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**ON COMMON FIXED POINT SETS OF COMMUTATIVE
MAPPINGS**

TECK CHEONG LIM

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Let C be a compact convex subset of a locally convex topological vector space X . Anzai and Ishikawa recently proved that if T_1, \dots, T_n is a finite commutative family of continuous affine self-mappings of C , then $F(\sum_{i=1}^n \lambda_i T_i) = \bigcap_{i=1}^n F(T_i)$ for every λ_i such that $0 < \lambda_i < 1$ and $\sum_{i=1}^n \lambda_i = 1$, where $F(T)$ denotes the fixed point set of T . It is natural to question whether the conclusion of their theorem is dependent on the topological properties of X, C and T_i —in this case, the linear topology, the compactness and the continuity. We shall see that this is not; the theorem can be formulated in an algebraic context.

Our theorem, when applied to Hausdorff topological vector spaces, yields a better version of Anzai-Ishikawa's theorem (see Corollary 2).

DEFINITION 1. A subset B of a real vector space is said to be (algebraically) bounded if $\bigcap_{\varepsilon > 0} \varepsilon(C - C) = \{0\}$, where $C = C_0(B)$, the convex hull of B .

Every bounded convex subset of a Hausdorff topological vector space is algebraically bounded. Every bounded subset of a locally convex Hausdorff topological vector space is algebraically bounded.

THEOREM 1. Let C be a convex subset of a real vector space X and T_1, \dots, T_n a finite commutative family of affine self-mappings of C . If the set $D = \{T_1^{m_1} T_2^{m_2} \dots T_n^{m_n} x : 0 \leq m_i < \infty, i = 1, \dots, n\}$ is bounded for each $x \in C$, then $F(\sum_{i=1}^n \lambda_i T_i) = \bigcap_{i=1}^n F(T_i)$ for every $0 < \lambda_i < 1$ with $\sum_{i=1}^n \lambda_i = 1$.

LEMMA 1. Let x_n be a sequence in a Banach space such that $x_n \rightarrow x$. Then the sequence y_n defined by

$$y_n = (1/2^n)(x_0 + {}_n C_1 x_1 + \dots + {}_n C_n x_n)$$

converges to x .

Proof. For an arbitrary $\varepsilon > 0$, choose m such that $\|x_i - x\| < \varepsilon/2$ for $i \geq m$. Choose $N \geq m$ such that

$$1/2^n(1 + {}_n C_1 + \dots + {}_n C_{m-1}) < \varepsilon/(2D)$$

for all $n \geq N$, where D is a number such that $\|x_i - x\| \leq D$ for all $i \geq 0$. Then

$$\begin{aligned} & \|y_n - x\| \\ &= \|(1/2^n)(x_0 - x + {}_n C_1(x_1 - x) + \dots + {}_n C_{m-1}(x_{m-1} - x) + \dots + x_n - x)\| \\ &\leq (1/2^n)(1 + {}_n C_1 + \dots + {}_n C_{m-1})D + (\varepsilon/2)(1/2^n)({}_n C_m + \dots + 1) \\ &< \varepsilon \end{aligned}$$

for all $n \geq N$.

REMARK 1. The above lemma is also a consequence on Silverman-Toeplitz's theorem on regular method of summability.

Proof of Theorem 1. We may assume that $n = 2$. The inclusion $\bigcap_1^n F(T_i) \subset F(\sum_1^n \lambda_i T_i)$ is obvious. Let $A = \lambda_1 I + \lambda_2 T_1, B = \lambda_2 I + \lambda_1 T_2$ and $T = (1/2)(A + B)$. Then $T = (1/2)(I + \lambda_1 T_1 + \lambda_2 T_2)$. Moreover, $F(T) = F(\lambda_1 T_1 + \lambda_2 T_2), F(A) = F(T_1)$ and $F(B) = F(T_2)$. Let $x \in F(\lambda_1 T_1 + \lambda_2 T_2) = F(T)$. For every n , we have

$$\begin{aligned} (1) \quad x &= \left(\frac{A + B}{2}\right)^n x \\ &= \frac{1}{2^n} (A^n x + {}_n C_1 A^{n-1} B x + \dots + {}_n C_i A^{n-i} B^i x + \dots + B^n x) \end{aligned}$$

and

$$(2) \quad T_1 x = \frac{1}{2^n} (T_1 A^n x + {}_n C_1 T_1 A^{n-1} B x + \dots + {}_n C_i T_1 A^{n-i} B^i x + \dots + T_1 B^n x),$$

where we make use of the commutativity of A and B and the affine property of T_1 .

Following Anzai-Ishikawa's computation [1], we have

$$\begin{aligned} A^m y - T_1 A^m y &= \sum_{i=0}^m {}_m C_i \lambda_1^{m-i} \lambda_2^i T_1^i y - \sum_{i=1}^{m+1} {}_m C_{i-1} \lambda_1^{m-i+1} \lambda_2^{i-1} T_1^i y \\ &= \sum_{i=0}^{m+1} ({}_m C_i \lambda_1^{m-i} \lambda_2^i - {}_m C_{i-1} \lambda_1^{m-i+1} \lambda_2^{i-1}) T_1^i y, \quad {}_m C_{-1} = {}_m C_{m+1} = 0 \\ &= \sum_{i=0}^{m_0} \mu_i T_1^i y - \sum_{i=m_0+1}^{m+1} (-\mu_i) T_1^i y \\ &= a_m \left(\sum_{i=0}^{m_0} \alpha_i T_1^i y - \sum_{i=m_0+1}^{m+1} \beta_i T_1^i y \right). \end{aligned}$$

Here, m_0 is the largest integer less than or equal to $\lambda_2(m + 1)$;

$$\mu_i = {}_m C_i \lambda_1^{m-i} \lambda_2^i - {}_m C_{i-1} \lambda_1^{m-i+1} \lambda_2^{i-1},$$

$\mu_i \geq 0$ for $0 \leq i \leq m_0$ and < 0 for $m_0 + 1 \leq i \leq m + 1$;

$$a_m = \sum_{i=0}^{m_0} \mu_i = \sum_{i=m_0+1}^{m+1} (-\mu_i) = {}_m C_{m_0} \lambda_1^{m-m_0} \lambda_2^{m_0} \longrightarrow 0$$

as $m \rightarrow \infty$; $\alpha_i = \mu_i/a_m \geq 0$ for $0 \leq i \leq m_0$, $\beta_i = -\mu_i/a_m \geq 0$ for $m_0 + 1 \leq i \leq m + 1$, $\sum_{i=0}^{m_0} \alpha_i = 1$ and $\sum_{i=m_0+1}^{m+1} \beta_i = 1$.

Let $E = C_0(D)$. By the convexity of E , $A^m y - T_1 A^m y \in a_m(E - E)$ provided $T_1^i y \in E$ for $i = 0, \dots, m + 1$.

Since T_1 and T_2 are affine, $T_1^i A^k B^j x \in E$ for $j, k = 0, \dots, n$; $i = 0, 1, \dots$. It follows from (1) and (2) that

$$\begin{aligned} x - T_1 x &= \frac{1}{2^n} ((A^n x - T_1 A^n x) + {}_n C_1 (A^{n-1} B x - T_1 A^{n-1} B x) + \dots + (B^n x - T_1 B^n x)) \\ &\in \frac{1}{2^n} (a_n(E - E) + {}_n C_1 a_{n-1}(E - E) + \dots + {}_n C_{n-1} a_1(E - E) + a_0(E - E)) \\ &\subseteq \frac{1}{2^n} (a_0 + {}_n C_1 a_1 + \dots + {}_n C_i a_i + \dots + a_n)(E - E), \end{aligned}$$

the last inclusion being a consequence of the convexity of $E - E$.

Since $E - E$ is convex and $0 \in E - E$, we have $\varepsilon_1(E - E) \subseteq \varepsilon_2(E - E)$ if $\varepsilon_1 < \varepsilon_2$. Hence by Lemma 1 and the boundedness of E , $\bigcap_1^n A(n)(E - E) = \{0\}$ where

$$A(n) = \frac{1}{2^n} (a_0 + {}_n C_1 a_1 + \dots + {}_n C_i a_i + \dots + a_n).$$

It follows that $x = T_1 x$. Similarly $x = T_2 x$. This completes the proof.

COROLLARY 1. *Let C be a bounded convex subset of a vector space and T_1, \dots, T_n a finite commutative family of affine mappings of C . Then $F(\sum_{i=1}^n \lambda_i T_i) = \bigcap_{i=1}^n F(T_i)$ for all positive numbers λ_i , $i = 1, \dots, n$ such that $\sum_{i=1}^n \lambda_i = 1$.*

COROLLARY 2. *Let C be a convex bounded (in the usual sense) subset of a Hausdorff topological vector space and T_1, \dots, T_n a finite commutative family of affine mappings of C . Then $F(\sum_i \lambda_i T_i) = \bigcap_1^n F(T_i)$ for all positive numbers λ_i , $i = 1, \dots, n$ such that $\sum_1^n \lambda_i = 1$.*

REMARK 2. We note that the boundedness condition cannot be removed. The mappings $T_1 x = x + a$, $T_2 x = x - a$, $a \neq 0$ defined on R^1 are commutative and affine, with $F(T_1) = F(T_2) = \phi$ and $F((1/2)T_1 + (1/2)T_2) = R^1$.

COROLLARY 3. *Let C , T_i , $i=1, \dots, n$ be defined as in Corollary 2. Assume that $T_1^p \dots T_n^p = T_1 \dots T_n$ for some $p \geq 2$ and that for each $x \in C$ and each $i = 1, \dots, n$, the set*

$$A_i x = \{T_{T_1}^{m_1} \dots \hat{T}_i^{m_i} \dots T_n^{m_n} X: 0 \leq m_j < \infty, j = 1, \dots, n\}$$

is bounded, where \wedge indicates that $T_i^{m_i}$ is missing. Then $F(\sum_1^n \lambda_i T_i) = \bigcap_1^n F(T_i)$ for all positive numbers $\lambda_i, i = 1, \dots, n$ such that $\sum_1^n \lambda_i = 1$.

Proof. If $m_i, i = 1, \dots, n$ are n natural numbers and $m_j = \min \{m_i, i = 1, \dots, n\}$, then

$$\begin{aligned} T_1^{m_1} \dots T_n^{m_n} X &= T_1^{m_1 - m_j} \dots \hat{T}_j^{m_j - m_j} \dots T_n^{m_n - m_j} T_1^{m_j} \dots T_n^{m_j} X \\ &= T_1^{m_1 - m_j} \dots T_n^{m_n - m_j} T_1^k \dots T_n^k X \in A_j(T_1^k \dots T_n^k X) \end{aligned}$$

where k is an integer satisfying $0 \leq k < p$. It follows that

$$\begin{aligned} Ax &= \{T_1^{m_1} \dots T_n^{m_n} x : 0 \leq m_i < \infty, i = 1, \dots, n\} \\ &= \bigcup \{A_i(T_1^k \dots T_n^k x) : i = 1, \dots, n, k = 0, \dots, p - 1\}. \end{aligned}$$

Hence Ax , being a finite union of bounded sets is bounded.

The special case when $n = 2$ and $p = 2$ can be given a simple direct proof. We shall illustrate it for the case $\lambda_1 = \lambda_2 = 1/2$. First we prove:

LEMMA 2. For each $n \geq 1$, there exists rational numbers (depending on n and not necessarily nonnegative) $\lambda_1, \dots, \lambda_{[n/2]}$ such that

$$\lambda_1 + \dots + \lambda_{[n/2]} = 1 - (1/2^{n-1})$$

and such that the equation

$$\begin{aligned} (3) \quad D^n &= (1/2^{n-1}) \left(\frac{A^n + B^n}{2} \right) + \lambda_1 ABD^{n-2} + \dots + \lambda_i A^i B^i D^{n-2i} + \dots \\ &+ \lambda_{[n/2]} A^{[n/2]} B^{[n/2]} D^{n-2[n/2]} \end{aligned}$$

is valid for any two commutative affine mappings A, B defined on a convex set, where $D = 1/2(A + B)$. ($[m]$ denotes the largest integer $\leq m$.)

Proof. If such rational numbers λ_i exist for a fixed n , then by putting $A = B = I$, we see that

$$\lambda_1 + \dots + \lambda_{[n/2]} = 1 - (1/2^{n-1}).$$

We shall prove by induction on n . For $n = 2, \lambda_1 = 1/2$. Assume that the lemma is true for $m \leq n$. Then

$$\begin{aligned} (4) \quad D^{n+1} &= D^n D = (1/2^n) \left(\frac{A^{n+1} + B^{n+1}}{2} \right) + \frac{1}{2} (1/2^{n-2}) \left(\frac{A^{n-1} + B^{n-1}}{2} \right) AB \\ &+ \lambda_1 ABD^{n-1} + \dots + \lambda_{[n/2]} A^{[n/2]} B^{[n/2]} D^{n-2[n/2]+1}. \end{aligned}$$

Making use of the induction hypothesis for $m = n - 1$, substitute

$$(1/2^{n-2})\left(\frac{A^{n+1} + B^{n+1}}{2}\right) = D^{n-1} - \mu_1 ABD^{(n+1)-4} - \dots \\ - \mu_{[(n-1)/2]} A^{[(n-1)/2]} B^{[(n-1)/2]} D^{n-1-2[(n-1)/2]}$$

into (4). The proof will be then complete by collecting similar terms and making use of $[(n - 1)/2] + 1 = [(n + 1)/2]$ and $[(n + 1)/2] - [n/2] = 0$ or 1.

COROLLARY 4. *Let C be defined as in Theorem 1 and A, B be two commutative affine self-mappings of C such that $A^2B^2 = AB$ and such that the sets $\{A^n x: n = 0, 1, 2, \dots\}$ and $\{B^n x: n = 0, 1, 2, \dots\}$ are bounded for each $x \in C$. Then $F((1/2)A + (1/2)B) = F(A) \cap F(B)$.*

Proof. Let $x \in F((1/2)A + (1/2)B)$. Using Lemma 2 and the condition $A^2B^2 = AB$, we have

$$(5) \quad x = \left(\frac{A + B}{2}\right)^n x = (1/2^{n-1})\left(\frac{A^n x + B^n x}{2}\right) + \left(1 - \frac{1}{2^{n-1}}\right)ABx.$$

Thus,

$$x - ABx = \frac{1}{2^{n-1}}\left(\frac{A^n x + B^n x}{2} - ABx\right).$$

By the boundedness condition, we see that $ABx = x$. By (3) for $n = 2$, we have $x = (A^2x + B^2x)/2$. By applying A to $x = (1/2)Ax + (1/2)Bx$ we have $Ax = (1/2)A^2x + (1/2)x$ and hence $A^2x - x = 2(Ax - x)$. Thus by repeatedly replacing A, B by A^2 and B^2 in the above argument, we obtain $A^{2^n}x - x = 2^n(Ax - x)$. This contradicts the boundedness of $\{A^n x: n = 0, 1, 2, \dots\}$ unless $Ax = x$. Similarly $Bx = x$, completing the proof.

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K. Adachi, <i>On the multiplicative Cousin problems for $N^P(D)$</i>	297
Howard Banilower, <i>Isomorphisms and simultaneous extensions in $C(S)$</i>	305
B. R. Bhonsle and R. A. Prabhu, <i>An inversion formula for a distributional finite-Hankel-Laplace transformation</i>	313
Douglas S. Bridges, <i>Connectivity properties of metric spaces</i>	325
John Patton Burgess, <i>A selection theorem for group actions</i>	333
Carl Claudius Cowen, <i>Commutants and the operator equations $AX = \lambda XA$</i>	337
Thomas Curtis Craven, <i>Characterizing reduced Witt rings. II</i>	341
J. Csima, <i>Embedding partial idempotent d-ary quasigroups</i>	351
Sheldon Davis, <i>A cushioning-type weak covering property</i>	359
Micheal Neal Dyer, <i>Nonminimal roots in homotopy trees</i>	371
John Erik Fornæss, <i>Plurisubharmonic defining functions</i>	381
John Fuelberth and James J. Kuzmanovich, <i>On the structure of finitely generated splitting rings</i>	389
Irving Leonard Glicksberg, <i>Boundary continuity of some holomorphic functions</i>	425
Frank Harary and Robert William Robinson, <i>Generalized Ramsey theory. IX. Isomorphic factorizations. IV. Isomorphic Ramsey numbers</i>	435
Frank Harary and Allen John Carl Schwenk, <i>The spectral approach to determining the number of walks in a graph</i>	443
David Kent Harrison, <i>Double coset and orbit spaces</i>	451
Shiro Ishikawa, <i>Common fixed points and iteration of commuting nonexpansive mappings</i>	493
Philip G. Laird, <i>On characterizations of exponential polynomials</i>	503
Y. C. Lee, <i>A Witt's theorem for unimodular lattices</i>	509
Teck Cheong Lim, <i>On common fixed point sets of commutative mappings</i>	517
R. S. Pathak, <i>On the Meijer transform of generalized functions</i>	523
T. S. Ravisankar and U. S. Shukla, <i>Structure of Γ-rings</i>	537
Olaf von Grudzinski, <i>Examples of solvable and nonsolvable convolution equations in \mathcal{K}'_p, $p \geq 1$</i>	561
Roy Westwick, <i>Irreducible lengths of trivectors of rank seven and eight</i>	575