Pacific Journal of Mathematics

ON CHARACTERIZATIONS OF EXPONENTIAL POLYNOMIALS

PHILIP G. LAIRD

Vol. 80, No. 2 October 1979

PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 2, 1979

ON CHARACTERIZATIONS OF EXPONENTIAL POLYNOMIALS

PHILIP G. LAIRD

This paper considers some characterizations of exponential polynomials in $C(G)$, the set of all continuous complex valued functions on a σ-compact locally compact Abelian group G. For $f \in C(G)$, U_f will denote the subspace of $C(G)$ obtained by taking finite linear combinations of translates of f. It is known that f is an exponential polynomial if and only if U_f is of finite dimension. Our main result is to show that f is an exponential polynomial when U_f is closed in $C(G)$ if $C(G)$ is given the topology of convergence uniform on all compact subsets of G.

Further characterizations of exponential polynomials are given when G is real Euclidean n-space, R^n.

A function $b \in C(G)$ is additive if $b(x+y) = b(x) + b(y)$ for all $x, y \in G$ and $g \in C(G)$ is an exponential if $g(x+y) = g(x)g(y)$ for all $x, y \in C(G)$. An exponential polynomial is a finite linear combination of terms $h = b_1^{q_1} b_2^{q_2} \cdots b_m^{q_m} g$ where $b_1, b_2, \cdots, b_m$ are additive, $q_1, q_2, \cdots, q_m$ are nonnegative integers and g is an exponential.

If f is an exponential polynomial, it is easy to see that U_f is finite dimensional. For if h is as above, then $T_\alpha h\colon x \to h(x-\alpha)$ is a finite linear combination of terms $b_1^{r_1} b_2^{r_2} \cdots b_m^{r_m} g$ for each $\alpha \in G$ where $r_j = 0, 1, \cdots, q_j$ for $j = 1, 2, \cdots, m$. A result of Engert [5] shows that if U_f is finite dimensional, then f is an exponential polynomial. The proof of this result when G is any σ-compact locally compact Abelian group is naturally more involved than when G is merely R or R^n. Proofs for the case of $C(R)$ may be found in Anselone and Korevaar [1] and Loewner [8] who also refers to $C(R^n)$.

Throughout this paper, the only topology considered on $C(G)$ is that of convergence uniform on all compact subsets of G. With G being σ-compact, let G be the countable union of compact sets K_p. Let $S_p(f) = \sup\{|f(x)|\colon x \in K_p\}$ and $d(f, g) = \sum_{p=1}^{\infty} 2^{-p} \min(1, S_p(f-g))$ for $f, g \in C(G)$. Then d is a metric for $C(G)$ and $C(G)$ is complete in this metric.

With such a topology for $C(G)$, if U_f is finite dimensional, it is closed. The converse to this is shown here (Theorem 3) so that in $C(G)$,

$$f \text{ is an exponential polynomial} \Longleftrightarrow U_f \text{ is finite dimensional}$$
$$\Longleftrightarrow U_f \text{ is closed in } C(G)\,.$$

In showing that when U_f is closed, it is then finite dimensional, the following notation shall be used throughout. As above, assume that $G = \bigcup_{p=1}^{\infty} K_p$ where each K_p is compact. For a given function f in $C(G)$, set

$$S_p = \Big\{ g \in C(G) \colon g = \sum_{k=1}^{p} a_k T_{\beta_k} f$$

$$\text{where } |a_k| \leqq p \text{ and } \beta_k \in K_p \text{ for } k = 1, 2, \cdots, p \Big\} .$$

It is clear that $U_f = \bigcup_{p=1}^{\infty} S_p$. The method of proof is one suggested by Edwards [4], pages 38–39 in establishing the result for functions on the circle group.

LEMMA 1. *S_p is pointwise equicontinuous in $C(G)$.*

Proof. Let $x \in G$ and $\varepsilon > 0$. Let B denote the set of all neighborhoods of 0 in G. It suffices to show that there is a $U \in B$ such that

$$|f(x - \alpha) - f(y - \alpha)| < \varepsilon/p^2 \text{ for all } \alpha \in K_p \text{ and all } y \text{ with } y - x \in B .$$

Then

$$|g(x) - g(y)| < \sum_{k=1}^{p} |a_k| \varepsilon/p^2 \leqq \varepsilon$$

whenever $y - x \in U$ and $g \in S_p$.

Set $F = x - K_p$ so if $\alpha \in K_p$, $\beta = x - \alpha \in F$. For each $\beta \in F$, there exists $V_\beta \subset B$ such that $|f(z) - f(\beta)| < \varepsilon/2p^2$ whenever $z - \beta \in V_\beta$. For this V_β, there is a $W_\beta \in B$ such that $W_\beta + W_\beta \subset V_\beta$. With $\{\beta + W_\beta \colon \beta \in F\}$ forming an open cover for the compact set F, select a finite subcover $\{\beta_j + W_{\beta_j}\}_{j=1}^{m}$. Let $W = \bigcap_{j=1}^{m} W_{\beta_j}$ and $U = W \cap (-W)$ so $U \in B$. If $\alpha \in K_p$ and $x - \alpha \in F$, $x - \alpha \in \beta_l + W_{\beta_l}$ say. Then

$$y - \alpha = y - x + x - \alpha \in U + x - \alpha \subset \beta_l + V_{\beta_l}$$

which also contains $x - \alpha$. Hence $f(x - \alpha)$ and $f(y - \alpha)$ differ from $f(\beta_l)$ by amounts in modulus less than $\varepsilon/2p^2$ and the result follows.

LEMMA 2. *S_p is compact in $C(G)$.*

Proof. Use is made of the condition that in $C(G)$, a closed equicontinuous set S is compact if $S[x] = \{f(x) \colon f \in S\}$ is compact in C (see, for example, [3], page 34 or [6], page 234). With f being continuous and $x \in G$, $\{f(x - \beta) \colon \beta \in K_p\}$ is compact whence $S_p[x]$ is compact in C. To show that S_p is closed, let $\{g_q\}$ be any Cauchy sequence in S_p with $g_q = \sum_{k=1}^{p} a_{q,k} T_{\beta_{q,k}} f$. Since $|a_{q,1}| \leqq p$ for all positive integers q, a convergent subsequence $a_{q',1}$ may be found with limit, say a_1, and $|a_1| \leqq p$. Continue in this manner to find convergent subsequences $\{a_{r,k}\}_{r=1}^{\infty}$ for $k = 1, 2, \cdots, p$ with respective limits a_k where $|a_k| \leqq p$. Now use $\{\beta_{r,k}\}_{r=1}^{\infty} \subset K_p$ for $k = 1, 2, \cdots, p$ and K_p is compact to find convergent subsequences $\{\beta_{v,k}\}_{v=1}^{\infty}$. With $a_{v,k} \to a_k$, $|a_k| \leqq p$ and $\beta_{v,k} \to \beta_k \in K_p$ as $v \to \infty$ for $k = 1, 2, \cdots, p$, it follows that $g_v \to g$ for some $g \in S_p$ So $g_q \to g$ as $q \to \infty$ showing that S_p is closed. Hence S_p is compact in $C(G)$.

THEOREM 3. *If U_f is closed in $C(G)$, then U_f is finite dimensional.*

Proof. Since $U_f = \bigcup_{p=1}^{\infty} S_p$ is closed in the metric space $C(G)$, it follows by Baire's category theorem applied to U_f that there must be as S_p that is not nowhere dense. As this S_p is closed, it must have a nonvoid interior. Hence U_f contains a compact neighbourhood of zero. So, by Riesz's theorem (see, for example [3], page 65) U_f is finite dimensional.

The remainder of this article, concerns exponential polynomials in $C(R^n)$. These functions in $C(R^n)$ are finite linear combinations of terms $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \exp(a_1 x_q + a_2 x_2 + \cdots + a_n x_n)$ where $x = (x_1, x_2, \cdots, x_n) \in R^n$, $p_1, p_2, \cdots, p_n$ are nonnegative integers and $a_1, a_2, \cdots, a_n$ are complex numbers. In restricting G to be R^n, little economy of the proof of Theorem 3 is gained except for Lemma 1. However, it is considerably easier to show for $C(R^n)$ compared with $C(G)$ that if U_f is finite dimensional, then f is an exponential polynomial. A new and simple proof is as follows.

Suppose that U_f has finite dimension m where $m > 1$. (If $m = 0$, $f = 0$ and if $m = 1$ a simpler version of the following suffices.) Let $g_1, g_2, \cdots, g_m$ be a basis of U_f and $g = (g_1, g_2, \cdots, g_m)$. Then $T_\alpha g = A(\alpha) g$ where $A(\alpha)$ is an $m \times m$ complex matrix. From $T_{\alpha+\beta} = T_\alpha T_\beta$, one finds that $A(\alpha + \beta) = A(\alpha) A(\beta)$ and $A(0) = I$, the unit matrix. Since $T_\alpha f \to T_\beta f$ as $\alpha \to \beta$, $A(\alpha)$ is continuous. So $z \in R^n$ near 0 may be chosen and fixed so that $A(z)$ is nonsingular. It is clear from

$$A(x) = \left(\int_{x_1}^{x_1+z_1} \cdots \int_{x_n}^{x_n+z_n} A(y) dy\right)(A(z))^{-1} ,$$

that each partial derivative of A exists. Letting $\{e_1, e_2, \cdots, e_n\}$ be the standard basis for R^n,

$$D_j g = \lim_{h \to 0} (A(-he_j) - A(0)) g/h = C_j g ,$$

where the matrix $C_j = D_j A(0)$. So $D_j(\exp(-C_j x_j) g) = 0$ showing that $g = \exp(C_j x_j)\phi_j$ where ϕ_j is independent of x_j for $j = 1, 2, \cdots, n$ and ϕ_j takes value in R^m.

From $\exp(C_1 x_1)\phi_1 = \exp(C_2 x_2)\phi_2$ with $x_1 = 0$ $\phi_1(x_2, x_3, \cdots, x_n) = \exp(C_2 x_2)\phi_2(0, x_3, x_4, \cdots, x_n)$. Successively equating $\exp(C_j x_j)\phi_j = \exp(C_{j+1} x_{j+1})\phi_{j+1}$ with $x_j = 0$ for $j = 1, 2, \cdots, n-1$, we find

$$g = \exp(C_1 x_1)\exp(C_2 x_2)\cdots\exp(C_n x_n)d$$

where $d \in R^n$ is constant. As it is well known that the elements of $\exp(Cx)$ are exponential polynomials in x ([2], page 46), it follows that the components of g are exponential polynomials. Hence f is an exponential polynomial in $C(R^n)$ when U_f is finite dimensional.

Other characterizations of exponential polynomials in $C(R^n)$ are now given. For $C(R)$, one such is that of the set of all solutions to all nontrivial linear ordinary differential equations with constant coefficients. For $C(R^n)$ with $n > 1$, one cannot identify the set of all exponential polynomials with the set of all solutions to all nontrivial linear partial differential equations with constant coefficients. However, a necessary and sufficient condition that $f \in C(R^n)$ be an exponential polynomial is that there exists n nonzero linear differential operators $L_j = L_j(D_j)$ with constant coefficients where each L_j only involves the jth partial derivative D_j and $L_j f = 0$ for $j = 1, 2, \cdots, n$. A proof of this given by Laird [7], page 816, is reproduced here for completeness. The necessity of the condition is obvious. Conversely, if $f \in C(R^n)$ and if $L_1 f = 0$, then f is a finite sum of terms $A(x_2, x_3, \cdots, x_n)x_1^{q_1}\exp ax_1$. With $L_2 f = 0$, $L_2 A = 0$ and so each A is a finite sum of terms $B(x_3, x_4, \cdots, x_n)x_2^{q_2}\exp bx_2$. Continuing in this manner, one finds that f is an exponential polynomial.

The following is an extension of the above result.

THEOREM 4. *Let $f \in C(R^n)$ and let $A = (a_{jk})$ be a real nonsingular $n \times n$ real matrix. Then a necessary and sufficient condition that f be an exponential polynomial is that there exist n nonzero polynomials $P_1, P_2, \cdots, P_n$, each of one variable, such that*

$$P_j(a_{j1}D_1 + a_{j2}D_2 + \cdots + a_{jn}D_n)f = 0$$

for $j = 1, 2, \cdots, n$.

Proof. Let $u_k = \sum_{m=1}^{n} b_{km}x_m$ for $k = 1, 2, \cdots, n$ and $f(x) = g(u)$. Then

$$D_m f(x) = \sum_{k=1}^{n} \frac{\partial g}{\partial u_k}\frac{\partial u_k}{\partial x_m}$$

so that

$$\sum_{m=1}^{n} a_{jm} D_m f = \frac{\partial g}{\partial u_j}$$

when $B = (b_{km})$ is chosen so that $B^T = A^{-1}$. The given condition is then $P_j(D_j)g = 0$ for $j = 1, 2, \cdots, n$ which is equivalent to g and so to f being an exponential polynomial.

THEOREM 5. *Let $a \in R^n$, $f \in C(R^n)$ and $U_f(a)$ denote the subspace in $C(R^n)$ obtained from finite linear combinations of terms $f(x - ta)$ for $t \in R$. A necessary and sufficient condition that f be an exponential is that $U_f(a_j)$ be finite dimensional for n linearly independent vectors $a_1, a_2, \cdots, a_n$ in R^n.*

Proof. The necessity is easily seen from $U_f(a) \subset U_f$ for all $a \in R^n$, and if f is an exponential polynomial, then U_f is finite dimensional.

The converse, which has been recognized by Loewner [8] when $\{a_1, a_2, \cdots, a_n\}$ is the standard basis, may be shown directly, or as follows. Let $f_j(t) = f(ta_j)$ for all $t \in R$ and $j = 1, 2, \cdots, n$. If each $U_f(a_j)$ is finite dimensional in $C(R^n)$, then U_{f_j} is finite dimensional in $C(R^n)$. So each f_j is an exponential polynomial and there is a nonzero polynomial P_j so that $P_j(D)f_j = 0$. With $Df_j = a.\,\mathrm{grad} f$, the conditions of the sufficiency part of Theorem 4 are satisfied. Hence f is an exponential polynomial in $C(R^n)$.

ACKNOWLEDGMENTS. The author would like to thank Dr. R. V. Nillsen of the University of Wollongong for several helpful discussions and also acknowledge the use of the Science Citation Indices.

REFERENCES

1. P. M. Anselone and J. Korevaar, *Translation invariant subspaces of finite dimension*, Proc. Amer. Math. Soc., **15** (1964), 747-752.

2. W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, 1965.

3. R. E. Edwards, *Functional Analysis: Theory and Applications*, Holt, Rinehart and Winston, New York, 1965.

4. ———, *Fourier Series, A Modern Introduction*, Volume II, Holt, Rinehart and Winston, New York, 1967.

5. M. Engert, *Finite dimensional translation invariant subspaces*, Pacific J. Math., **32** (1970), 333-343.

6. J. L. Kelly, *General Topology*, Van Nostrand, Princeton, 1955.

7. P. G. Laird, *Entire mean periodic functions*, Canad. J. Math., **27** (1975), 805-818.

8. C. Loewner, *On some transformations invariant under Euclidean or non-Euclidean isometries*, J. Math. Mech., **8** (1959), 393-409.

Received January 4, 1977 and in revised form July 12, 1978.

UNIVERSITY OF WOLLONGONG
WOLLONGONG, N. S. W. 2500
AUSTRALIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, CA 90024

CHARLES W. CURTIS
University of Oregon
Eugene, OR 97403

C. C. MOORE
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM
Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1978 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Pacific Journal of Mathematics

Vol. 80, No. 2 October, 1979