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**COMMON FIXED POINTS AND ITERATION OF COMMUTING
NONEXPANSIVE MAPPINGS**

SHIRO ISHIKAWA

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The following result is shown. Let $T_i (i = 1, 2, \dots, \nu)$ be commuting nonexpansive self-mappings on a compact convex subset D of a Banach space and let x be any point in D . Then the sequence

$$\left\{ \left[\prod_{n_{\nu-1}=1}^{n_{\nu}} \left[S_{\nu} \prod_{n_{\nu-2}=1}^{n_{\nu-1}} \left[\dots \left[S_3 \prod_{n_1=1}^{n_2} \left[S_2 \prod_{n_0=1}^{n_1} S_1 \right] \right] \right] \dots \right] \right] x \right\}_{n_{\nu}=1}^{\infty}$$

converges to a common fixed point of $\{T_i\}_{i=1}^{\nu}$, where $S_i = (1 - \alpha_i)I + \alpha_i T_i$, $0 < \alpha_i < 1$, I is the identity mapping.

In [2], DeMarr proved that if $T_i (i \in J, J$ is an index set) are commuting nonexpansive self-mappings on a compact convex subset D of a Banach space (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in D , and $T_i T_j = T_j T_i$ for all $i, j \in J$), then $T_i (i \in J)$ have a common fixed point in D .

The problem we shall consider in this paper is that of constructing a sequence of points $\{x_n\}_{n=1}^{\infty}$ in D that converges to the common fixed point of $T_i (i \in J, J$ is a finite index set).

If a Banach space is strictly convex (i.e., $\|\alpha x + (1 - \alpha)y\| < \max\{\|x\|, \|y\|\}$ for $x \neq y, 0 < \alpha < 1$), the problem was solved in [5].

Throughout this paper, we denote an identity mapping by I and the set of fixed points of T by $F[T]$. And we define $\prod_{i=1}^{n+1} T_i = T_{n+1}(\prod_{i=1}^n T_i)$ for any positive integer n and $\prod_{i=1}^1 T_i = T_1$.

We have the following main theorem.

THEOREM. *Let $T_i (i = 1, 2, \dots, \nu)$ be commuting nonexpansive mappings from a compact convex subset D of a Banach space into itself, and let x be any point in D .*

Then $\bigcap_{i=1}^{\nu} F[T_i]$ is nonempty and the sequence $\{x_n^{\infty}\}$ converges to a point in $\bigcap_{i=1}^{\nu} F[T_i]$, where $x_{n_{\nu}}$ is defined for each positive integer n_i by

$$\left[\prod_{n_{\nu-1}=1}^{n_{\nu}} \left[S_{\nu} \prod_{n_{\nu-2}=1}^{n_{\nu-1}} \left[S_{\nu-1} \dots \left[S_3 \prod_{n_1=1}^{n_2} \left[S_2 \prod_{n_0=1}^{n_1} S_1 \right] \right] \right] \dots \right] \right] x$$

where $S_i = (1 - \alpha_i)I + \alpha_i T_i, 0 < \alpha_i < 1 (i = 1, 2, \dots, \nu)$.

Before proving the theorem, we first prove the following lemmas on which the proof of theorem is based.

LEMMA 1. *Let T and P be nonexpansive mappings from a*

bounded convex subset D of a Banach space into itself that satisfy the conditions

$$(1) \quad P(D) = F[P] \quad \text{and} \quad T(P(D)) \subset P(D).$$

Let x_0 be any point in D and let α be any number such that $0 < \alpha < 1$. Then the sequences $\{x_n - Tx_n\}_{n=0}^{\infty}$ and $\{x_n - Px_n\}_{n=0}^{\infty}$ respectively converge to zero, where x_n is defined for each positive integer n by

$$(2) \quad x_n = (1 - \alpha)y_n + \alpha Ty_n, \quad y_n = Px_{n-1},$$

that, is $x_n = (SP)^n x_0$, where $S = (1 - \alpha)I + \alpha T$.

Proof. We see from (1) that for all $n \geq 1$

$$(3) \quad y_n = Py_n \quad \text{and} \quad Ty_n = PTy_n.$$

Since T and P are nonexpansive mappings, we have, from (2) and (3), for all $n \geq 0$

$$\|y_{n+1} - Ty_{n+1}\| = \|Px_n - PTy_{n+1}\| \leq \|x_n - Ty_{n+1}\|$$

and, from (2) and (3), for all $n \geq 1$

$$\begin{aligned} \|x_n - Ty_{n+1}\| &\leq \|x_n - Ty_n\| + \|Ty_n - Ty_{n+1}\| \\ &\leq (1 - \alpha)\|y_n - Ty_n\| + \|y_n - y_{n+1}\| \\ &\leq (1 - \alpha)\|y_n - Ty_n\| + \|Py_n - Px_n\| \\ &\leq (1 - \alpha)\|y_n - Ty_n\| + \|y_n - x_n\| \\ &\leq (1 - \alpha)\|y_n - Ty_n\| + \alpha\|y_n - Ty_n\| = \|y_n - Ty_n\| \end{aligned}$$

from which, we obtain

$$\|y_{n+1} - Ty_{n+1}\| \leq \|x_n - Ty_{n+1}\| \leq \|y_n - Ty_n\| \quad \text{for all } n \geq 1.$$

Hence the sequence $\{\|y_n - Ty_n\|\}_{n=1}^{\infty}$, which is nonincreasing and bounded below, has a limit.

Suppose that $\lim \|y_n - Ty_n\| = r > 0$, that is, for any $\varepsilon > 0$, there is an integer m such that

$$(4) \quad r \leq \|y_n - Ty_n\| \leq (1 + \varepsilon)r \quad \text{for all } n \geq m.$$

Also, from the boundedness of D , we can choose M such that

$$(5) \quad L \leq (M - m)r < 2L, \quad \text{where } L \text{ is a diameter of } D.$$

We have from (3), (2) and (4) that for any $n \geq m$ and $k \geq 0$

$$\begin{aligned} \|y_n - y_{n+k+1}\| &\leq \|y_n - y_{n+1}\| + \|y_{n+1} - y_{n+2}\| + \cdots + \|y_{n+k} - y_{n+k+1}\| \\ &\leq \|Py_n - Px_n\| + \|Py_{n+1} - Px_{n+1}\| + \cdots + \|Py_{n+k} - Px_{n+k}\| \\ &\leq \|y_n - x_n\| + \|y_{n+1} - x_{n+1}\| + \cdots + \|y_{n+k} - x_{n+k}\| \end{aligned}$$

$$(6) \quad \leq \alpha(k+1)(1+\varepsilon)r.$$

Now we shall prove by induction that

$$(7) \quad (1+\alpha k)(1+\varepsilon)r - (1-\alpha)^{-k}\varepsilon r \leq \|Ty_M - y_{M-k}\|$$

for any k such that $0 \leq k \leq M-m$.

When $k=0$, the result is trivial. Now we assume that (7) is true for some k such that $0 \leq k \leq M-m-1$. We see, from (3) and (2), that

$$\begin{aligned} \|Ty_M - y_{M-k}\| &= \|PTy_M - Px_{M-(k+1)}\| \leq \|Ty_M - x_{M-(k+1)}\| \\ &= \|(1-\alpha)(Ty_M - y_{M-(k+1)}) + \alpha(Ty_M - Ty_{M-(k+1)})\| \\ &\leq (1-\alpha)\|Ty_M - y_{M-(k+1)}\| + \alpha\|y_M - y_{M-(k+1)}\| \end{aligned}$$

from which and (6), it follows that

$$\|Ty_M - y_{M-k}\| \leq (1-\alpha)\|Ty_M - y_{M-(k+1)}\| + \alpha^2(k+1)(1+\varepsilon)r.$$

From this and the assumption by induction, we have

$$\begin{aligned} (1+\alpha k)(1+\varepsilon)r - (1-\alpha)^{-k}\varepsilon r \\ \leq (1-\alpha)\|Ty_M - y_{M-(k+1)}\| + \alpha^2(k+1)(1+\varepsilon)r \end{aligned}$$

and it is clear that this inequality is equal to (7) with $k+1$ for k . Hence, by induction, it follows that (7) is true for any k such that $0 \leq k \leq M-m$.

Since $\log(1+t) \leq t$ for all $t \in (-1, \infty)$, we have from (5) that

$$\begin{aligned} (1-\alpha)^{-(M-m)} &= \exp\left[(M-m)\log\left(1+\frac{\alpha}{1-\alpha}\right)\right] \\ &\leq \exp\left[(M-m)\frac{\alpha}{1-\alpha}\right] \leq \exp\left(\frac{2L}{(1-\alpha)r}\right). \end{aligned}$$

Thus it follows from (7) with $M-m$ for k that

$$\begin{aligned} \|Ty_M - y_m\| &\geq (1+\alpha(M-m))(1+\varepsilon)r - \varepsilon r \exp\left(\frac{2L}{(1-\alpha)r}\right) \\ &\geq (r+L) - \varepsilon r \exp\left(\frac{2L}{(1-\alpha)r}\right). \end{aligned}$$

Since ε is any positive number, this inequality is incompatible with the definition of L . Hence we obtain that $r=0$, that is,

$$(8) \quad \lim_{n \rightarrow \infty} \|y_n - Ty_n\| = 0.$$

Now since T and P are nonexpansive mappings, we have from (2) and (3) that, for all $n \geq 1$,

$$\begin{aligned}
\|x_n - Tx_n\| &= \|(1 + \alpha)y_n + \alpha Ty_n - T((1 - \alpha)y_n + \alpha Ty_n)\| \\
&= \|(1 - \alpha)y_n - (1 - \alpha)Ty_n + Ty_n - T((1 - \alpha)y_n + \alpha Ty_n)\| \\
&\leq (1 - \alpha)\|y_n - Ty_n\| + \alpha\|y_n - Ty_n\| \\
&= \|y_n - Ty_n\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_n - Px_n\| &= \|(1 - \alpha)y_n + \alpha Ty_n - P((1 - \alpha)y_n + \alpha Ty_n)\| \\
&= \|(1 - \alpha)[Py_n - P((1 - \alpha)y_n + \alpha Ty_n)] \\
&\quad + \alpha[PTy_n - P((1 - \alpha)y_n + \alpha Ty_n)]\| \\
&\leq 2\alpha(1 - \alpha)\|y_n - Ty_n\|.
\end{aligned}$$

Therefore we obtain that from (8) that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = \lim_{n \rightarrow \infty} \|x_n - Px_n\| = 0.$$

LEMMA 2. Let T and P be nonexpansive mappings from a compact convex subset D of a Banach space into itself such that

$$(9) \quad P(D) = F[P] \quad \text{and} \quad T(P(D)) \subset P(D).$$

Let x_0 be any point in D . Define $x_n = \bar{P}_n x_0$ for each positive integer n , where $\bar{P}_n = (SP)^n$, $S = (1 - \alpha)I + \alpha T$, $0 < \alpha < 1$. Then it follows that

$$(10) \quad \text{for any } x_0 \text{ in } D, \lim_{n \rightarrow \infty} (SP)^n x_0 = Px_0 \text{ exists, which is, denoted by } \bar{P}x_0,$$

$$(11) \quad \bar{P}(D) = F[\bar{P}] = F[T] \cap F[P]$$

and

$$(12) \quad \{\bar{P}_n\}_{n=1}^{\infty} \text{ converges uniformly to } \bar{P}.$$

Proof. Since D is compact, there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}$ that converges to a point u in D . From the boundedness of D , Lemma 1 is applicable, so we have,

$$\begin{aligned}
\|u - Tu\| &\leq \lim_{i \rightarrow \infty} \{\|u - x_{n_i}\| + \|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tu\|\} \\
&\leq \lim_{i \rightarrow \infty} \{2\|x_{n_i} - n\| + \|x_{n_i} - Tx_{n_i}\|\} = 0,
\end{aligned}$$

and similarly $\|u - Pu\| = 0$.

From this, it follows that

$$(13) \quad u \in F[T] \cap F[P].$$

Since (9) implies (3), we see from (13) and (3) that for all $n \geq 0$,

$$\begin{aligned} \|u - x_{n+1}\| &= \|u - ((1 - \alpha)y_{n+1} + \alpha Ty_{n+1})\| \\ &\leq (1 - \alpha)\|u - y_{n+1}\| + \alpha\|Tu - Ty_{n+1}\| \\ &\leq \|u - y_{n+1}\| = \|Pu - Px_n\| \leq \|u - x_n\|. \end{aligned}$$

From this, we obtain that $\lim_{n \rightarrow \infty} \|u - x_n\| = \lim_{n_i \rightarrow \infty} \|u - x_{n_i}\| = 0$. Hence we have proved that (10) is true, that is, for any x_0 in D , $\bar{P}(x_0) = \lim_{n \rightarrow \infty} (SP)^n x_0$ is well-defined. From (13), we see that $\bar{P}(x_0) \in F[T] \cap F[P]$ for all x_0 in D , that is,

$$(14) \quad \bar{P}(D) \subset F(T) \cap F(P).$$

And we have that, for any v in $F[T] \cap F[P]$,

$$v = (SP)^n v = \lim_{n \rightarrow \infty} (SP)^n v = \bar{P}v,$$

so we see that

$$(15) \quad F[T] \cap F[P] \subset F[\bar{P}].$$

Also, clearly $w = \bar{P}w \in \bar{P}(D)$ for all w in $F[\bar{P}]$. From this, (14) and (15), we get (11).

Finally we shall prove (12). Let ε be any positive number. Since D is compact, there are finite points $\{x_0^1, x_0^2, \dots, x_0^k\}$ such that, for any x in D ,

$$(16) \quad \min \{\|x - x_0^i\| : 1 \leq i \leq k\} < \frac{\varepsilon}{3}.$$

From (10), we can choose N such that

$$(17) \quad \|(SP)^n x_0^i - \bar{P}x_0^i\| < \frac{\varepsilon}{3} \quad \text{for all } n \geq N \text{ and } 1 \leq i \leq k.$$

Let x_0 be any point in D . From (16), we can take x_0^j such that

$$(18) \quad \|x_0 - x_0^j\| < \frac{\varepsilon}{3}.$$

Since SP is nonexpansive, clearly \bar{P} is also nonexpansive. Hence we obtain from (17) and (18) that, for all $n \geq N$,

$$\begin{aligned} \|(SP)^n x_0 - \bar{P}x_0\| &\leq \|(SP)^n x_0 - (SP)^n x_0^j\| + \|(SP)^n x_0^j - \bar{P}x_0^j\| + \|\bar{P}x_0^j - \bar{P}x_0\| \\ &\leq 2\|x_0 - x_0^j\| + \|(SP)^n x_0^j - \bar{P}x_0^j\| \leq \varepsilon \end{aligned}$$

which implies (12).

LEMMA 3. *Let T and P_n ($n = 1, 2, \dots$) be nonexpansive mappings from a compact convex subset D of a Banach space into itself. Assume*

that the following conditions are satisfied:

(19) for any x in D , $\lim_{n \rightarrow \infty} P_n x = Px$ exists ,

(20) $P(D) = F[P] \subset F[P_n]$ for all $n \geq 1$

(21) P_n converges uniformly to P

and

(22) $T(P(D)) \subset P(D)$.

Then it follows that

(23) for any x in D , $\lim_{n \rightarrow \infty} \hat{P}_n x = \hat{P}x$ exists, where $\hat{P}_n = \prod_{i=1}^n (SP_i)$,
 $S = (1 - \alpha)I + \alpha T, 0 < \alpha < 1$,

(24) $\hat{P}(D) = F[\hat{P}] = F[T] \cap F[P] \subset F[\hat{P}_n]$ for all $n \geq 1$

and

(25) \hat{P}_n converges uniformly to \hat{P} .

Proof. Let ε be any positive number. Since P satisfies the conditions of P in Lemma 2, from (12), we can choose N such that

(26) $\|(SP)^N y - \bar{P}y\| < \frac{\varepsilon}{2}$ for all y in D ,

where \bar{P} is defined as in Lemma 2.

From (21), there exists M such that

$$\|SPx - SP_n x\| \leq \|Px - P_n x\| \leq \frac{\varepsilon}{2N}$$

for all $n \geq M$ and all x in D .

This implies that, for all n such that $n \geq M$

$$\begin{aligned} & \|\hat{P}_n x - (SP)^N \hat{P}_{n-N} x\| \\ & \leq \|(SP_n) \hat{P}_{n-1} x - (SP) \hat{P}_{n-1} x\| + \|(SP) \hat{P}_{n-1} x - (SP)(S)^{N-1} \hat{P}_{n-N} x\| \\ & \leq \frac{\varepsilon}{2N} + \|\hat{P}_{n-1} x - (SP)^{N-1} \hat{P}_{n-N} x\| \\ & \leq 2 \frac{\varepsilon}{2N} + \|\hat{P}_{n-2} x - (SP)^{N-2} \hat{P}_{n-N} x\| \leq \dots \leq \frac{\varepsilon}{2} . \end{aligned}$$

From this and (26), we have that, for all n such that $n \geq \max\{N, M\}$,

$$\begin{aligned} \|\hat{P}_n x - \bar{P}(\hat{P}_{n-N} x)\| & \leq \|\hat{P}_n x - (SP)^N \hat{P}_{n-N} x\| + \|(SP)^N \hat{P}_{n-N} x - \bar{P}(\hat{P}_{n-N} x)\| \\ & \leq \varepsilon . \end{aligned}$$

Since Lemma 2 says that $\bar{P}(D) = F[T] \cap F[P]$, this implies that there exists a subsequence $\{\hat{P}_{n_i}x\}_{i=1}^\infty$ that converges to a point u in $F[T] \cap F[P]$. Also we see, from (20), for all $n \geq 1$,

$$\|\hat{P}_{n+1}x - u\| = \|SP_{n+1}\hat{P}_n x - SP_{n+1}u\| \leq \|\hat{P}_n x - u\|.$$

Hence we get that $\lim_{n \rightarrow \infty} \|\hat{P}_n x - u\| = \lim_{i \rightarrow \infty} \|\hat{P}_{n_i} x - u\| = 0$, that is, $\hat{P}_n x$ converges to a point in $F[T] \cap F[P]$ for any x in D . This implies (23), and

$$(27) \quad \hat{P}(D) \subset F[T] \cap F[P].$$

If $v \in F[T] \cap F[P]$, then $v = \hat{P}_n v = \lim_{n \rightarrow \infty} \hat{P}_n v = \hat{P}v$, so we see

$$(28) \quad F[T] \cap F[P] \subset F[\hat{P}].$$

Since clearly $F[\hat{P}] \subset \hat{P}(D)$ and $F[T] \cap F[P] \subset F[\hat{P}_n]$ for all $n \geq 1$, (24) follows from (27) and (28).

Now we shall prove (25). Let ε be any positive number. As in the proof of Lemma 2, we can choose finite points $\{x_0^1, x_0^2, \dots, x_0^k\}$ from D satisfying (16). From (23), we can choose N' such that

$$(29) \quad \|\hat{P}_n x_0^i - \hat{P}x_0^i\| \leq \frac{\varepsilon}{3} \quad \text{for all } n \geq N' \text{ and } 1 \leq i \leq k.$$

Let x_0 be any point in D . By (16), we can take x_0^j that satisfies (18).

Since \hat{P} is nonexpansive, we obtain from (18) and (29) that, for all $n \geq N'$,

$$\begin{aligned} \|\hat{P}_n x_0 - \hat{P}x_0\| &\leq \|\hat{P}_n x_0 - \hat{P}_n x_0^j\| + \|\hat{P}_n x_0^j - \hat{P}x_0^j\| + \|\hat{P}x_0^j - \hat{P}x_0\| \\ &\leq 2\|x_0 - x_0^j\| + \|\hat{P}_n x_0^j - \hat{P}x_0^j\| \leq \varepsilon. \end{aligned}$$

This implies (25).

LEMMA 4. *Let $T_i (i = 1, 2, \dots, k)$ be a commuting family of mappings. Then it follows that*

$$T_k \left(\bigcap_{i=1}^{k-1} F[T_i] \right) \subset \bigcap_{i=1}^{k-1} F[T_i].$$

Proof. Let x be any point in $\bigcap_{i=1}^{k-1} F[T_i]$. We see that $T_k x = T_k T_i x = T_i T_k x$ for all i such that $1 \leq i \leq k-1$, which implies that $T_k x$ belongs to $F[T_i]$ for all $1 \leq i \leq k-1$.

Proof of theorem. For all i such that $1 \leq i \leq \nu$, put

$$\left[\prod_{n_{i-1}=1}^{n_i} \left[S_i \prod_{n_{i-2}=1}^{n_{i-1}} \left[S_{i-1} \cdots \left[S_2 \prod_{n_0=1}^{n_1} S_1 \right] \cdots \right] \right] \right] x = P_{n_i}^{(i)} x.$$

We shall prove the theorem by induction. Let us assume that the following conditions are true for some integer j such that $1 \leq j \leq \nu - 1$:

$$(30) \quad \text{for any } x \text{ in } D, \lim_{n_j \rightarrow \infty} P_{n_j}^{(j)} x = P^{(j)} x \text{ exists,}$$

$$(31) \quad P^{(j)}(D) = F[P^{(j)}] = \bigcap_{i=1}^j F[T_i] \subset F[P_{n_j}^{(j)}] \text{ for all integers } n_j \geq 1,$$

$$(32) \quad \{P_{n_j}^{(j)}\}_{n_j=1}^{\infty} \text{ converges uniformly to } P^{(j)}$$

and

$$(33) \quad T(P^{(j)}(D)) \subset P^{(j)}(D).$$

Since $P_{n_{j+1}}^{(j+1)} x = [\prod_{n_j=1}^{n_{j+1}} (S_{j+1} P_{n_j}^{(j)})] x$, we can apply Lemma 3 by regarding $T_{j+1}, S_{j+1}, P^{(j)}, P_{n_j}^{(j)}, P_{n_{j+1}}^{(j+1)}, P^{(j+1)}$ and conditions (30)-(33) as T, S, P, P_n, P_n, P and conditions (19)-(22). Hence we have,

$$(34) \quad \text{for any } x \text{ in } D, \lim_{n_{j+1} \rightarrow \infty} P_{n_{j+1}}^{(j+1)} x = P^{(j+1)} x \text{ exists,}$$

$$(35) \quad P^{(j+1)}(D) = F[P^{(j+1)}] = \bigcap_{i=1}^{j+1} F[T_i] \subset F[P_{n_{j+1}}^{(j+1)}] \text{ for all } n_{j+1} \geq 1$$

and

$$(36) \quad \{P_{n_{j+1}}^{(j+1)}\}_{n_{j+1}=1}^{\infty} \text{ converges uniformly to } P^{(j+1)}.$$

Moreover, if $j + 2 \leq \nu$, Lemma 4 shows from (35) that

$$(37) \quad T_{j+1}(P^{(j+1)}(D)) \subset P^{(j+1)}(D).$$

When $j = 1$, conditions (30)-(32) immediately follow by regarding P in Lemma 2 as an identity mapping. Also from (31) and Lemma 4, we get (33).

Therefore, by induction, it follows that $\lim_{n_\nu \rightarrow \infty} P_{n_\nu}^{(\nu)} x = P^{(\nu)} x \in P^{(\nu)}(D) = \bigcap_{i=1}^{\nu} F[T_i]$. This completes the proof of the theorem.

From the finite intersection property, we have the following result due to DeMarr [2]. And note that we do not assume Zorn's lemma in our proof.

COROLLARY 1. *Let $T_i (i \in J, J \text{ is an index set})$ be commuting nonexpansive mapping from a compact convex subset of a Banach space into itself. Then there exists a point u in D such that $T_i u = u$ for all $i \in J$.*

When $\nu = 1$ and $\alpha_1 = 1/2$, we have the following corollary, which is essentially equal to the result we have obtained as a Corollary 2 in [3].

COROLLARY 2. *Let T be a nonexpansive mapping from a compact convex subset D of a Banach space into itself. Then $\{(I + T)/2\}_{n=1}^{\infty}x$ converges to a fixed point of T .*

The author would like to thank the referee for letting me know about reference [5].

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