposition that $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ and A, B, C satisfy the conditions (i)–(iii) of Theorem 3.6.

PROPOSITION 3.11. B_A *is injective if and only if* B *is a left* Q'-*module.*

Proof. If B_A is injective, then since $T = \text{End}_A(S^0B) = \text{End}_A(B)$, B is a left T-module. Hence B is a left Q'-module.

Let B be a left Q'-module. Then by [16, Theorem 2.12] all nonsingular right Q'-modules are projective. Thus the proofs of Lemmas L, M, N and O of [17] can be modified to fit our situation since A_A is finite dimensional, and it follows that B_A is injective.

The following theorem shows that a semihereditary condition on R guarantees that B_A is injective.

THEOREM 3.12. *Let* R *be a right finite dimensional semihereditary ring with zero socle and FGSP. Then* R *is isomorphic to*

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

where

(i) A *is a right finite dimensional semihereditary semiprime ring with FGSP;*

(ii) C *is a semihereditary ring with a two-sided Artinian two-sided classical quotient ring* Q *such that* $Q' \subseteq T = \text{End}_A(S^0B)$ *and* T *is the two-sided maximal quotient ring of* C;

(iii) B *is a* $Q' - A$ *bimodule such that* B_A *is nonsingular.*

Conversely, any such ring is a right finite dimensional semihereditary ring with zero scole such that R *has FGSP.*

Proof. If R has FGSP, then conditions (i) and (ii) follow from Theorem 3.6 and the fact that R is semihereditary. By [10, Lemma 3.4], B is a left Q'-module since R has a right Artinian classical right quotient ring by Gordon and Small [21].

Conversely, by the remarks made prior to Proposition 3.11 and by Proposition 3.11, any ring satisfying the conditions of this theorem has FGSP, is right finite dimensional and has zero socle. Now by [10, Theorem 3.7] R is semihereditary since $_CB$ is flat by Lemma 3.7 and conditions (i) and (ii).

Another ring condition which forces B_A to be injective is the

commutativity of A.

COROLLARY 3.13. *Let* $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ *be a formal matrix ring satisfying* (i)–(iii) *of Theorem* 3.6. *If* A *is commutative and if* R *has FGSP, then* R *is semihereditary.*

Proof. By Theorem 3.12, we need to show that B is a left Q'-module. It suffices to show that $cB = B$ for all regular elements c of C. If $cB \neq B$, then there exists $\beta \in B$ such that $c^{-1}\beta \notin B$. Since $S^0 A$ is the classical quotient ring of A, there exists $\beta' \in B$ and b a regular element in A such that $c^{-1}\beta = \beta' b^{-1}$; hence $\beta b - c\beta' = 0$. Thus $\begin{bmatrix} 0 & 0 \\ \beta & -c \end{bmatrix}\begin{bmatrix} b & 0 \\ \beta' & 0 \end{bmatrix} = 0$.

Let H be the right annihilator of $x = \begin{bmatrix} 0 & 0 \\ \beta & -c \end{bmatrix}$ in R. H is a closed right ideal of R. For if $r \in R$ is in the closure of H in R, then $rL \subseteq H$ for some essential right ideal L of R. Thus $xrL = 0$; so $xr \in Z(R_R) = 0$. Hence $r \in H$ and H is closed.

Let $K = \begin{bmatrix} b^2 & 0 \\ \beta'b & 0 \end{bmatrix} R$. Since $\begin{bmatrix} b & 0 \\ \beta' & 0 \end{bmatrix} \in H$, we have $K \subseteq H$. Consider R/K. $H/K \neq 0$ as $\begin{bmatrix} b & 0 \\ \beta' & 0 \end{bmatrix} \in H - K$. To see this, suppose $\begin{bmatrix} b & 0 \\ \beta' & 0 \end{bmatrix} = \begin{bmatrix} b^2 & 0 \\ \beta'b & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \in K$. Then $b = b^2 x$; so $1 = bx$ as b is a regular element of A. Hence b is a unit in A; so $c^{-1}\beta = \beta b^{-1} \in B$, a contradiction.

Now

$$\left(\begin{bmatrix} b & 0 \\ \beta' & 0 \end{bmatrix} + K\right)/K \in Z(R/K) \subseteq H/K .$$

The second inclusion follows since H is closed in R. The first containment follows since bA is essential in A; so $\begin{bmatrix} b & 0 \\ \beta' & 0 \end{bmatrix}\begin{bmatrix} bA & 0 \\ B & 0 \end{bmatrix} \subseteq K$ and $\begin{bmatrix} bA & 0 \\ B & 0 \end{bmatrix}$ is an essential right ideal of R.

We now show that $Z(R/K)$ is not a direct summand of R/K. For if $Z(R/K)$ is a direct summand of R/K, then there exists $\begin{bmatrix} u & 0 \\ v & 0 \end{bmatrix} \in H$ and $\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \in R$ such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} b^2 a & 0 \\ \beta'ba & 0 \end{bmatrix} = \begin{bmatrix} u & 0 \\ v & 0 \end{bmatrix} + \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}.$$

It is immediate that $z = 1$. Also x is not in H' where H' is the projection of H in A. For if $x \in H'$, then $1 \in H'$; so there is an $\alpha \in B$ such that $\begin{bmatrix} 1 & 0 \\ \alpha & 0 \end{bmatrix} \in H$. Then

$$\begin{bmatrix} 0 & 0 \\ \beta & -c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \beta - c\alpha & 0 \end{bmatrix} = 0 ;$$

so $c^{-1}\beta = \alpha \in B$, a contradiction.

Now we show that $(bA + b^2A)/b^2A$ is essential in A/b^2A. Since A is a finite direct product of Prüfer domains, the ideal lattice of A is distributive by [22] or [1]. Hence if $a \in A$ is such that $(b^2A + aA) \cap bA = b^2A$, then $(b^2A \cap bA) + (aA + bA) = b^2A$ by distributivity. But $b^2A \cap bA = b^2A$; so $b^2A + (aA \cap bA) = b^2A$ and $aA \cap bA \subsetneqq b^2A$. Thus $ba = b^2a'$ for $a' \in A$; so $a = ba'$. Hence $aA \subsetneqq bA$. Consequently $(bA + b^2A)/b^2A$ is essential in A/b^2A.

Since $x + b^2A \notin (bA + b^2A)/b^2A$, there is a regular element d of A such that $xd + b^2A$ is a nonzero element of $(bA + b^2A)/b^2A$. As $z = 1$, there exists $\begin{bmatrix} d & 0 \\ w & 0 \end{bmatrix}$ such that $\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \begin{bmatrix} d & 0 \\ w & 0 \end{bmatrix} \in \begin{bmatrix} b & 0 \\ \beta' & 0 \end{bmatrix} R/K$. Thus $\left(\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} + K \right) R$ has nonzero intersection with $Z(R/K)$ which contradicts the assumption that $Z(R/K)$ is a direct summand of R/K. Thus we conclude that if R has FGSP, B is a left Q'-module.

If R satisfies Theorem 3.6, B_A need not be injective, for let A be a two-sided Noetherian prime ring with zero socle such that every singular right A-module is injective; that is, A is a SI-ring in the sense of Goodearl [16]. Then A has a two-sided classical (maximal) quotient ring and A is hereditary by Goodearl [16, Proposition 3.3]. Examples of such rings are given by Cozzens [7]. Then the ring $\begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ satisfies Theorem 3.6 but A_A is not injective.

We now investigate the situation when $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ satisfies (i)-(iii) of Theorem 3.6 and A is a two-sided hereditary Noetherian prime (HNP) ring. Let σ be an idempotent kernel functor for the category of right A-modules such that A is σ-torsion free; i.e., $\sigma(A) = 0$. Then σ has property T by Goldman [12, Theorem 4.4 and Theorem 4.5]. This means that if Q_σ is the ring of quotients of A relative to σ and $Q_\sigma(M)$ is the module of quotients for an A-module M, then $Q_\sigma(M) \cong M \otimes Q_\sigma$. A right A-module E is σ-injective if it has the property: if M is any right A-module and N is a submodule of M such that M/N is σ-torsion, then every A-homomorphism from N to E extends to a homomorphism of M to E. Any Q_σ-module is σ-injective by [12, Theorem 4.3]. For further details see [12] or [30].

Let $\mathscr{C} = \{S : S$ is a simple right A-module and $\text{Ext}_A^1 = (S, S_1) \neq 0$ for some simple right A-module $S_1\}$ and let σ be the torsion theory generated by the family $\mathscr{C}$. With this notation, we have the following proposition.

PROPOSITION 3.14. *Let* $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ *satisfy conditions* (i)-(iii) *of Theorem* 3.6 *and let* A *be a HNP ring. If* R *has FGSP, then* $B = Q_\sigma(B)$.

Proof. We need to show that S^0B/B is σ-torsion free; that is, S^0B/B has no simple submodule isomorphic to $S \in \mathscr{C}$. For if S^0B/B contains such a simple submodule, then there exists a $\beta \in S^0B/B$ such that $\beta J \subseteq B$ for J a maximal right ideal of A such that $A/J \cong S \in \mathscr{C}$. $J \subseteq \{a \in A : \beta a \in B\}$ and $\{a \in A : \beta a \in B\}$ is a proper right ideal of A; so $J = \{a \in A : \beta a \in B\}$.

Consider

$$H = \left\{ \begin{bmatrix} a & 0 \\ \beta a & 0 \end{bmatrix} : a \in J \right\} .$$

H is a closed right ideal of R. For suppose that $\begin{bmatrix} u & 0 \\ v & w \end{bmatrix}\begin{bmatrix} K & 0 \\ L & 0 \end{bmatrix} \subseteq H$ where K_A is essential in A_A and L_A is essential in B_A. Then $wL = 0$. Since L_A is essential in B_A and $w \in \mathrm{End}_A(S^0B)$, $w(S^0B) = 0$. Hence $w = 0$. Thus $\begin{bmatrix} u & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} K & 0 \\ L & 0 \end{bmatrix} \subseteq H$; so

$$\begin{bmatrix} u & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} uk & 0 \\ vk & 0 \end{bmatrix} \in H$$

for k a regular element of K. Thus $\beta uk = vk$; so $\beta u = v$ as k is a regular element of K. Thus $\begin{bmatrix} u & 0 \\ v & 0 \end{bmatrix} = \begin{bmatrix} u & 0 \\ \beta u & 0 \end{bmatrix} \in H$.

Let $N = R/H$. Then $N/NR_{23} \cong A/J \cong S$. Since R has FGSP, $0 = \mathrm{Ext}_A^1(N/NR_{23}, S_1) \cong \mathrm{Ext}_A^1(A/J, S_1)$ for all simple right R-modules S_1 by Theorem 3.6 (iv). This is a contradiction; so $B = Q_\sigma(B)$.

We have the following partial converse of Proposition 3.14.

LEMMA 3.15. *Let* $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ *satisfy conditions* (i)-(iii) *of Theorem* 3.6 *and* A *be a HNP ring. If* $B = Q_\sigma(B)$, *then* N/NR_{23} *is* σ-*torsion free for any finitely generated nonsingular right* R-*module* N.

Proof. If N/NR_{23} is not σ-torsion free, then there is a submodule N' of N such that $NR_{23} \subseteq N' \subseteq N$ and $N'/NR_{23} \cong S \in \mathscr{C}$. Since R_{23} is a direct summand of R_R and N is finitely generated, then NR_{23} is finitely generated. Therefore N' can be chosen to be finitely generated.

Let $N' = F/H$ where $F = R^n$ for some positive integer n and $H \in L^*(F)$. $H = \begin{bmatrix} U & 0 \\ V & J \end{bmatrix}$ where $U \subseteq A^n$, $V \subseteq B^n$ and $J \subseteq C^n$. F/H is nonsingular; so Fe/He is nonsingular where $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. For if $\begin{bmatrix} u & 0 \\ v & 0 \end{bmatrix} L \subseteq He \subseteq H$ where L essential in R_R, then $\begin{bmatrix} u & 0 \\ v & 0 \end{bmatrix} \in H$; so $\begin{bmatrix} u & 0 \\ v & 0 \end{bmatrix} \in He$. Thus we may assume that $H = \begin{bmatrix} U & 0 \\ V & 0 \end{bmatrix}$.

Let $Z = \left\{ \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix} : \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix} \in H \right\}$. Z is a closed submodule of F since $Z = H \cap FR_{23}$ and both H and FR_{23} are closed in F. Since $A^n/U \cong S$, pick $y \in A^n/U$ and set $J = \{a \in A : ua \in U\}$. Then $A/J \cong A^n/U \cong S$. For $a \in J$, define $f(a) = v + Z$ where $\begin{bmatrix} ya & 0 \\ v & 0 \end{bmatrix} \in H$. By the definition of Z, f is a well defined homomorphism of J into $V/Z \subseteq B^n/Z$. B^n is σ-injective and B^n/Z is σ-torsion free since F/Z is nonsingular. Hence by [12, Theorem 4.5], B^n/Z is σ-injective; so f can be extended to A. Therefore f is given as a left multiplication by an element $x \in B^n/Z$. Thus for $a \in J$, $f(a) = (x + Z)a \in V/Z$. Therefore $\left(\begin{bmatrix} y & 0 \\ x & 0 \end{bmatrix} + Z \right) J \subseteq H/Z$. $y \in U$; so $\begin{bmatrix} y & 0 \\ x & 0 \end{bmatrix} + Z \in H/Z$. But

$$\begin{bmatrix} y & 0 \\ x & 0 \end{bmatrix} + Z \in Z((F/Z)/(H/Z))$$

since

$$\left(\begin{bmatrix} y & 0 \\ x & 0 \end{bmatrix} + Z \right) \begin{bmatrix} J & 0 \\ B & 0 \end{bmatrix} \subseteq H/Z .$$

But $(F/Z)/(H/Z) \cong F/H$ is nonsingular, a contradiction. Thus N/NR_{23} is σ-torsion free.

Given a right ideal M in a ring T, the *idealizer* of M in T is the largest subring of T which contains M as a two-sided ideal. Let A be the idealizer of a HNP SI-ring A^* relative to a semimaximal right ideal of A^* (see [14] or [25] for details). By Goodearl [14, Theorem 9] A is a splitting ring. It follows from [25, Corollary 2.4] that if U is a simple right A-module, then there is a maximal right ideal M of A^* such that U is one of the following three types: (I) $U \cong A^*/M$; (II) $U \cong A^*/(A + M)$ or (III) $U \cong (A + M)/M$. Let $X = \{S : S$ is a simple right A-module of type I or III$\}$. Then $A^* = Q_\tau$ where τ is the idempotent kernel functor generated by the cyclic modules of the form A/I where A/I has a composition series with no composition factors isomorphic to $S \in X$. For details see [14].

It can be checked that the only nonsplit exact sequence of right

A-modules

$$0 \longrightarrow S_1 \longrightarrow Y \longrightarrow S \longrightarrow 0$$

where S_1 and S are simple *A*-modules occurs where S_1 has type III and S has type II. Hence a simple *A*-module $S \in \mathscr{C}$ if and only if S is τ-torsion. We now have the following proposition.

PROPOSITION 3.16. *Let* $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ *which satisfies conditions* (i)-(iii) *of Theorem* 3.6. *If A is an idealizer of a HNP SI-ring A^* relative to a semimaximal right ideal of A^*, then R has FGSP if and only if $B = Q_\sigma(B)$.*

Proof. If R has FGSP, then $Q_\sigma(B) = B$ by Proposition 3.14.

Conversely, let $Q_\sigma(B) = B$. We need to show that the exact sequence of right *R*-modules

$$E: 0 \longrightarrow W \longrightarrow V \longrightarrow N \longrightarrow 0$$

is split exact where V is finitely generated and $W = Z(V_R)$. As in the proof of the converse of Theorem 3.6, we may assume $WR_{12} = 0$ or $WR_{23} = 0$. The proof of the case when $WR_{12} = 0$ is identical to that given in the proof of Theorem 3.6. So assume that $WR_{23} = 0$. Lemma 3.10 is still valid; hence we have a second short exact sequence

$$E^*: 0 \longrightarrow W \longrightarrow V/VR_{23} \longrightarrow N/NR_{23} \longrightarrow 0 .$$

Since $A \cong R/R_{23}$ is a HNP ring, W has a composition series. Hence we can further assume that W is a simple singular *A*-module. By Lemma 3.15, N/NR_{23} is σ-torsion free; so by the remarks made prior to this proposition, N/NR_{23} is τ-torsion free. Thus $0 \to N/NR_{23} \to Q_\tau(N/NR_{23})$ is monic; hence $\operatorname{Ext}_A^1(Q_\tau(N/NR_{23}), W) \to \operatorname{Ext}_A^1(N/NR_{23}, W) \to 0$ is exact. Hence if $\operatorname{Ext}_A^1(Q_\tau(N/NR_{23}), W) = 0$, then $\operatorname{Ext}_A^1(N/NR_{23}, W) = 0$.

Since W is *A*-singular and $Q_\tau = A^*$ is nonsingular, $\operatorname{Ext}_A^1(A^*, W) = 0$ as A is a splitting ring. Hence by [2, Proposition 4.14, p. 118], $\operatorname{Ext}_A^1(Q_\tau(N/NR_{23}), W) \cong \operatorname{Ext}_A^1 * (Q_\tau(N/NR_{23}), \operatorname{Hom}_A(A^*, W))$. Since all the faithful simple *A*-modules are injective by [14, Theorem 9], we can assume that W is bounded; say $WI = 0$ for a nonzero ideal I of A. As A is the idealizer of A^*, we may assume that I is a right ideal of A^* and $I^2 = I$ by [14, Lemma D]. $\operatorname{Hom}_A(A^*, W)$ is a right A^*-module via $(fa)(x) = f(ax)$ for $a, x \in A^*$. Thus for $f \in \operatorname{Hom}_A(A^*, W)$, we have

$$(fI)(x) = f(Ix) \subsetneqq f(I) = f(I^2) = f(I)I = 0 .$$

Therefore $\mathrm{Hom}_A(A^*, W)$ is a bounded A^*-module. But A^* is a simple ring; hence $\mathrm{Hom}_A(A^*, W) = 0$. Thus it follows that $\mathrm{Ext}_A^1(Q_\tau(N/NR_{23}), W) = 0$. But by the previous paragraph, $\mathrm{Ext}_A^1(N/NR_{23}, W) = 0$ and E^* splits. Thus E is a split exact sequence and R has FGSP.

Letting A and A^* be as in Proposition 3.16, where $A \neq A^*$ it follows that $\begin{bmatrix} A & 0 \\ A & A \end{bmatrix}$ does not have FGSP whereas $\begin{bmatrix} A & 0 \\ A^* & A \end{bmatrix}$ has FGSP.

4. **Essential products of rings having FGSP.** Given two right nonsingular rings R_1 and R_2, an *essential product* of R_1 and R_2 is any subdirect product R of R_1 and R_2 which contains an essential right ideal of $R_1 \times R_2$. Goodearl has shown [13, Theorem 3.10] that any right nonsingular ring R is an essential product of nonsingular rings R_1 and R_2 where R_1 has essential socle and R_2 has zero socle. In this section we shall characterize right nonsingular rings R with FGSP which either are right finite dimensional or possess a semiprimary classical right quotient ring in terms of an essential product of R_1 which has essential socle and R_2 which has zero socle.

Throughout this section we shall assume that the right nonsingular ring R is an essential product of right nonsingular rings R_1 and R_2 where R_1 has essential socle and R_2 has zero socle. Then $S^0R = S^0R_1 \times S^0R_2$ by [13, Proposition 2]; so let $\pi_i: S^0R \to S^0R_i$ be the natural projection maps for $i = 1, 2$. Then $\pi_i(R) = R_i$ for $i = 1, 2$ and we set $E_1 \times 0 = R \cap (R_1 \times 0)$ and $0 \times E_2 = R \cap (0 \times R_2)$. E_1 and E_2 are two-sided ideals of R_1 and R_2 respectively and they are essential as right ideals. We begin with the following lemma.

LEMMA 4.1. *R_1 and R_2 have FGSP if R has FGSP.*

Proof. $R_1 \cong R/(0 \times E_2)$ and $0 \times E_2$ is a closed right ideal of R by [13, Propositions 7 and 8]; hence by [16, Proposition 1.11], R_1 has FGSP. Similarly R_2 has FGSP.

If R_R is finite dimensional, then R_1 and R_2 are right finite dimensional rings. If, in addition, R has FGSP, by Theorems 2.5 and 3.6 we have that

$$R_1 \cong \begin{bmatrix} A_1 & 0 \\ B_1 & C_1 \end{bmatrix} \text{ and } R_2 \cong \begin{bmatrix} A_2 & 0 \\ B_2 & C_2 \end{bmatrix}$$

where the A_i's, B_i's, and C_i's satisfy the conditions of the respective

theorems. Let $e_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ where 1 is the identity of A_i for $i = 1, 2$ and $f_i = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ where 1 is the identity of C_i for $i = 1, 2$. Then $R_1 e_1 = \mathrm{soc}\,(R_1) = \mathrm{soc}\,(R)$ as R_2 has zero socle. Hence $R_1 e_1 \subsetneqq E_1$; so $e_1 \in R$. Thus we have

LEMMA 4.2. $e_1 \in R$.

The following lemma and its proof is contained in [13, Theorem 12].

LEMMA 4.3. *Let R have FGSP. Then the following statements hold.*

(a) *If L_i is any essential right R_i-submodule of E_i, then E_i/L_i is a direct summand of R_i/L_i for $i = 1, 2$.*

(b) $E_i^2 = E_i$ *for $i = 1, 2$.*

Let $E_i' = f_i E_i f_i$ for $i = 1, 2$. Then we have the following lemma.

LEMMA 4.4. *Let R have FGSP. Then E_1' is a direct summand of C_1 as a right ideal and $E_2' = C_2$.*

Proof. Let $e = (e_1, e_2)$ and $f = (f_1, f_2)$. Then e and f are idempotents of $R_1 \times R_2$. Since $R_1 e_1$ is essential in R_1 as a right ideal and $R_2 e_2$ is essential in R_2 as a right ideal $(R_1 \times R_2)\,e \cap R$ is an essential right ideal of R. Let $M = R \cap f(R_1 \times R_2)$. Since $ef = 0$, it follows easily that M is a two-sided ideal of $L^*(R)$ whose left annihilator is an essential right ideal of R. By the proof of Lemma 3.1, $M^2 = gR$ for g an idempotent element of R. Let $\pi_1(g) = g_1$ and $\pi_2(g) = g_2$. Let $E = E_1 \times E_2$ and $E' = E \cap [f(R_1 \times R_2)f]$. $(E')^2 = E'$ since $E^2 = E$ by Lemma 4.3. $E' \subseteq M$; so $E' = (E')^2 \subseteq M^2 = gR$. Let $E_1^* = (g_1 R g_1 \times 0) \cap gRg = g_1 E_1' g_1$ and $E_2^* = (0 \times g_2 R g_2) \cap gRg = g_2 E_2' g_2$. Since R, R_1, and R_2 are finite dimensional rings, the techniques used in the proof of Lemma 3.3 can be applied to show that $gRg, g_1 R_1 g_1$, and $g_2 R_2 g_2$ are all semihereditary rings. Now gR is a two-sided ideal of R; so $R \cong \begin{bmatrix} g'Rg' & 0 \\ gRg' & gRg \end{bmatrix}$ when $g' = 1 - g$ and, in addition, Rg' is an essential right ideal of R. Thus by Lemma 2.2, $g(S^0 R)g$ is a right flat overring of gRg. Similarly $g_1(S^0 R)g_1$ and $g_2(S^0 R)g_2$ are right flat overrings of $g_1 R_1 g_1$ and $g_2 R_2 g_2$ respectively. $g(S^0 R)g = g_1(S^0 R)g_1 \times g_2(S^0 R)g_2$; so by [9, Proposition 3.8], E_1^* and E_2^* are direct summands of $g_1 R g_1$ and $g_2 R g_2$ respectively as right ideals.

Let $g_1' = f_1 g_1 f_1$. Since C_1 is a semihereditary ring with a semi-

primary classical left quotient ring by Theorem 2.5, C_1 has a complete set of orthogonal idempotents $\{h_1, \cdots, h_n\}$ such that $g_1'C_1 = \sum_{i>k} h_iC_1$ by [9, Lemma 3.3]. This implies that $\{h_{k+1}, \cdots, h_n\}$ is a complete set of orthogonal idempotents for $g_1'C_1g_1'$. By a similar argument $E_1^* = \sum_{i>l} h_i(g_1'C_1g_1')$ where $l \geqq k$. Claim $E_1' = \sum_{i>l} h_iC_1$. Let $X = E_1' \cap h_jC_1$ where $j \leqq l$. Note the $j > k$ since $E_1' \subsetneqq g_1'C_1$. Now $X \subseteq h_jN(C_1)$ since E_1^* is a direct summand of $g_1'C_1g_1'$. If $X \neq 0$, then $L = h_jR_1e_1$ is a proper essential R_1-submodule of h_jR_1 and $(X + L)/L$ is a direct summand of h_jR_1/L by Lemma 4.3. But this is impossible as $(X + L)/L \subseteq (h_jN(C_1) + L)/L$ and $(X + L)/L$ is a small submodule of h_jR_1/L. Hence $X = 0$ and $E_1' = \sum_{i>l} h_iC_1$.

Now let $g_2' = f_2g_2f_2$. By arguments similar to those given in the previous paragraph, C_2 has a complete set of orthogonal idempotents $\{h_1', \cdots, h_m'\}$ such that $E_2' = \sum_{i>t} h_i'C_2$. Claim $E_2' = C_2$. For if $E_2' \neq C_2$, then $h_j' \notin E_2'$ for some $j \leqq t$. Since R_2 has zero socle, $h_j'E_2$ contains a proper essential right submodule H by [16, Proposition 1.2]. Let $L = e_2E_2 \oplus h_1'E_2 \oplus \cdots \oplus h_{j-1}'E_2 \oplus H \oplus h_{j+1}'E_2 \oplus \cdots \oplus h_m'E_2$. L is a proper essential submodule of E_2. Thus E_2/L is a direct summand of R_2/L by Lemma 4.3. However, E_2/L is exactly $h_j'E_2e_2/H$; hence $h_j'E_2e_2/H$ is a direct summand of $h_j'R_2/H$. But this is impossible since $h_j'E_2e_2 \subseteq h_j'N(R_2)$ which forces $h_j'E_2e_2/H$ to be a small submodule of $h_j'R_2/H$.

If $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ is a formal triangular matrix ring satisfying Theorem 2.5, we shall say that R satisfies (α). If $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ is a formal triangular matrix ring satisfying Theorem 3.6, we shall say that R satisfies (β). With this notation, we have the first main result of this section.

THEOREM 4.5. *Let R be a right nonsingular right finite dimensional ring with FGSP. Then R is a subdirect product of R_1 and R_2 where*

$$(1) \quad R_1 = \begin{bmatrix} A_1 & 0 \\ B_1 & C_1 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

satisfies (α) and $E_1 = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & 0 & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$;

$(2) \quad R_2 = \begin{bmatrix} A_2 & 0 \\ B_2 & C_2 \end{bmatrix}$ *satisfies (β) and* $E_2 = \begin{bmatrix} X & 0 \\ B_2 & C_2 \end{bmatrix}$ *where X is a two-sided idempotent ideal of A_2 essential as a right ideal having the property: if L is an essential submodule of the right A_2-module*

X, *then* X/L *is a direct summand of* A_2/L.

Conversely, any subdirect product of ring R_1 *and* R_2 *satisfying* (1) *and* (2) *is a right nonsingular right finite dimensional ring with* $FGSP$.

Proof. If R has FGSP, then R_1 and R_2 satisfy (1) and (2) by Lemmas 4.1, 4.2, 4.3, and 4.4.

To prove the converse we will need to prove some lemmas. We shall assume that R is an essential product of R_1 and R_2 satisfying (1) and (2) in Lemmas 4.6 and 4.7.

LEMMA 4.6. *If* N *is a finitely generated nonsingular right* R-*module, then there is an exact sequence of* R-*modules*

$$0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0$$

where N_1 *is a finitely generated nonsingular right* R_1-*module and* N_2 *is a finitely generated nonsingular right* R_2-*module.*

Proof. Given any nonsingular R-module N, there is an exact sequence of R-modules

$$0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0$$

where N_i is R_i-nonsingular for $i = 1, 2$ by [13, Proposition 3]. Thus if N is finitely generated as well, N_2 is a finitely generated nonsingular R_2-module.

It remains to show that N_1 is finitely generated. First we will show that N_2 is AFR as an R-module. Since N_2 is a finitely generated nonsingular R_2-module and R_2 has FGSP, N_2 is AFR as an R_2-module by [11, Corollary 2]. Let $H = \mathrm{soc}\,(R_R)$ and $S = R/H$. By [16, page 65] it is enough to show that N_2 is finitely related as an S-module. Now $E_1' = E_1 \cap f_1 R_1 f_1$ is a direct summand of S as a right ideal. Furthermore $S/E_1' \cong R_2$. Since N_2 is AFR as a R_2-module and R_2 has zero socle, there is an exact sequence of R_2-modules

$$0 \longrightarrow K \longrightarrow R_2^n \longrightarrow N_2 \longrightarrow 0$$

where n is a positive integer and K is finitely generated. Then there exists an exact sequence

$$0 \longrightarrow K \oplus (E_1')^n \longrightarrow (R_2 \oplus E_1')^n \longrightarrow N_2 \longrightarrow 0$$

and N_2 is finitely related as an S-module. Thus N_2 is AFR as an R-module.

By [16, Lemma 4.7(a)] N_1 is AFG as an R-module. Since N_1 is a submodule of N and soc (N_R) is finitely generated, N_1 is finitely generated.

The following lemma is due to Goodearl and using the fact that N is finitely generated, the proof is virtually identical to the proof of Case II of [13, Theorem 12, page 501].

LEMMA 4.7. *Let N be a finitely generated right nonsingular R_i-module and W a singular right R_j-module for $i \neq j$. Then $\mathrm{Ext}_R^1(N, W) = 0$.*

We now return to the proof of the converse of Theorem 4.5.

Proof. (Converse of Theorem 4.5.) That R is right nonsingular and right finite dimensional follows from the fact that R is an essential product of R_1 and R_2. To show that R has FGSP, it is sufficient to show that any exact sequence of right R-modules

$$0 \longrightarrow W \longrightarrow Y \longrightarrow N \longrightarrow 0$$

is split exact where W is R-singular and N is finitely generated nonsingular. By [13, Proposition 3] and Lemma 4.6 there are exact sequences of R-modules

$$0 \longrightarrow W_1 \longrightarrow W \longrightarrow W_2 \longrightarrow 0$$
$$0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0$$

where W_i is R_i-singular for $i = 1, 2$ and N_i is finitely generated R_i-nonsingular for $i = 1, 2$.

To show that $\mathrm{Ext}_R^1(N, W) = 0$ it suffices to show that $\mathrm{Ext}_R^1(N_i, W_j) = 0$ for $i = 1, 2$ and $j = 1, 2$. Now $\mathrm{Ext}_R^1(N_i, W_j) = 0$ for $i \neq j$ by Lemma 4.7.

Consider an exact sequence

$$D: 0 \longrightarrow W_1 \longrightarrow Y \longrightarrow N_1 \longrightarrow 0$$

of R-modules. Since W_1 and N_1 are R_1-modules and $E_2^2 = E_2$, D is an exact sequence of R_1-modules. Thus D splits as R_1-modules since R_1 has FGSP. From this it follows that the sequence splits as R-modules so $\mathrm{Ext}_R^1(N_1, W_1) = 0$.

By a similar argument, $\mathrm{Ext}_R^1(N_2, W_2) = 0$. Thus R has FGSP.

Goodearl has given an example in [13, Example 1, page 503] which shows that the essential product given in Theorem 4.5 need

not be a direct product. In fact Theorem 4.5 shows that the example given there is the "canonical" way to construct essential products which are not direct products. In order to construct an essential product as described in Theorem 4.5 it is necessary that the ring A_2 have a two-sided idempotent ideal X essential as a right ideal such that if L is an essential submodule of the right A_2-module X, X/L must be a direct summand of A_2/L. Proposition 4.8 gives a class of rings possessing such an ideal.

A subring R of T is an *iterated idealizer* from T provided there is a chain of subrings $R_0 = R \subseteq R_1 \subseteq \cdots \subseteq R_n = T$ such that each R_i is the idealizer of a semimaximal right ideal of R_{i+1}.

PROPOSITION 4.8. *Let A be a HNP ring which is an iterated idealizer of a HNP SI-ring. Then A has a minimal two-sided ideal M which is idempotent and has the property that if L is an essential submodule of M_A, M/L is a direct summand of A/L.*

Proof. By [14, Lemma D] A possesses a minimal two-sided ideal M which is idempotent. First we show that $\text{Ext}_A^1(A/M, S) = 0$ for all simple right A-modules S. By [14, Theorem 10], if S is a faithful simple A-module, then A injective; so $\text{Ext}_A^1(A/M, S) = 0$. If S is bounded, that is, if $r_A(S) \neq 0$, and if $\text{Ext}_A^1(A/M, S) \neq 0$, then there is an exact sequence

$$0 \longrightarrow S \longrightarrow X \longrightarrow A/M \longrightarrow 0$$

which is nonsplit and X can be taken to be cyclic. Since A/M and S are both bounded, X is bounded as A is a prime ring. Thus $MX = 0$ as M is the minimal two-sided ideal of A. Therefore X is a homomorphic image of A/M; so X has composition series length less than or equal to that of A/M. But this is impossible; so $\text{Ext}_A^1(A/M, S) = 0$. It follows that $\text{Ext}_A^1(A/M, W) = 0$ for all finitely generated singular A-modules W.

Hence if L is an essential submodule of M_A, then $\text{Ext}_A^1(A/M, M/L) = 0$; so that M/L is a direct summand of A/L.

We now turn to studying rings R with FGSP which possess a semiprimary classical right quotient ring Q. Let $R_1, R_2, \pi_1, \pi_2, E_1,$ and E_2 be as defined in the beginning of this section. Since R is right nonsingular, Q is right nonsingular and $\text{soc}(Q_Q) = Qe$ where e is an idempotent element of Q. Set $\pi_i(Q) = Q_i, \pi_i(e) = e_i$ and $\pi_i(1 - e) = f_i$ for $i = 1, 2$. Then

$$Q_i = \begin{bmatrix} e_i Q_i e_i & 0 \\ f_i Q_i e_i & f_i Q_i f_i \end{bmatrix}$$

for $i = 1, 2$. By Lemma 4.2, $e_1 \in R$ as R_1 has essential socle and R_2 has zero socle. Thus it follows that $f_1 \in R_1$. Furthermore, we have the following lemma in terms of this notation.

LEMMA 4.9. *Let R have FGSP and possess a semiprimary classical right quotient ring Q. Then $f_2 \in R_2$; so $e_2 \in R_2$.*

Proof. Let $M = R_2 \cap f_2 Q_2$. Then it follows from Lemma 3.1 and the remarks following the lemma that $M = f_2 R_2$ where $f_2 \in R_2$.

LEMMA 4.10. *Let R have FGSP and possess a semiprimary classical right quotient ring Q. Then $e_2 R_2 e_2$ is semiprime and is right finite dimensional.*

Proof. First note that $f_2 R_2$ is a maximal two-sided ideal of $L^*(R_2)$ whose left annihilator is essential is a right ideal of R. For suppose $f_2 R_2 \subseteq M$ where M is a two-sided ideal of $L^*(R_2)$ and $H = l_{R_2}(M)$ is essential as a right ideal of R_2. By Lemma 3.1, $M = gR_2$ for g an idempotent element of R_2. Then $MQ_2 = gQ_2$ is a two-sided ideal of $L^*(Q_2)$ whose left annihilator contains H. Hence gQ_2 has a right essential left annihilator in Q_2. Therefore $e_2 g = 0$; so since $f_2 R_2 \subseteq M$, $g \in f_2 R_2$, and $M = f_2 R_2$.

Claim $e_2 R_2 e_2$ is semiprime. Let I be a nilpotent ideal of $e_2 R_2 e_2$. If P is the closure of I in $e_2 R_2 e_2$, then P is a two-sided ideal of $L^*(e_2 R_2 e_2)$ whose left annihilator is right essential in $e_2 R_2 e_2$ by [17, Lemma A]. Then it follows that

$$\begin{bmatrix} P & 0 \\ f_2 R_2 e_2 & f_2 R_2 f_2 \end{bmatrix}$$

is a two-sided ideal of $L^*(R_2)$ whose left annihilator is right essential in R_2. This forces I to be zero as $f_2 R_2$ is the maximal two-sided ideal of R_2 with this property. Thus $e_2 R_2$ is semiprime.

Since Q is an essential extension of R, it follows that $R \cap eQ$ is essential in eQ. Therefore $R \cap (0 \times e_2 Q_2)$ is essential in $0 \times e_2 Q e_2$; so $e_2 R_2 e_2$ is essential in $e_2 Q_2 e_2$. Thus $e_2 Q_2 e_2 = S^0(e_2 R_2 e_2)$ and $e_2 R_2 e_2$ is finite dimensional since $e_2 Q_2 e_2$ is semisimple Artinian.

THEOREM 4.11. *A right nonsingular ring R which possesses a semiprimary classical right quotient ring has FGSP if and only if $R = R_1 \times R_2$ where R_1 has essential socle, R_2 has zero socle and both R_1 and R_2 have FGSP with semiprimary classical right quotient rings.*

Proof. Let R have FGSP. By the choice of e_1 and f_1, $R_1 e_1$ is

essential as a right ideal of R_1 and R_1 has FGSP by Lemma 4.1. Since R_1 has essential socle, the techniques of § 2 can be applied to show that

$$R_1 = \begin{bmatrix} e_1 R_1 e_1 & 0 \\ f_1 R_1 e_1 & f_1 R_1 f_1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_1 & C_1 \end{bmatrix}$$

where A_1 is semisimple Artinian and C_1 is a semihereditary subring of $T = \text{End}_A(B)$ such that T_C is flat.

Since R_2 has FGSP by Lemma 4.1, we have shown in Lemmas 4.9 and 4.10 that $R_2 = \begin{bmatrix} A_2 & 0 \\ B_2 & C_2 \end{bmatrix}$ where $A_2 = e_2 R_2 e_2$ is a right finite dimensional semiprime ring. By a slight modification of the proof of Lemma 3.3, we see that $C_2 = f_2 R_2 f_2$ is a semihereditary ring possessing a two-sided maximal quotient ring T_2 which is right C_2-flat.

The proof of Lemma 4.4 can be modified to yield that E_1' is a direct summand of C_1 and $E_2' = C_2$.

$E_2' = C_2 = f_2 R_2 f_2$; so $f_2 \in R$. Thus by [18, Theorem A] $f_2 Q f_2 = f_2 Q_2 f_2$ is the semiprimary classical right quotient ring of C_2. Also $f_2 Q_2 f_2 \subseteq f_2 (S^0 R) f_2$; so by Lemmas 2.6 and 4.10, R_2 has a semiprimary classical right quotient ring Q_2. Therefore R_2 is right finite dimensional by Proposition 3.5. Now it follows that

$$R_1 = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad E_1 = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & 0 & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

and

$$R_2 = \begin{bmatrix} A_2 & 0 \\ B_2 & C_2 \end{bmatrix}, \quad E_2 = \begin{bmatrix} X & 0 \\ B_2 & C_2 \end{bmatrix}$$

where A_{ij}, A_2, B_2, C_2, and X are as in Theorem 4.5.

If R is not the direct product of R_1 and R_2, $X \neq A_2$. If

$$\left(\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ b & c \end{bmatrix} \right) \in R$$

where $a_{ij} \in A_{ij}$, $b \in B_2$ and $c \in C_2$, then $a_{22} = 0$. Claim

$$P = \left(\begin{bmatrix} 0 & 0 & 0 \\ A_{21} & 0 & 0 \\ A_{31} & A_{32} & N(A_{33}) \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ B_2 & N(C_2) \end{bmatrix} \right)$$

is the prime radical of R. P is nilpotent; so $P \subseteq N(R)$. Let

$$U = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \right),$$

and suppose that $U \subset N(R)$. $N(R)$ is nilpotent so $U^n = 0$ for some positive integer n. Thus $I^n = 0$ and $J^n = 0$. Since A_2 is semiprime, then $J = 0$. Then it follows that $I = 0$. From this it is immediate that $P = N(R)$.

Since A_2 possesses a semisimple classical right quotient ring and $X \neq A_2$ is essential as a right ideal of A_2, there is a regular element of d of A_2 such that $d \in X$. Consider the element

$$x = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix} \right) \in R$$

and let $\bar{x} = x + N(R)$. Now $\bar{x}$ is a regular element of $R/N(R)$. For if

$$\bar{y} = \left(\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \quad \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \right) + N(R)$$

is such that $\bar{x}\bar{y} = 0$, then $a_{11} = 0$, $a_{33} \in N(A_{33})$, $c \in N(C_2)$ and $da = 0$. Since d is a regular element of A_2, $a = 0$. Hence $a_{22} = 0$; so $\bar{y} = \bar{0}$. Similarly $\bar{y}\bar{x} = \bar{0}$ implies $\bar{y} = \bar{0}$. By the proof of [29, Theorem 2.12], x must be a regular element of R. However if $a_{21} \neq 0$ is in A_{21},

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = 0 ,$$

and x is not a regular element of R. This contradiction yields that $X = A_2$ and $R = R_1 \times R_2$.

The converse is obvious.

REMARK. Another situation in which the essential product of Theorem 4.5 is actually a direct product occurs when A_2 is commutative. Then A_2 is a direct product of Prüfer domains. If X is as in Theorem 4.5, then there is a nonzero divisor $d \in X$. If $X = dA_2$, then X is a finitely generated idempotent ideal of A_2. But it is well known that such an ideal is a direct summand of A_2 which is impossible. If $dA_2 \subsetneq X$, then X/d_2A_2 is a direct summand of A_2/d_2A_2. But again this forces X to be finitely generated which is impossible. Thus $X = A_2$ and the product is direct.

5. **Applications to rings with SP and BSP.** Let M be a singular R-module and I an essential right ideal of R. We say that M has *bounded order* I if M can be embedded in a module which has a set of generators all annihilated by I. M has *bounded order* if M has bounded order for some essential right ideal I of R. A ring R has *bounded splitting property* (BSP), if $Z(M)$ is a direct summand of M for every right R-module M such that $Z(M)$ has bounded order.

We now make some observations on how the results and techniques developed in this paper can be applied to rings with SP and BSP. If a ring R has BSP or SP, then R has FGSP. Also by the proof of [16, Theorem 5.3] if R has BSP or SP and H is a two-sided ideal of R which is essential as a right ideal, then R/H is right perfect.

Let R be a right nonsingular right finite dimensional ring with zero socle. If R has BSP, then R has FGSP; so $R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ where A, B, and C satisfy the conditions of Theorem 3.6. Since $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ is essential in R_R, then C is a semiprimary ring. Hence by Proposition 3.11, B_A is injective. Thus we have the following result.

THEOREM 5.1. *Let R be a right nonsingular right finite dimensional ring with zero socle and BSP. Then R is isomorphic to*

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

where

(1) *A is a semiprime right nonsingular right finite dimensional ring with zero socle and BSP;*

(2) *B is a $C - A$ bimodule which is a finite dimensional nonsingular injective A-module;*

(3) *C is a semiprimary ring with a two-sided maximal quotient ring $T = \mathrm{End}_A(B)$.*

Conversely, any such matrix ring is a right nonsingular right finite dimensional ring with zero socle and has BSP.

Proof. We only need to prove the converse. But this follows by a slight modification of the arguments used to prove [17, Theorem 6].

Thus the ring $\begin{bmatrix} Z & 0 \\ Q & Z \end{bmatrix}$ has FGSP by Theorem 3.6 and Corollary

3.13 whereas $\begin{bmatrix} Z & 0 \\ Q & Q \end{bmatrix}$ has BSP. Z denotes the integers and Q the rational numbers.

If R has essential socle and is either right finite dimensional or a right order in a semiprimary classical right quotient ring, then BSP is equivalent to SP and the structure of these rings are given in [8].

We now turn to the problem of expressing a right nonsingular ring R with BSP or SP as an essential product of R_1 and R_2 where R_1 has essential socle and R_2 has zero socle. Two conditions were critical in order to obtain Theorem 4.5. First of all, each R_i was isomorphic to a formal triangular matrix ring $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ where $\begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}$ is an essential right ideal. Secondly, the E_i's must satisfy Lemma 4.3. The second condition holds if R has SP or BSP. If R has SP with no infinite sets of orthogonal idempotents in its socle, then R_1 and R_2 has the desired triangular form by [8, Theorem 3.5] and [17, Theorem 7]. Furthermore, if R is right finite dimensional and has BSP, R_1 and R_2 have the required triangular form by Theorem 2.5 (or [8, Theorem 3.5]) and Theorem 2.6. Thus in these cases a theorem similar to Theorem 4.5 can be proven.

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