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Suppose f is an entire function of infinite order with zeros restricted to a finite number of rays through the origin. It is shown for $p > 1$ that $N(r, 0) = o(m_p^+(r, f))$ where $m_p^+(r, f)$ is the L^p norm of $\log^+ |f(re^{i\theta})|$ and in addition that $N(r, 0) = o(T(r, f))$ as r tends to infinity omitting values in an exceptional set E of zero logarithmic density. The set E is shown by example in general to be nonempty, even for functions with zeros on a single ray and arbitrarily slow infinite rate of growth. These results settle certain questions arising from previous work of Edrei, Fuchs, and Hellerstein and of Hellerstein and Shea.

Introduction. In this paper we prove two theorems involving the rate of growth of an entire function f, the angular distribution of its zeros, and the Nevanlinna deficiency $d(0, f)$ of zero, defined to be

$$d(0, f) \equiv 1 - \limsup_{r \to \infty} N(r, 0)/T(r, f) ,$$

where $N(r, 0)$ is the usual integrated counting function of the zeros of f and $T(r, f)$ is the Nevanlinna characteristic. Conditions on the rate of growth of f and on the arguments of its zeros sufficient to imply $d(0, f) > 0$ have been known for some time [1, Theorem 2]. Of particular interest here is the following result of Edrei, Fuchs, and Hellerstein [3, Theorem 2].

THEOREM A. *Suppose f is an entire function with zeros restricted to the K distinct rays* $\arg z = \alpha_j$, $1 \leq j \leq K$. *There exists* $K' = K'(\alpha_1, \cdots, \alpha_K)$ *and an absolute constant* $A \in (0, 1)$ *such that if f has finite order $\lambda > K'$ then $d(0, f) > B_\lambda$ for some $B_\lambda > A$.*

Later Hellerstein and Shea [7] showed that in Theorem A the quantity B_λ can be chosen so that $B_\lambda \to 1$ as $\lambda \to \infty$, and in addition obtained a sharp asymptotic bound for B_λ in the case that the zeros of f are real. (For other related results, see [4], [5, Chapter 6], [8], and [11].)

In view of Theorem A and the above result of Hellerstein and Shea, it is natural to ask [6, Problem 1.12] if $d(0, f) > 0$ or even $d(0, f) = 1$ for entire f of infinite order with zeros on only a finite number of rays through the origin. We answer this question in the negative and explore certain related questions by proving the following two theorems. (We recall that a nondecreasing function $\varphi\colon(-\infty,\infty)\to[0,\infty)$ is strongly convex if it is convex and $\varphi(x)/x\to\infty$ as $x\to\infty$.)

THEOREM 1. *Suppose f is entire of infinite order with zeros restricted to a finite number of rays through the origin. Then*

$$\lim_{r\to\infty}\int_0^{2\pi}\varphi\left(\frac{\log^+|f(re^{i\theta})|}{N(r,0)}\right)d\theta=\infty \tag{1}$$

for every strongly convex function φ and in particular for $p>1$

$$\lim_{r\to\infty}\frac{N(r,0)}{m_p^+(r,f)}=0 \tag{2}$$

where

$$m_p^+(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}(\log^+|f(re^{i\theta})|)^p d\theta\right)^{1/p}.$$

Furthermore there exists a set $E\subset[1,\infty)$ having logarithmic density zero such that

$$\lim_{\substack{r\to\infty\\ r\notin E}}\frac{N(r,0)}{T(r,f)}=0. \tag{3}$$

In general under the above hypotheses $N(r,0)/T(r,f)$ does not tend to zero as r tends to infinity without restriction, even for functions with zeros on a single ray and arbitrarily slow infinite rate of growth, as is shown by

THEOREM 2. *Suppose $\kappa\colon(0,\infty)\to(0,\infty)$ is such that $\kappa(r)\to\infty$ as $r\to\infty$. Associated with κ there exists an entire f having infinite lower order and positive zeros which satisfies $d(0,f)=0$ and*

$$\frac{\log T(r,f)}{\log r}<\kappa(r) \tag{4}$$

for sufficiently large r.

Our approach to both Theorem 1 and Theorem 2 is to study f via the Fourier series of $\log|f(re^{i\theta})|$. We prove (3) by in fact showing that as r tends to infinity through values not in E, the ratio of $N(r,0)$ to the maximum term of the Fourier series of $\log|f(re^{i\theta})|$ tends to zero. In our proof of Theorem 2 we achieve $d(0,f)=0$ by constructing f so that, for an appropriate sequence

r_n tending to ∞, the Fourier series of $\log|f(r_ne^{i\theta})|$ is approximated, in a suitable sense, by the Fourier series of the product of $N(r_n, 0)$ and a certain Poisson kernel. Because of the intricate nature of this construction, we provide an overview of the proof of Theorem 2 at the beginning of §3.

We assume familiarity with the notation of Nevanlinna theory. Throughout the remainder of the paper we abbreviate $n(r, 0)$ by $n(r)$ and $N(r, 0)$ by $N(r)$. It is not intended that the constant m_0 have the same value with each occurrence.

1. Preliminaries. We recall for entire f the formulas, apparently first noticed by F. Nevanlinna [10], for the Fourier coefficients $c_m(r, f)$ of $\log|f(re^{i\theta})|$. If $f(0) = 1$ and $\log f(z) = \sum a_m z^m$ near 0, then for $m = 1, 2, 3, \cdots$

$$c_m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|e^{-im\theta}d\theta \tag{1.1}$$
$$= \frac{a_m}{2}r^m + \frac{1}{2m}\sum_{|z_\nu|\leqq r}\left(\left(\frac{r}{z_\nu}\right)^m - \left(\frac{\bar{z}_\nu}{r}\right)^m\right),$$

where $\{z_\nu\}$ denotes the sequence of zeros of f repeated according to multiplicity. Clearly

$$c_m(r, f) = \overline{c_{-m}(r, f)}\,, \quad m = -1, -2, -3, \cdots,$$

and $c_0(r, f) = N(r)$. A proof of these identities can be found in many places, including [9].

The following lemma is used in the proof of Theorem 1. Its essential idea is due to Weyl [13] and it appears in a form similar to that given below in [3, pp. 149–151]. We include its proof for completeness.

LEMMA 1.1. *Suppose $\alpha_1, \alpha_2, \cdots, \alpha_K$ are distinct elements of $[0, 2\pi)$. For real x, let x^* denote the unique number in $[-\pi, \pi)$ congruent to x modulo 2π. There exists an increasing sequence $I = \{n_q\}$ of positive integers such that I has positive density and*

$$(n_q\alpha_j)^* \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \tag{1.2}$$

for $1 \leqq j \leqq K$ and $q = 1, 2, 3, \cdots$.

Proof. Without loss of generality we assume no α_j is zero. Let $\omega_j = \alpha_j/2\pi$ for $1 \leqq j \leqq K$. Let $M(\leqq K)$ be the maximum number of the ω_j which are linearly independent over the integers. Renumbering if necessary, we assume $\omega_1, \omega_2, \cdots, \omega_M$ are linearly independent over the integers.

If $M = K$, we let $B = 1$. If $M < K$, we define B as follows. For $M < p \leqq K$, there exists an integer $\sigma > 0$ and integers m_{pj} such that

$$\sigma\omega_p = \sum_{j=1}^{M} m_{pj}\omega_j \,. \tag{1.3}$$

Set

$$B_p = \sum_{j=1}^{M} |m_{pj}| \,, \quad M < p \leqq K \,,$$

and

$$B = \sup\,(\sigma, B_{M+1}, B_{M+2}, \cdots, B_K) \,.$$

By a theorem of Weyl [13, Satz 16], since $\omega_1, \omega_2, \cdots, \omega_M$ are linearly independent over the integers, there exists a sequence I' of positive integers u_q having positive density such that for $q = 1, 2, 3, \cdots$

$$|u_q\omega_j - L_{qj}| < \frac{1}{12B} \,, \quad 1 \leqq j \leqq M \,, \tag{1.4}$$

for some integers L_{qj}. Thus in the case that $M = K$, the proof is finished by (1.4) upon setting $I = I'$ and $n_q = u_q$.

Suppose $M < K$. We note for all $q = 1, 2, 3, \cdots$

$$|\sigma u_q\omega_j - \sigma L_{qj}| < \frac{\sigma}{12B} \leqq \frac{1}{12} \,, \quad 1 \leqq j \leqq M \,. \tag{1.5}$$

If $p > M$, then for all q by (1.3) and (1.4)

$$\sigma u_q\omega_p = \sum_{j=1}^{M} m_{pj}u_q\omega_j = \sum_{j=1}^{M} m_{pj}(L_{qj} + \delta_{qj})$$

for some δ_{qj} with $|\delta_{qj}| < (12B)^{-1}$ for $1 \leqq j \leqq M$ and $q = 1, 2, 3, \cdots$. For $M < p \leqq K$ and all q, we set

$$C_{pq} = \sum_{j=1}^{M} m_{pj}L_{qj}$$

and notice that

$$\begin{aligned} |\sigma u_q\omega_p - C_{pq}| &\leqq \sum_{j=1}^{M} |m_{pj}|\,|\delta_{qj}| \\ &< \frac{B_p}{12B} \leqq \frac{1}{12} \,. \end{aligned} \tag{1.6}$$

From (1.5) and (1.6), we see that $I = \{n_q\}$ with $n_q = \sigma u_q$ satisfies all requirements of the lemma.

Our proof of Theorem 1 also requires

LEMMA 1.2. *If $\{n_q\}$ is an increasing sequence of positive integers which has positive density, then there exists a subsequence $n_{q_k} = m_k$ such that*

$$(1.7) \qquad \text{(i)} \quad m_{k+1}/m_k \to 1$$

and

$$\text{(ii)} \quad \sum_{k=1}^{\infty} \frac{1}{m_{k+1} - m_k} < \infty .$$

Proof. The fact that $\{n_q\}$ has positive density implies $n_{q+1}/n_q \to 1$. We let

$$\gamma_q = \max\{n_{p+1}/n_p \colon p \geqq q\}$$

and note that $\gamma_q \to 1$. For each q and each $a > 1$ it follows that there exists an integer $p \geqq q$ such that

$$(1.8) \qquad a \leqq n_p/n_q \leqq a\gamma_q .$$

We let $n_{q_1} = m_1$ be arbitrary and see from (1.8) that there exists a subsequence $n_{q_k} = m_k$ such that for $k = 1, 2, 3, \cdots$

$$(1.9) \qquad \left(1 + \frac{3}{k}\right) \leqq \frac{m_{k+1}}{m_k} \leqq \left(1 + \frac{3}{k}\right)\gamma_{q_k} ,$$

establishing (1.7i). Certainly (1.9) guarantees

$$\log m_k > 3 \log k - 0(1) ,$$

which in conjunction with (1.9) yields

$$\frac{1}{m_{k+1} - m_k} < \frac{1}{m_{k+1}\left(1 - \left(1 + \frac{3}{k}\right)^{-1}\right)} = O\left(\frac{1}{k^2}\right) ,$$

establishing (1.7ii).

2. Proof of Theorem 1. We begin with

LEMMA 2.1. *Suppose f is entire of infinite order with zeros on the distinct rays $\arg z = \alpha_j \in [0, 2\pi)$, $1 \leqq j \leqq K$. If $r_n \to \infty$ such that*

$$(2.1) \qquad \liminf_{n\to\infty} \frac{N(r_n)}{T(r_n, f)} > 0 ,$$

then there exists $\eta_1, \eta_2, \cdots, \eta_K$ *in* $[0, 1]$ *with* $\sum_{j=1}^{K} \eta_j = 1$ *and there exists a subsequence of* r_n *(still denoted by* r_n*) such that*

$$\lim_{n\to\infty} \frac{c_m(r_n, f)}{N(r_n)} = \sum_{j=1}^{K} \eta_j e^{-im\alpha_j} \tag{2.2}$$

for all integers m.

Proof. Without loss of generality we suppose $f(0) = 1$. Let $N_j(t)$ be the integrated counting function of the zeros of f on the ray $\arg z = \alpha_j$. By passing to a subsequence if necessary, we may assume

$$N_j(r_n)/N(r_n) \longrightarrow \eta_j \in [0, 1] \tag{2.3}$$

with $\sum_{j=1}^{K} \eta_j = 1$.

We write

$$f(z) = e^{h(z)} \prod_{j=1}^{K} f_j(z)$$

with

$$f_j(z) = \prod_{\nu} E\left(\frac{z}{z_{\nu j}}, \nu\right)$$

where $z_{\nu j}$ is the sequence of zeros of f on $\arg z = \alpha_j$ repeated according to multiplicity and arranged in order of increasing modulus. If $h(z) = \sum a_m z^m$, then for $m = 1, 2, 3, \cdots$

$$c_m(r, f) = \frac{a_m}{2} r^m + \sum_{j=1}^{K} c_m(r, f_j) ,$$

where by (1.1)

$$c_m(r, f_j) = \frac{1}{2m} \sum_{|z_{\nu j}| \le r} \left(\left(\frac{r}{z_{\nu j}}\right)^m - \left(\frac{\bar{z}_{\nu j}}{r}\right)^m \right) - \frac{1}{2m} \sum_{\nu < m} \left(\frac{r}{z_{\nu j}}\right)^m .$$

Two integrations by parts yield

$$c_m(r, f) = \frac{a_m}{2} r^m + \sum_{j=1}^{K} e^{-im\alpha_j} (g_{jm}(r) + N_j(r) + d_{jm} r^m) \tag{2.4}$$

where

$$d_{jm} = -\frac{1}{2m} \sum_{\nu < m} \frac{1}{|z_{\nu j}|^m}$$

and

$$g_{jm}(r) = \frac{m}{2} \int_0^r \left(\left(\frac{r}{t}\right)^m - \left(\frac{t}{r}\right)^m \right) \frac{N_j(t)}{t} dt .$$

We set

$$g_m(r) = \sum_{j=1}^{K} g_{jm}(r) .$$

Certainly the lower order of f is infinite. This fact (first established in [2]) can be deduced as follows. If $N(r)$ has finite order, then f, an entire function of infinite order, can be represented as the product of an entire function of finite order and a zero-free entire function, trivially implying the lower order of f is infinite. Suppose on the other hand that $N(r)$ has infinite order and let I be the sequence of integers of Lemma 1.1. By (2.4) for each fixed $m \in I$ we have as $r \to \infty$

$$\text{(2.5)} \qquad \operatorname{Re} \frac{c_m(r, f)}{r^m} \geqq \frac{\sqrt{3}}{2}(r^{-m}g_m(r) + r^{-m}N(r)) + O(1)$$
$$\geqq \frac{m\sqrt{3}}{8}\int_0^{r/2} \frac{N(t)}{t^{m+1}}dt + O(1) .$$

By Nevanlinna's First Fundamental Theorem,

$$\text{(2.6)} \qquad |c_m(r, f)| \leqq 2T(r, f)$$

for all m. Since $N(r)$ has infinite order, we conclude from (2.5) and (2.6) that f has infinite lower order.

From (2.1) we thus conclude

$$\text{(2.7)} \qquad \lim_{n\to\infty} \frac{r_n^m}{N(r_n)} = 0 , \quad m = 1, 2, 3, \cdots .$$

We next establish

$$\text{(2.8)} \qquad g_m(r_n) = o(N(r_n)) , \quad m = 1, 2, 3, \cdots .$$

If (2.8) were false, there would exist a positive integer m_0, $\varepsilon > 0$, and a subsequence of r_n (still denoted by r_n) such that

$$g_{m_0}(r_n) > \varepsilon N(r_n)$$

for all n. Since $g_m(r)/m$ is an increasing function of m for each fixed $r > 0$, for $m > m_0$ and $m \in I$ we have

$$g_m(r_n) > \frac{m}{m_0} g_{m_0}(r_n) > \frac{m\varepsilon}{m_0} N(r_n) , \quad n = 1, 2, 3, \cdots$$

and hence by (2.5) and (2.7)

$$\begin{aligned} \operatorname{Re} c_m(r_n, f) &\geqq \frac{\sqrt{3}}{2}(g_m(r_n) + N(r_n)) + O(r_n^m) \\ &\geqq \frac{\sqrt{3}}{2}\Big(\frac{m\varepsilon}{m_0} + 1\Big)N(r_n) + O(r_n^m) \\ &\geqq \frac{\sqrt{3}}{2}\Big(\frac{m\varepsilon}{m_0} + 1 + o(1)\Big)N(r_n)\,. \end{aligned} \tag{2.9}$$

Since $m \in I$ may be chosen arbitrarily large, (2.6) and (2.9) contradict (2.1), establishing (2.8).

For an arbitrary positive integer m, we now set $r = r_n$ in (2.4), divide by $N(r_n)$, and appeal to (2.3), (2.7), and (2.8) to deduce (2.2). For negative indices, (2.2) is established by conjugation. Its truth for $m = 0$ is obvious. This proves Lemma 2.1.

We now prove (1). If (1) were false, there would exist a strongly convex φ and a sequence $r_n \to \infty$ such that

$$\sup_n \int_0^{2\pi} \varphi\Big(\frac{\log^+ |f(r_n e^{i\theta})|}{N(r_n)}\Big)d\theta < \infty\,. \tag{2.10}$$

Thus (2.1) would hold for r_n, and by Lemma 2.1 we may consider a subsequence (still denoted by r_n) for which (2.2) holds. We seek a contradiction.

Nevanlinna's First Fundamental Theorem and (2.1) imply that the sequence of measures on the unit circle T defined by

$$d\mu_n = \frac{\log |f(r_n e^{i\theta})|}{2\pi N(r_n)}d\theta \tag{2.11}$$

is bounded in total variation norm, say by L. We show that the measures (2.11) converge weakly to the measere on T with point mass at $e^{i\alpha_j}$ having weight η_j. Suppose g is a continuous function on T and let P be a trigonometric polynomial. We have

$$\begin{aligned} w_n &\equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\log |f(r_n e^{i\theta})|}{N(r_n)}g(e^{i\theta})d\theta - \sum_{j=1}^{K} \eta_j g(^{i\alpha_j}) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\log |f(r_n e^{i\theta})|}{N(r_n)}(g(e^{i\theta}) - P(e^{i\theta}))d\theta \\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\log |f(r_n e^{i\theta})|}{N(r_n)}P(e^{i\theta})d\theta - \sum_{j=1}^{K} \eta_j g(e^{i\alpha_j}) \\ &\equiv x_n + y_n - \sum_{j=1}^{K} \eta_j g(e^{i\alpha_j})\,, \end{aligned}$$

where

$$|x_n| \leqq L||g - P||_\infty$$

and, by (2.2),

$$\lim_{n\to\infty} y_n = \sum_{j=1}^{K} \eta_j P(e^{i\alpha_j}) .$$

Since $\sum_{j=1}^{K} \eta_j = 1$, these last two observations imply

$$|w_n| \leqq (L+1)\|g-P\|_\infty + o(1) , \quad (n \longrightarrow \infty) .$$

Since the trigonometric polynomials are dense in the continuous functions on T, the asserted weak convergence is established.

Without loss of generality we suppose $\eta_1 > 0$. Let $\delta > 0$ be such that the arc $J = \{e^{i\theta}: |\theta - \alpha_1| < \delta\}$ contains no point $e^{i\alpha_j}$ for $2 \leqq j \leqq K$. Let $g: T \to [0, 1]$ be a continuous function vanishing on $T - J$ with $g(e^{i\alpha_1}) = 1$. In view of the weak convergence of the measures $d\mu_n$, for $n > n_0(g)$

$$\begin{aligned} \frac{\eta_1}{2} &< \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\log|f(r_n e^{i\theta})|}{N(r_n)} g(e^{i\theta})d\theta \\ &\leqq \frac{1}{2\pi}\int_{\alpha_1-\delta}^{\alpha_1+\delta} \frac{\log^+|f(r_n e^{i\theta})|}{N(r_n)} d\theta . \end{aligned}$$

Thus $\{\log^+|f(r_n e^{i\theta})|/N(r_n)\}$ is not a uniformly integrable family and it follows by standard arguments [12, pp. 37–38] that (2.10) cannot hold, giving the desired contradiction.

For $p > 1$ the choice in (1) of $\varphi(t) = t^p$ if $t \geqq 0$ and $\varphi(t) = 0$ if $t < 0$ establishes (2).

We now turn to the proof of (3) and again assume with no loss in generality that $f(0) = 1$. In view of Lemmas 1.1 and 1.2, we may now let $I = \{m_k\}$ be an increasing sequence of positive integers satisfying (1.7i and ii) and, in addition,

$$(2.12) \qquad (m_k\alpha_j)^* \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$$

for $1 \leqq j \leqq K$ and $k = 1, 2, 3, \cdots$.

Since f has infinite lower order (note the discussion leading to (2.5) does not use hypothesis (2.1)), we may assume $N(r)$ has infinite order. For $m = 1, 2, 3, \cdots$ we define a nondecreasing unbounded sequence s_m by

$$s_m = \inf\{t \geqq e: \log n(t)/\log t \geqq m/2\} .$$

Thus

$$(2.13) \qquad n(t) < t^{m/2} , \quad e \leqq t < s_m .$$

Again letting $z_{\nu j}$ be the zeros of f on $\arg z = \alpha_j$ repeated according to multiplicity, we represent f as

$$f(z) = e^{H(z)} \prod_{j=1}^{K} G_j(z)$$

where $H(z) = \sum B_m z^m$ is entire and

$$G_j(z) = \prod_{\nu} E\left(\frac{z}{z_{\nu j}}, q_\nu\right) \tag{2.14}$$

where $q_\nu = m$ if $s_m \leqq |z_{\nu j}| < s_{m+1}$ and $q_\nu = 0$ if $|z_{\nu j}| < s_1$. We show the product (2.14) converges to an entire function by establishing

$$\sum_{\nu} \left(\frac{r}{|z_{\nu j}|}\right)^{q_\nu + 1} < \infty \tag{2.15}$$

for every $r > 0$. Letting $\tilde{n}_j(t)$ be the number of zeros of G_j in $|z| < t$, we have from (2.13) for $m = 1, 2, 3, \cdots$

$$\begin{aligned} \sum_{s_m \leqq |z_{\nu j}| < s_{m+1}} \left(\frac{r}{|z_{\nu j}|}\right)^{q_\nu+1} &= \int_{s_m}^{s_{m+1}} \left(\frac{r}{t}\right)^{m+1} d\tilde{n}_j(t) \\ &\leqq \frac{r^{m+1}\tilde{n}_j(s_{m+1})}{s_{m+1}^{m+1}} + (m+1)\int_{s_m}^{s_{m+1}} \left(\frac{r}{t}\right)^{m+1} \frac{\tilde{n}_j(t)}{t} dt \\ &\leqq \left(\frac{r}{s_{m+1}^{1/2}}\right)^{m+1} + 2\left(\frac{r}{s_m^{1/2}}\right)^{m+1} \leqq 3\left(\frac{r}{s_m^{1/2}}\right)^{m+1} . \end{aligned} \tag{2.16}$$

Thus if $s_{m_0} > 4r^2$, then

$$\sum_{|z_{\nu j}| \geqq s_{m_0}} \left(\frac{r}{|z_{\nu j}|}\right)^{q_\nu+1} \leqq \frac{3}{2^{m_0}} ,$$

establishing (2.15).

Certainly for positive m

$$c_m(r, f) = \frac{B_m}{2} r^m + \sum_{j=1}^{K} c_m(r, G_j) , \tag{2.17}$$

where by (1.1)

$$\begin{aligned} c_m(r, G_j) = e^{-im\alpha_j} \Bigg\{ &\frac{1}{2m} \sum_{|z_{\nu j}| \leqq r} \left(\left(\frac{r}{|z_{\nu j}|}\right)^m - \left(\frac{|z_{\nu j}|}{r}\right)^m\right) \\ &- \frac{1}{2m} \sum_{q_\nu < m} \left(\frac{r}{|z_{\nu j}|}\right)^m \Bigg\} . \end{aligned} \tag{2.18}$$

Since $q_\nu < m$ is equivalent to $|z_{\nu j}| < s_m$, integration by parts yields

$$\begin{aligned} c_m(r, G_j) = e^{-im\alpha_j} \Bigg\{ &r^m \left(\frac{-\tilde{n}_j(s_m)}{2m s_m^m} - \frac{N_j(s_m)}{2 s_m^m}\right) \\ &+ N_j(r) - \frac{m}{2}\int_0^r \left(\frac{t}{r}\right)^m \frac{N_j(t)}{t} dt + \frac{m}{2}\int_{s_m}^r \left(\frac{r}{t}\right)^m \frac{N_j(t)}{t} dt \Bigg\} . \end{aligned} \tag{2.19}$$

Since $N(r) \leqq n(r) \log r + 0(1)$, for large m we have $N(s_m) < s_m^{3m/4}$ by (2.13). Combining (2.13), (2.17), and (2.19) we obtain

$$c_m(r, f) = \gamma_m r^m + \beta_m(r)N(r) + \frac{m}{2}\sum_{j=1}^{K} e^{-im\alpha_j}\int_{s_m}^{r}\left(\frac{r}{t}\right)^m \frac{N_j(t)}{t}dt \tag{2.20}$$

for a sequence of constants γ_m with $|\gamma_m|^{1/m} \to 0$ and a function $\beta_m(r)$ with $|\beta_m(r)| < 1$ for all $r > 0$ and all $m = 1, 2, 3, \cdots$.

For $\gamma_m \neq 0$ we set $\gamma_m = |\gamma_m| e^{i\rho_m}$, $0 \leqq \rho_m < 2\pi$, and let V be the set of $m \in I$ with $\gamma_m = 0$ or

$$\left|\rho_m - \frac{q\pi}{2}\right| < \frac{\pi}{4} \quad \text{for } q = 1 \text{ or } q = 3 . \tag{2.21}$$

For $m \in V$ it follows from (2.12), (2.20), (2.21), and elementary trigonometry that for $r > 0$

$$\begin{aligned} &|c_m(r, f) - \beta_m(r)N(r)| \\ &\quad = \left|\gamma_m r^m + \frac{m}{2}\sum_{j=1}^{K} e^{-im\alpha_j}\int_{s_m}^{r}\left(\frac{r}{t}\right)^m \frac{N_j(t)}{t}dt\right| \\ &\quad \geqq \left(\sin\frac{\pi}{12}\right)\frac{m}{2}\left|\sum_{j=1}^{K} e^{-im\alpha_j}\int_{s_m}^{r}\left(\frac{r}{t}\right)^m \frac{N_j(t)}{t}dt\right| \\ &\quad \geqq \frac{\sqrt{3}\, m\left(\sin\frac{\pi}{12}\right)}{4}\left|\int_{s_m}^{r}\left(\frac{r}{t}\right)^m \frac{N(t)}{t}dt\right| . \end{aligned} \tag{2.22}$$

For $m \in V$, we set $b_m = s_m$.

For $m \in I - V$ we set

$$\varphi_m(t) = \begin{cases} 0 & N(t) = 0 \\ \dfrac{1}{N(t)}\displaystyle\sum_{j=1}^{K} e^{-i(m\alpha_j + \rho_m)}N_j(t) & N(t) > 0 \end{cases}$$

and note that for $N(t) > 0$ the continuous function φ_m satisfies

$$\sin\frac{\pi}{12} \leqq |\mathrm{Re}\,\varphi_m(t)| \leqq 1 . \tag{2.23}$$

Thus from (2.20) for $r > 0$

$$\begin{aligned} &\mathrm{Re}\,(e^{-i\rho_m}(c_m(r, f) - \beta_m(r)N(r))) \\ &\quad = |\gamma_m| r^m + \frac{m}{2}\int_{s_m}^{r}\left(\frac{r}{t}\right)^m (\mathrm{Re}\,\varphi_m(t))\frac{N(t)}{t}dt . \end{aligned} \tag{2.24}$$

Since $N(t)$ has infinite order and $|\gamma_m|^{1/m} \to 0$, it follows from (2.23) that for $m \in I - V$ there exists a sequence $b_m \to \infty$ such that $N(b_m) > 0$ and

$$|\gamma_m| = \frac{m}{2}\int_{b_m}^{s_m} (\mathrm{Re}\,\varphi_m(t))\frac{N(t)}{t^{m+1}}dt \tag{2.25}$$

for $m > m_0$. Thus by (2.23), (2.24), and (2.25) for $m > m_0$ and $r > 0$

$$\begin{aligned} \frac{|c_m(r, f) - \beta_m(r)N(r)|}{\sin\frac{\pi}{12}} \\ \geqq \frac{m}{2\sin\frac{\pi}{12}}\left|\int_{b_m}^{r}\left(\frac{r}{t}\right)^m(\operatorname{Re}\varphi_m(t))\frac{N(t)}{t}dt\right| \\ \geqq \frac{m}{2}\left|\int_{b_m}^{r}\left(\frac{r}{t}\right)^m\frac{N(t)}{t}dt\right| . \end{aligned} \tag{2.26}$$

Without loss of generality we may suppose m_0 is so large that $m_0 < m_k \in I$ implies $m_{k+1}/m_k < 2$.

Let $I_0 = I \cap (m_0, \infty)$. From (2.6), (2.22), and (2.26) we see that to prove (3), it is sufficient to demonstrate the existence of a set $E \subset [1, \infty)$ with logarithmic density zero such that

$$\lim \frac{N(r)}{\sup_{m \in I_0}\left|m\int_{b_m}^{r}\left(\frac{r}{t}\right)^m\frac{N(t)}{t}dt\right|} = 0 \tag{2.27}$$

as r tends to infinity through values not in E.

For

$$r \geqq x_0 \equiv \min\{b_m\colon m \in I_0\} ,$$

let

$$\nu(r) = \max\{m\colon m \in I_0 \text{ and } b_m \leqq r\} .$$

We denote by S the range of the nondecreasing integer-valued function ν. For $m = m_k \in S$, we let $m' = m_{k+1}$. From the definition of ν we have

$$J_m \equiv \nu^{-1}\{m\} \subset [b_m, b_{m'}) . \tag{2.28}$$

Furthermore

$$[x_0, \infty) = \bigcup_{m \in S} J_m , \tag{2.29}$$

where the right side is a union of disjoint intervals. For $m \in S$ we define $0 < \varepsilon_m < 1$ by

$$1 + \varepsilon_m = m'/m \tag{2.30}$$

and note by (1.7i) that $\varepsilon_m \to 0$ as m tends to infinity through values in S.

Letting m_l denote logarithmic measure and letting S_1 be the set of $m \in S$ such that

$$m_l(J_m) < \frac{1}{m\varepsilon_m},$$

we see from (1.7ii) and (2.30) that if

$$E_1 = \bigcup_{m \in S_1} J_m,$$

then

$$m_l(E_1) \leqq \sum_{m \in S_1} \frac{1}{m' - m} < \infty . \tag{2.31}$$

For $m \in S - S_1$ we consider an arbitrary interval $\tilde{J}_m \subset J_m$ with

$$m_l(\tilde{J}_m) = \frac{1}{m\varepsilon_m} . \tag{2.32}$$

Since $d(\log N(t))/d(\log t) = n(t)/N(t)$ assumes a given value (m) at only a finite number of points in any bounded interval on which $n(t) > 0$, we see that there exists a real $y_m = y_m(\tilde{J}_m)$ such that

$$A_m \equiv \{t \in \tilde{J}_m : \log N(t) > m \log t + y_m\}$$

satisfies

$$m_l(A_m) = \frac{1}{m\varepsilon_m^{1/2}} . \tag{2.33}$$

We note that $r \in \tilde{J}_m - A_m$ and $t \in A_m$ imply

$$N(t) > N(r)(t/r)^m . \tag{2.34}$$

Given $r \in \tilde{J}_m - A_m$, let

$$A_m^*(r) = A_m^* = A_m \cap (0, r)$$

and

$$A_m^{**}(r) = A_m^{**} = A_m \cap (r, \infty) .$$

Thus either

$$m_l(A_m^*) \geqq \frac{1}{2m\varepsilon_m^{1/2}}$$

or

$$m_l(A_m^{**}) \geqq \frac{1}{2m\varepsilon_m^{1/2}} .$$

In the former case by (2.28) and (2.34)

$$(2.35)\qquad m\int_{b_m}^{r}\left(\frac{r}{t}\right)^m\frac{N(t)}{t}dt \geqq m\int_{A_m^*}\left(\frac{r}{t}\right)^m\frac{N(t)}{t}dt$$
$$\geqq mN(r)m_l(A_m^*) \geqq \frac{N(r)}{2\varepsilon_m^{1/2}} .$$

In the latter case by (2.28), (2.30), and (2.34)

$$m'\int_{r}^{b_m'}\left(\frac{r}{t}\right)^{m'}\frac{N(t)}{t}dt \geqq m'N(r)\int_{A_m^{**}}\left(\frac{t}{r}\right)^{m-m'}\frac{dt}{t}$$
$$(2.36)\qquad \geqq m'N(r)m_l(A_m^{**})\min_{t\in A_m^{**}}\left(\frac{t}{r}\right)^{-m\varepsilon_m}$$
$$\geqq \frac{m'N(r)}{2em\varepsilon_m^{1/2}} = \frac{(1+\varepsilon_m)N(r)}{2e\varepsilon_m^{1/2}} ,$$

since for $t\in A_m^{**}$

$$\log\left(\frac{t}{r}\right)^{-m\varepsilon_m} = -m\varepsilon_m(\log t - \log r)$$
$$\geqq -m\varepsilon_m m_l(\tilde{J}_m) = -1 .$$

From (2.32), (2.33), (2.35), and (2.36) we see there exists a set

$$E_2 \subset E_3 \equiv \bigcup_{m\in S-S_1} J_m$$

for which

$$m_l(E_2\cap[x_0, r)) = o(m_l(E_3\cap[x_0, r))) = o(\log r)$$

and such that (2.27) holds as r tends to infinity through values in $E_3 - E_2$. Combined with (2.29) and (2.31), this establishes (2.27) with $E = E_1 \cup E_2$ and thus proves (3).

3. **Proof of Theorem 2.** Due to the complicated nature of our construction, we begin with a brief outline of the proof of Theorem 2. We first construct an entire g with zero counting function $N(r)$ having the property that $\log N(r)$ is approximately a piecewise-linear convex function of $\log r$ (see (3.10)) such that, for a sequence r_n tending to infinity, $d(\log N(t))/d(\log t)$ evaluated at $t=r_n$ is much larger (approximately M_n^2) than is $\log N(r_n)/\log r_n$ (approximately M_n). (See (3.18) and the remarks immediately preceding it.) This key property enables us to construct polynomials h_n so that an initial segment of the Fourier series of $e^{h_n}g$ differs in L^2 norm from the corresponding portion of the Fourier series of

$$\left(\operatorname{Re}\frac{1+\beta_n e^{i\theta}}{1-\beta_n e^{i\theta}}\right)N(r_n) > 0$$

by $o(N(r_n))$. (See (3.26) and (3.28).) Because the sequence r_n is sufficiently well spaced, from the polynomials h_n we are able to construct an entire h (see (3.38)) so that the Fourier series of $\log|f| = \log|e^h g|$ behaves on $|z| = r_n$ much like that of $\log|e^{h_n} g|$, leading to (3.53) and implying $d(0, f) = 0$.

It would seem a fair observation that the essential difference between the infinite order and finite order situations is that in the former case infinitely many coefficients of $h(z) = \sum a_m z^m$ are at our disposal, subject only to the condition $|a_m|^{1/m} \to 0$, and that they may in fact be so chosen as to achieve $N(r_n)/T(r_n, f) \to 1$ on a widely spaced sequence; on the other hand, for f of finite order, only finitely many nonzero a_m are at our disposal and the approach employed below is clearly unavailable. Finally, we remark that much of the intricacy of the construction is a result of the requirement that f grow slowly in the sense of (4).

We now turn to the details of the proof and begin with

LEMMA 3.1. *Suppose* $\gamma\colon (0, \infty) \to (0, \infty)$ *is a nondecreasing function with* $\gamma(x)/x \to \infty$ *as* $x \to \infty$. *For* $x > 4$, *let* $\gamma_1(x) = \gamma((x-4)/4)$. *There exist sequences of positive integers* M_n *and* x_n *tending to infinity, a positive sequence* β_n *tending upward to* 1, *and a piecewise-linear convex function* $\varphi\colon [x_1, \infty) \to [1, \infty)$ *such that*

(3.1) (i) $M_{n+1} \geqq n M_n^4$, $n = 1, 2, 3, \cdots$;

(ii) $\dfrac{\beta_n^{2M_{n+1}}}{1 - \beta_n} < \dfrac{1}{n}$, $n = 1, 2, 3, \cdots$;

(iii) $\dfrac{x_{n+1}}{x_n} = 16 j_n$, $j_n =$ *integer*, $n = 1, 2, 3, \cdots$;

(iv) $1 - \beta_{n+1}^m < e^{-4m x_n}$, $M_{n+1} \leqq m \leqq 2M_{n+1}$, $n = 1, 2, 3, \cdots$;

(v) $\varphi(x) \leqq \gamma_1(x)$, $x \geqq x_1$;

(vi) φ *is convex on* $[x_n, x_{n+1}]$, *linear on* $[x_n, 8x_n]$, *and linear on each segment* $[8x_n + 4k, 8x_n + 4(k+1)]$ *contained in* $[x_n, x_{n+1}]$, $k =$ *integer*, $n = 1, 2, 3, \cdots$;

(vii) $\varphi'(x) \leqq (\varphi'(x-4))^4$, $x \geqq x_1 + 4$, *where* φ' *denotes the right derivative of* φ;

(viii) $M_n^2 = \varphi'(x_n) \leqq \varphi'(x) \leqq M_{n+1}^{1/2}$, $x_n \leqq x < x_{n+1}$, $n = 1, 2, 3, \cdots$;

(ix) $\dfrac{\varphi(x_n)}{x_n} \leqq M_n^{1/2}$, $n = 1, 2, 3, \cdots$;

(x) $\varphi(x_n) + M_{n+1}^{1/2}(x - x_n) + M_{n+2}^{1/2}(x - x_{n+1}) < \gamma_1(x)$ *for* $x \geqq x_{n+1}$, $n = 1, 2, 3, \cdots$;

and

(xi) $4M_{n+1}x < \gamma_1(x)$, $x \geqq x_n$, $n = 1, 2, 3, \cdots$.

Proof of Lemma 3.1. We let $M_1 = 2$ and let M_2 be an arbitrary integer greater than M_1^4. Let $0 < \beta_1 < 1$ be such that (3.1 ii) holds with $n = 1$. Let x_1 be an integer greater than 4 so large that (3.1 xi) holds with $n = 1$. Such an x_1 exists since $\gamma_1(x)/x \to \infty$. Define $\varphi(x_1) = 1$. We note (3.1 i), (3.1 ii), (3.1 ix), and (3.1 xi) are satisfied for $n = 1$ and (3.1 v) holds with $x = x_1$.

We now suppose for some positive integer p that we have a sequence of positive integers $M_1, M_2, \cdots, M_{p+1}$, a second sequence of positive integers $x_1, x_2, \cdots, x_p$, an increasing sequence $\beta_1, \beta_2, \cdots, \beta_p$ of positive numbers less than 1, and a function $\varphi\colon [x_1, x_p] \to [1, \infty)$. In addition we suppose (3.1 i), (3.1 ii), (3.1 ix), and (3.1 xi) hold for $n \leqq p$, (3.1 iii), (3.1 iv), (3.1 vi), (3.1 viii), (3.1 x) hold for positive $n \leqq p - 1$, that (3.1 v) holds for $x_1 \leqq x \leqq x_p$, and that (3.1 vii) holds for $x_1 + 4 \leqq x < x_p$. These hypotheses are satisfied in the case $p = 1$, vacuously in the case of (3.1 iii), (3.1 iv), (3.1 vi), (3.1 viii), (3.1 x), and (3.1 vii).

We define numbers β_{p+1}, M_{p+2}, and x_{p+1} and extend the definition of φ to $(x_p, x_{p+1}]$ in the following manner. We choose $\beta_{p+1} \in (\beta_p, 1)$ such that (3.1 iv) holds with $n = p$. We then let M_{p+2} be an integer such that (3.1 i) and (3.1 ii) hold with $n = p + 1$. We next choose

$$x_{p+1} > 8x_p + 8\left(\frac{1}{2}\log M_{p+1} - 2\log M_p\right) \tag{3.2}$$

such that (3.1 iii) and (3.1 x) hold with $n = p$ and (3.1 xi) holds with $n = p + 1$.

We now define φ on $(x_p, x_{p+1}]$. Recalling that φ' denotes the right derivative, we specify

$$\varphi'(x_p) = M_p^2 \tag{3.3}$$

and

$$\varphi'(8x_p + 4k) = 2^k M_p^2\,, \quad k = 0, 1, 2, \cdots, k_p\,, \tag{3.4}$$

where k_p is the largest integer k such that

$$2^{k+1} M_p^2 < M_{p+1}^{1/2}\,. \tag{3.5}$$

We note from (3.2) and (3.5) that

$$8x_p + 4(k_p + 1) < 8x_p + 8\left(\frac{1}{2}\log M_{p+1} - 2\log M_p\right) < x_{p+1}\,.$$

We define φ on $(x_p, x_{p+1}]$ to be the unique function satisfying (3.1 vi) with $n = p$, (3.3), (3.4), and

$$\varphi'(x) = M_{p+1}^{1/2}\,, \quad 8x_p + 4(k_p + 1) \leqq x < x_{p+1}\,. \tag{3.6}$$

Thus (3.1 viii) holds with $n = p$ and (3.1 ix) holds with $n = p + 1$.

In the case $p = 1$, we observe that (3.1 v) holds for $x_p \leqq x \leqq x_{p+1}$ by virtue of (3.1 xi) with $n = 1$ and (3.1 viii) with $n = 1$, since $x_1 \leqq x \leqq x_2$ implies

$$\varphi(x) \leqq 1 + M_2^{1/2}(x - x_1) \leqq 4M_2 x \leqq \gamma_1(x) .$$

If $p \geqq 2$, (3.1 viii) with $n = p - 1$ implies

$$\varphi(x_p) \leqq \varphi(x_{p-1}) + M_p^{1/2}(x_p - x_{p-1}) ,$$

which in conjunction with (3.1 x) with $n = p - 1$ and (3.1 viii) with $n = p$ implies (3.1 v) holds for $x_p \leqq x \leqq x_{p+1}$.

Finally we observe that (3.1 vii) holds for $x_1 + 4 \leqq x < x_{p+1}$. If $p = 1$, this is a result of (3.4) and (3.6) with $p = 1$. For $p \geqq 2$, inequality (3.1 vii) holds for $x_p \leqq x < x_{p+1}$ by (3.4) and (3.6), with equality holding for $x_p \leqq x < x_p + 4$.

This finishes the inductive step of the proof. We have (3.1 i), (3.1 ii), (3.1 ix), and (3.1 xi) holding with $n = p + 1$ and (3.1 iii), (3.1 iv), (3.1 vi), (3.1 viii), and (3.1 x) holding with $n = p$. In addition (3.1 v) holds for $x_1 \leqq x \leqq x_{p+1}$ and (3.1 vii) holds for $x_1 + 4 \leqq x < x_{p+1}$. Finally we notice that the convexity of φ follows from (3.1 vi) and (3.1 viii), and that $\beta_n \to 1$ by (3.1 iv). This completes the proof of Lemma 3.1. In what follows we shall make no use of (3.1 x). It is included only as an aid in the inductive step of the proof of the lemma.

We now use the lemma to prove the theorem. It is elementary that corresponding to κ of Theorem 2, there exists a nondecreasing $\gamma\colon (0, \infty) \to (0, \infty)$ and $x' > 0$ such that

$$\frac{\gamma(x)}{x} < \kappa(e^x) , \quad x > x' , \tag{3.7}$$

and $\gamma(x)/x \to \infty$ as $x \to \infty$. We apply Lemma 3.1 to this γ and define

$$n(t) = \begin{cases} 0 & 0 \leqq t < e^{x_1} \\ [e^{\varphi(\log t)}\varphi'(\log t)] & e^{x_1} \leqq t . \end{cases}$$

We note that $n(t)$ is nondecreasing and continuous from the right on $[0, \infty)$. We shall construct an entire f with positive zeros and $n(t, 1/f) = n(t)$.

We define $N(r) = \int_0^r (n(t)/t)dt$ and note for $\log r > x_1$ that

$$N(r) = (1 - \theta(r)) \exp (\varphi(\log r)) \tag{3.8}$$

for some $0 < \theta(r) < 1$ with $\theta(r) \to 0$ as $r \to \infty$. It follows immediately from (3.8) and the definition of $n(t)$ that

$$\frac{d(\log N(t))}{d(\log t)} = \frac{n(t)}{N(t)} = (1 + o(1))\varphi'(\log t) \tag{3.9}$$

and

$$\varphi(\log t) = \log N(t) + o(1) . \tag{3.10}$$

From (3.1 vii) and the convexity of φ we thus obtain

$$\begin{aligned} n(t)/N(t) &= (1 + o(1))\varphi'(\log t) \leqq (1 + o(1))(\varphi'(-1 + \log t))^4 \\ &\leqq (1 + o(1))(\varphi(\log t))^4 \leqq (1 + o(1))(\log N(t))^4 . \end{aligned} \tag{3.11}$$

Let $\alpha(x) = \varphi(x)/x$. The convexity of φ together with $\varphi'(x_1) > \alpha(x_1)$ implies α is continuous and strictly increasing on $[x_1, \infty)$. By (3.1 viii) certainly α is unbounded on $[x_1, \infty)$. Thus, for $m = 1, 2, 3, \cdots$, we may define a strictly increasing, unbounded sequence s_m by specifying s_m to be the unique solution of

$$\alpha(\log t) = m/2 . \tag{3.12}$$

From (3.8) and (3.12) it follows that

$$N(t) < t^{m/2} , \quad 0 < t \leqq s_m . \tag{3.13}$$

In view of (3.11) and (3.13)

$$n(s_m) \leqq (1 + o(1))N(s_m)(\log N(s_m))^4 = o(s_m^{3m/4}) ,$$

implying the existence of $A > 0$ such that for $m = 1, 2, 3, \cdots$

$$\frac{n(s_m) + N(s_m)}{s_m^m} < \frac{A}{s_m^{m/4}} . \tag{3.14}$$

We let $0 < z_1 \leqq z_2 \leqq z_3 \leqq \cdots$ be the nondecreasing sequence of positive numbers with counting function $n(t)$. For $z_\nu \leqq s_1$, we let $q_\nu = 0$. For $m \geqq 2$, we let $q_\nu = m - 1$ if $s_{m-1} < z_\nu \leqq s_m$. We define

$$g(z) = \prod_\nu E\left(\frac{z}{z_\nu}, q_\nu\right) .$$

In view of (3.11), (3.13), and the choice of q_ν, an argument (with $n(t)$ replacing $\tilde{n}_j(t)$) virtually identical to that leading to (2.15) shows g to be entire.

We now define a sequence r_n tending to infinity. For $n \geqq 1$ we select

$$x'_n \in (x_n, 2x_n) \tag{3.15}$$

satisfying

$$\alpha(x'_n) = M_n . \tag{3.16}$$

Since

$$\alpha(2x_n) > \frac{\varphi(2x_n) - \varphi(x_n)}{2x_n} = \frac{1}{2}\varphi'(x_n) = \frac{M_n^2}{2}$$

by (3.1 vi) and (3.1 viii), we see from (3.1 ix) and the continuity of α that such an x'_n exists.

We let

$$r_n = \exp x'_n \,. \tag{3.17}$$

For notational convenience we let $\rho_n = \varphi'(x'_n)$ and note by (3.1 vi) and (3.1 viii) that $\rho_n = M_n^2$. From (3.8), (3.16), and (3.17) we have

$$N(r_n) = (1 - o(1))r_n^{M_n} \,. \tag{3.18}$$

For each $n = 1, 2, 3, \cdots$, we now define a finite sequence a_{mn}, $1 \leqq m \leqq M_{n+1}$, as follows. For $1 \leqq m \leqq 2M_n$, let

$$a_{mn} = \frac{n(s_m)}{ms_m^m} + \frac{N(s_m)}{s_m^m} + \frac{2N(r_n)}{r_n^m}(\beta_n^m - 1) \,. \tag{3.19}$$

For $2M_n < m \leqq M_{n+1}$, define

$$\begin{aligned} a_{mn} = {} & \frac{n(s_m)}{ms_m^m} + \frac{N(s_m)}{s_m^m} + \frac{2N(r_n)}{r_n^m}(\beta_n^m - 1) \\ & + m\int_{r_n}^{s_m} \frac{N(t)}{t^{m+1}}dt + m\int_0^{r_n} \left(\frac{t}{r_n^2}\right)^m \frac{N(t)}{t}dt \,. \end{aligned} \tag{3.20}$$

We note that $s_m > r_n$ if and only if $m > 2M_n$. This is a consequence of the monotonicity of α and the fact that $\alpha(\log s_m) = m/2$ and $\alpha(\log r_n) = M_n$.

We now estimate the size of a_{mn} for $n \geqq 2$. For $1 \leqq m \leqq M_n$, by (3.1 iv) and (3.14)

$$\begin{aligned} |a_{mn}| & < \frac{A}{s_m^{m/4}} + \frac{2N(r_n)}{r_n^m}(1 - \beta_n^{M_n}) \\ & < \frac{A}{s_m^{m/4}} + \frac{2N(r_n)}{r_n^m}e^{-4mx_{n-1}} \,. \end{aligned} \tag{3.21}$$

For $M_n < m \leqq 2M_n$, from (3.1 iv), (3.14), and (3.18)

$$|a_{mn}| < \frac{A}{s_m^{m/4}} + 2(1 - \beta_n^m) \leqq \frac{A}{s_m^{m/4}} + 2e^{-4mx_{n-1}} \,. \tag{3.22}$$

For $2M_n < m \leqq M_{n+1}$, (3.18) implies

$$\frac{N(r_n)}{r_n^m}(1 - \beta_n^m) \leqq r_n^{M_n - m} < r_n^{-m/2} \,. \tag{3.23}$$

Elementary integration and (3.13) imply

$$(3.24) \qquad m\int_{r_n}^{s_m}\frac{N(t)}{t^{m+1}}dt + m\int_0^{r_n}\left(\frac{t}{r_n^2}\right)^m\frac{N(t)}{t}dt \leqq \frac{8}{3}r_n^{-m/2}\,.$$

From (3.14), (3.23), and (3.24) we conclude for $2M_n < m \leqq M_{n+1}$ that

$$(3.25) \qquad |a_{mn}| < \frac{A}{s_m^{m/4}} + \frac{5}{r_n^{m/2}}\,.$$

Our choice of a_{mn} is motivated by the fact that if

$$h_n(z) = \sum_{m=1}^{M_{n+1}} a_{mn}z^m\,,$$

then

$$(3.26) \qquad c_m(r_n, e^{h_n}g) = \begin{cases}\beta_n^m N(r_n) + A_{mn} & 1 \leqq m \leqq 2M_n\\ \beta_n^m N(r_n) & 2M_n < m \leqq M_{n+1}\end{cases}$$

where for $1 \leqq m \leqq 2M_n$

$$A_{mn} = \frac{m}{2}\int_{s_m}^{r_n}\left(\frac{r_n}{t}\right)^m\frac{N(t)}{t}dt - \frac{m}{2}\int_0^{r_n}\left(\frac{t}{r_n}\right)^m\frac{N(t)}{t}dt\,.$$

In fact if $F(z) = e^{H(z)}g(z)$ where $H(z) = \sum b_m z^m$, then, since $q_\nu < m$ is equivalent to $x_\nu \leqq s_m$, calculations similar to those involved in (2.18) and (2.19) show that

$$(3.27) \qquad \begin{aligned} c_m(r, F) &= r^m\left\{\frac{b_m}{2} - \frac{n(s_m)}{2ms_m^m} - \frac{N(s_m)}{2s_m^m}\right\} + N(r)\\ &\quad + \frac{m}{2}\int_{s_m}^{r}\left(\frac{r}{t}\right)^m\frac{N(t)}{t}dt - \frac{m}{2}\int_0^{r}\left(\frac{t}{r}\right)^m\frac{N(t)}{t}dt\,.\end{aligned}$$

In view of (3.19) and (3.20), (3.26) is a special case of (3.27).

We now show

$$(3.28) \qquad \left(\sum_{m=1}^{2M_n} A_{mn}^2\right)^{1/2} = o(N(r_n))\,.$$

We begin by recalling, from the remarks following (3.20), that $1 \leqq m \leqq 2M_n$ is equivalent to $s_m \leqq r_n$. From (3.1 vi), (3.1 viii), (3.10), and (3.15) it follows that uniformly on the interval $e^{x_n} \leqq t \leqq r_n$ we have as n tends to infinity

$$(3.29) \qquad N(t) = (1 + o(1))N(r_n)(t/r_n)^{\rho_n}\,.$$

First suppose $m \leqq 2M_n$ is such that $e^{x_n} \leqq s_m \leqq r_n$. We write $A_{mn} = B_{mn} + C_{mn}$ where

$$B_{mn} = \frac{m}{2}\int_{s_m}^{r_n}\left(\frac{r_n}{t}\right)^m\frac{N(t)}{t}dt - \frac{m}{2}\int_{e^{x_n}}^{r_n}\left(\frac{t}{r_n}\right)^m\frac{N(t)}{t}dt$$

and

$$C_{mn} = -\frac{m}{2}\int_0^{e^{x_n}} \left(\frac{t}{r_n}\right)^m \frac{N(t)}{t} dt \ .$$

Elementary integration and (3.29) imply uniformly for the values of m under consideration

$$-(1 + o(1))\frac{m}{2(\rho_n + m)} N(r_n) \leqq B_{mn} \leqq (1 + o(1))\frac{m}{2(\rho_n - m)} N(r_n) \ . \tag{3.30}$$

In addition, uniformly in m

$$|C_{mn}| \leqq \frac{N(e^{x_n})}{2} \leqq (1 + o(1))N(r_n)^{1/2} \ , \tag{3.31}$$

where the last inequality is a result of (3.1 ix), (3.10), (3.15), (3.17), and (3.18). Since $\rho_n = M_n^2$, (3.18), (3.30), and (3.31) imply

$$\begin{aligned} \left(\sum_{e^{x_n} \leqq s_m \leqq r_n} A_{mn}^2\right)^{1/2} &\leqq \left(\sum_{e^{x_n} \leqq s_m \leqq r_n} B_{mn}^2\right)^{1/2} + \left(\sum_{e^{x_n} \leqq s_m \leqq r_n} C_{mn}^2\right)^{1/2} \\ &\leqq \left(\frac{1}{2} + o(1)\right)\left(\sum_{m=1}^{2M_n} \left(\frac{m}{\rho_n - m}\right)^2\right)^{1/2} N(r_n) + (1 + o(1))(2M_n N(r_n))^{1/2} \\ &= O\left(\frac{M_n^3}{\rho_n^2}\right)^{1/2} N(r_n) + o(N(r_n)) = o(N(r_n)) \ . \end{aligned} \tag{3.32}$$

For m such that $s_m < e^{x_n}$, we write $A_{mn} = B'_{mn} + C'_{mn}$, where

$$B'_{mn} = \frac{m}{2}\int_{s_m}^{r_n} \left(\frac{r_n}{t}\right)^m \frac{N(t)}{t} dt - \frac{m}{2}\int_{s_m}^{r_n} \left(\frac{t}{r_n}\right)^m \frac{N(t)}{t} dt$$

and

$$C'_{mn} = -\frac{m}{2}\int_0^{s_m} \left(\frac{t}{r_n}\right)^m \frac{N(t)}{t} dt \ .$$

From (3.29) we have uniformly in m

$$\begin{aligned} 0 \leqq B'_{mn} &\leqq \frac{m}{2}\int_{s_m}^{e^{x_n}} \left(\frac{r_n}{t}\right)^m \frac{N(e^{x_n})}{t} dt \\ &+ \frac{m}{2}\int_{e^{x_n}}^{r_n} \left(\frac{r_n}{t}\right)^m \frac{N(t)}{t} dt \leqq \frac{N(e^{x_n}) r_n^m}{2} + (1 + o(1))\frac{mN(r_n)}{2(\rho_n - m)} \ . \end{aligned} \tag{3.33}$$

We note that $m \geqq 2M_n^{1/2}$ implies by (3.1 ix)

$$\alpha(x_n) \leqq M_n^{1/2} \leqq m/2 = \alpha(\log s_m) \ . \tag{3.34}$$

Thus for $s_m < e^{x_n}$ we have $m < 2M_n^{1/2}$, and hence by (3.18) and the right half of (3.31)

$$N(e^{x_n}) r_n^m = o(N(r_n))^{3/4} \ . \tag{3.35}$$

As before,

$$(3.36)\qquad |C'_{mn}| \leqq \frac{N(e^{x_n})}{2} \leqq (1 + o(1))N(r_n)^{1/2} .$$

As in (3.32), the combination of (3.33), (3.35), and (3.36) yields

$$\Big(\sum_{s_m < e^{x_n}} A^2_{mn}\Big)^{1/2} = o(N(r_n)) ,$$

which in conjunction with (3.32) establishes (3.28).

We note that the combination of (3.26) and (3.28) gives

$$(3.37)\qquad \Big(\sum_{m=1}^{M_{n+1}} |c_m(r_n, e^{h_n}g) - \beta^m_n N(r_n)|^2\Big)^{1/2} = o(N(r_n)) .$$

We now define f. We let

$$(3.38)\qquad a_m = \begin{cases} a_{m1} & 1 \leqq m \leqq M_2 \\ a_{mn} & M_n < m \leqq M_{n+1},\ n \geqq 2 . \end{cases}$$

Letting $h(z) = \sum a_m z^m$, we note from (3.22) and (3.25) that h is entire. We define

$$f(z) = e^{h(z)}g(z) .$$

In order to show $N(r_n)/T(r_n, f) \to 1$, and hence $d(0, f) = 0$, we need an additional property of g, namely

$$(3.39)\qquad \Big(\sum_{m > M_{n+1}} |c_m(r_n, g)|^2\Big)^{1/2} = o(N(r_n)) .$$

We first note from (3.1 iii), (3.15), and (3.34) that $m > M_{n+1}$ implies

$$(3.40)\qquad s_m^{1/4} > e^{(x_{n+1})/4} > 2r_n .$$

We consider (3.27) with $b_m = 0$ and $r = r_n$. From (3.14) and (3.40) we have for $m > M_{n+1}$

$$(3.41)\qquad r_n^m\Big(\frac{n(s_m)}{2ms_m^m} + \frac{N(s_m)}{2s_m^m}\Big) \leqq A\Big(\frac{r_n}{s_m^{1/4}}\Big)^m < \frac{A}{2^m} .$$

In addition by (3.9), (3.10), and the convexity of φ there exists a positive constant t_0 independent of n such that as $n \to \infty$

$$N(t) \geqq N(r_n)\Big(\frac{t}{r_n}\Big)^{\rho_n(1+o(1))}$$

uniformly for $t_0 \leqq t \leqq r_n$. Consequently, uniformly for $m > M_{n+1}$ as $n \to \infty$

$$
\begin{aligned}
0 &\leqq \frac{N(r_n)}{2} - \frac{m}{2}\int_0^{r_n}\left(\frac{t}{r_n}\right)^m \frac{N(t)}{t}dt \\
&\leqq \frac{N(r_n)}{2} - \frac{mN(r_n)}{2}\int_{t_0}^{r_n}\left(\frac{t}{r_n}\right)^{m+\rho_n(1+o(1))}\frac{dt}{t} \\
&\leqq (1+o(1))\frac{\rho_n(N(r_n))}{2(\rho_n+m)} + \frac{1}{2^m},
\end{aligned}
\tag{3.42}
$$

where we have used (3.18).

For $m > M_{n+1}$, we have $s_m > e^{8x_n}$ by (3.1 iii) and (3.34), and consequently by (3.13) and (3.15)

$$
\frac{m}{2}\int_{e^{8x_n}}^{s_m}\left(\frac{r_n}{t}\right)^m \frac{N(t)}{t}dt \leqq \left(\frac{r_n}{e^{4x_n}}\right)^m < \frac{1}{2^m}. \tag{3.43}
$$

Uniformly for $m > M_{n+1}$ we have by (3.1 vi), (3.9), and (3.15) as $n \to \infty$

$$
\begin{aligned}
&\frac{m}{2}\int_{r_n}^{e^{8x_n}}\left(\frac{r_n}{t}\right)^m \frac{N(t)}{t}dt \\
&\quad = \frac{mN(r_n)}{2}\int_{r_n}^{e^{8x_n}}\left(\frac{r_n}{t}\right)^{m-\rho_n(1+o(1))}\frac{dt}{t} \\
&\quad = \frac{mN(r_n)}{2(m-\rho_n(1+o(1)))} + o\left(\frac{1}{2^m}\right).
\end{aligned}
\tag{3.44}
$$

Combining (3.43) and (3.44), we obtain uniformly for $m > M_{n+1}$

$$
\left|\frac{N(r_n)}{2} - \frac{m}{2}\int_{r_n}^{s_m}\left(\frac{r_n}{t}\right)^m \frac{N(t)}{t}dt\right| \leqq (1+o(1))\frac{\rho_n N(r_n)}{2(m-\rho_n)} + \frac{1}{2^{m-1}}. \tag{3.45}
$$

The combination of (3.42) and (3.45) yields uniformly for $m > M_{n+1}$

$$
\begin{aligned}
&\left|N(r_n) - \frac{m}{2}\int_{r_n}^{s_m}\left(\frac{r_n}{t}\right)^m \frac{N(t)}{t}dt - \frac{m}{2}\int_0^{r_n}\left(\frac{t}{r_n}\right)^m \frac{N(t)}{t}dt\right| \\
&\quad \leqq (1+o(1))\frac{N(r_n)}{2}\left(\frac{\rho_n}{m-\rho_n} + \frac{\rho_n}{m+\rho_n}\right) + \frac{1}{2^{m-2}}.
\end{aligned}
\tag{3.46}
$$

Since $\rho_n = M_n^2$, we see from (3.1 i), (3.27), (3.41), (3.46) and the Schwarz inequality that as $n \to \infty$

$$
\left(\sum_{m>M_{n+1}} |c_m(r_n, g)|^2\right)^{1/2} = O\left(\frac{\rho_n}{M_{n+1}^{1/2}}N(r_n)\right) + o(1) = o(N(r_n)),
$$

establishing (3.39).

We next observe that

$$
\left\|\log|f(r_n e^{i\theta})| - \operatorname{Re}\frac{1+\beta_n e^{i\theta}}{1-\beta_n e^{i\theta}}N(r_n)\right\|_1^2
$$

$$(3.47)\qquad \begin{aligned} &\leqq \left\|\log|f(r_ne^{i\theta})| - \operatorname{Re}\frac{1+\beta_n e^{i\theta}}{1-\beta_n e^{i\theta}}N(r_n)\right\|_2^2 \\ &= 2\sum_{m=1}^{M_{n+1}}|c_m(r_n, f) - \beta_n^m N(r_n)|^2 + 2\sum_{m>M_{n+1}}|c_m(r_n, f) - \beta_n^m N(r_n)|^2 \\ &\equiv 2I_n + 2II_n\,. \end{aligned}$$

To analyze II_n, we first note from (3.15), (3.22), (3.25), (3.38), and (3.40) that

$$(3.48)\qquad |a_m| < B(2r_n)^{-m}$$

for some constant $B > 0$ independent of n for all $m > M_{n+1}$. Thus by (3.1 ii), (3.39), and (3.48)

$$(3.49)\qquad \begin{aligned} II_n^{1/2} &\leqq \Big(\sum_{m>M_{n+1}}|c_m(r_n, g)|^2\Big)^{1/2} \\ &+ \frac{B}{2}\Big(\sum_{m>M_{n+1}}2^{-2m}\Big)^{1/2} + \Big(\sum_{m>M_{n+1}}\beta_n^{2m}\Big)^{1/2}N(r_n) = o(N(r_n))\,. \end{aligned}$$

From the definitions of h_n and I_n we have

$$I_n^{1/2} = \left(\sum_{m=1}^{M_{n+1}}\left|c_m(r_n, e^{h_n}g) - \beta_n^m N(r_n) + \left(\frac{a_m - a_{mn}}{2}\right)r_n^m\right|^2\right)^{1/2}.$$

By (3.37) and (3.38) we have

$$(3.50)\qquad I_n^{1/2} \leqq o(N(r_n)) + \frac{1}{2}\Big(\sum_{m=1}^{M_n}|a_m|^2 r_n^{2m}\Big)^{1/2} + \frac{1}{2}\Big(\sum_{m=1}^{M_n}|a_{mn}|^2 r_n^{2m}\Big)^{1/2}.$$

From (3.18) and (3.21) we have

$$(3.51)\qquad \begin{aligned} &\Big(\sum_{m=1}^{M_n}|a_{mn}|^2 r_n^{2m}\Big)^{1/2} \\ &\leqq o(N(r_n)) + A\Big(\sum_{m=1}^{M_n/2}\frac{r_n^{2m}}{s_m^{m/2}}\Big)^{1/2} + A\Big(\sum_{m=M_n/2}^{M_n}\frac{r_n^{2m}}{s_m^{m/2}}\Big)^{1/2} \\ &\leqq o(N(r_n)) + O(M_n^{1/2}r_n^{M_n/2}) + o(N(r_n)) = o(N(r_n))\,. \end{aligned}$$

Similarly

$$(3.52)\qquad \begin{aligned} \Big(\sum_{m=1}^{M_n}|a_m|^2 r_n^{2m}\Big)^{1/2} &\leqq O(M_n^{1/2}r_{n_n}^{M_n/2}) \\ &+ \Big(\sum_{m=M_n/2}^{M_n}|a_m|^2\Big)^{1/2}r_n^{M_n} = o(N(r_n))\,. \end{aligned}$$

The combination of (3.47), (3.49), (3.50), (3.51), and (3.52) yields

$$(3.53)\qquad \left\|\log|f(r_ne^{i\theta})| - \operatorname{Re}\frac{1+\beta_n e^{i\theta}}{1-\beta_n e^{i\theta}}N(r_n)\right\|_1 = o(N(r_n))\,,$$

trivially implying $m(r_n, 1/f) = o(N(r_n))$ and hence $d(0, f) = 0$.

For the remainder of the proof, we reserve the letter r for a value satisfying

$$(3.54) \qquad x_n \leqq \log r = x_n + 4q \leqq x_{n+1}/4$$

for some integers q and n. We must show

$$(3.55) \qquad \log T(\tilde{r}, f) < \gamma(\log \tilde{r}) , \quad \tilde{r} > R_0 ,$$

which in conjunction with (3.7) establishes (4).

We consider $c_m(r, f)$ given by (3.27) with $b_m = a_m$. For $m \leqq 2M_{n+1}$, from (3.14) and the fact that $a_m \to 0$ we conclude

$$(3.56) \qquad \left| r^m\left(\frac{a_m}{2} - \frac{n(s_m)}{2ms_m^m} - \frac{N(s_m)}{2s_m^m}\right)\right| = O(r^m) = O(r^{2M_{n+1}}) .$$

Noting

$$\frac{m}{2}\int_0^{s_m}\left(\frac{r}{t}\right)^m \frac{N(t)}{t}dt < r^m$$

by (3.13), we see from (3.10), (3.18), and the monotonicity of α that for $1 \leqq m \leqq 2M_{n+1}$

$$(3.57) \qquad \begin{aligned} &\left|N(r) + \frac{m}{2}\int_0^r\left(\left(\frac{r}{t}\right)^m - \left(\frac{t}{r}\right)^m\right)\frac{N(t)}{t}dt - \frac{m}{2}\int_0^{s_m}\left(\frac{r}{t}\right)^m\frac{N(t)}{t}dt\right| \\ &\qquad \leqq r^m N(r) \leqq r^{3M_{n+1}} . \end{aligned}$$

By (3.1 xi), (3.56), and (3.57)

$$(3.58) \qquad \left(\sum_{|m| \leqq 2M_{n+1}} |c_m(r, f)|^2\right)^{1/2} = o(r^{4M_{n+1}}) = o(e^{\gamma_1(\log r)}) .$$

From the definition of a_m, (3.1 iv), (3.23), (3.24), and (3.54) we have for $m > 2M_{n+1}$

$$(3.59) \qquad \left| r^m\left(\frac{a_m}{2} - \frac{n(s_m)}{2ms_m^m} - \frac{N(s_m)}{2s_m^m}\right)\right| < \frac{5}{2}\left(\frac{r}{r_{n+1}^{1/2}}\right)^m < \frac{1}{2^{m+1}} .$$

We have

$$s_m > e^{x_{n+1}} > e^4 r$$

for $m > 2M_{n+1}$ by (3.34). By (3.1 vi), (3.1 viii), (3.9), and (3.54) uniformly for $m > 2M_{n+1}$

$$(3.60) \qquad \begin{aligned} &\frac{N(r)}{2} - \frac{m}{2}\int_r^{e^4 r}\left(\frac{r}{t}\right)^m \frac{N(t)}{t}dt \\ &= \frac{N(r)}{2}\left(1 - m\int_r^{e^4 r}\left(\frac{r}{t}\right)^{m-\rho_r(1+o(1))}\frac{dt}{t}\right) \\ &= -(1+o(1))\frac{\rho_r}{2(m-\rho_r)}N(r) + o\left(\frac{N(r)}{e^{3m}}\right) = -(1+o(1))\frac{\rho_r N(r)}{2m} \end{aligned}$$

where $\rho_r = \varphi'(\log r) \leqq (m/2)^{1/2}$.

Since $e^4 r < t < e^{x_{n+1}}$ implies

$$N(t) < N(e^4 r)(t/e^4 r)^{M_{n+1}^{1/2}(1+o(1))}$$

by (3.1 viii) and (3.9), we conclude by elementary integration

$$\frac{m}{2}\int_{e^4 r}^{e^{x_{n+1}}} \left(\frac{r}{t}\right)^m \frac{N(t)}{t} dt \leqq \frac{N(e^4 r)}{e^{4m(1-o(1))}} \tag{3.61}$$

uniformly for $m > 2M_{n+1}$ as r tends to infinity through values satisfying (3.54). Finally from (3.13) and (3.54), for $m > 2M_{n+1}$

$$\frac{m}{2}\int_{e^{x_{n+1}}}^{s_m} \left(\frac{r}{t}\right)^m \frac{N(t)}{t} dt < \frac{r^m}{e^{m(x_{n+1})/2}} < \frac{1}{2^m}\,. \tag{3.62}$$

Combining (3.60), (3.61), and (3.62), we conclude from (3.1 viii)

$$\begin{aligned} &\left(\sum_{|m|>2M_{n+1}} \left|\frac{N(r)}{2} - \frac{m}{2}\int_r^{s_m} \left(\frac{r}{t}\right)^m \frac{N(t)}{t} dt\right|^2\right)^{1/2} \\ &\qquad = O(N(r)) + o(N(e^4 r)) = o(N(e^4 r))\,. \end{aligned} \tag{3.63}$$

Since $N(r) < r^{m/2}$ for $m > 2M_{n+1}$, a calculation similar to (3.42) shows uniformly for $m > 2M_{n+1}$

$$\begin{aligned} 0 &\leqq \frac{N(r)}{2} - \frac{m}{2}\int_0^r \left(\frac{t}{r}\right)^m \frac{N(t)}{t} dt \\ &\leqq (1 + o(1))\frac{\rho_r N(r)}{2m} + \frac{1}{2^m}\,, \end{aligned}$$

implying

$$\begin{aligned} &\left(\sum_{|m|>2M_{n+1}} \left|\frac{N(r)}{2} - \frac{m}{2}\int_0^r \left(\frac{t}{r}\right)^m \frac{N(t)}{t} dt\right|^2\right)^{1/2} \\ &\qquad = O(N(r)) = o(N(e^4 r))\,. \end{aligned} \tag{3.64}$$

Combining (3.27), (3.59), (3.63), and (3.64) with the Schwarz inequality, we conclude

$$\left(\sum_{|m|>2M_{n+1}} |c_m(r, f)|^2\right)^{1/2} = o(N(e^4 r)) = o(e^{\varphi(4+\log r)})\,, \tag{3.65}$$

where we use (3.10) in the second equality.

From (3.1 v), (3.58), and (3.65) we have

$$\log m_2^+(r, f) < \gamma_1(4 + \log r) \tag{3.66}$$

for sufficiently large r satisfying (3.54). For sufficiently large $\tilde{r}$ there thus exists r with

$$\log r \in [\log \tilde{r}, 4 \log \tilde{r}]$$

for which (3.66) holds. Thus for all $\tilde{r} > R_0$,

$$\begin{aligned} T(\tilde{r}, f) &\leqq T(r, f) \leqq m_2^+(r, f) \\ &\leqq e^{\gamma_1(4+\log r)} = e^{\gamma((\log r)/4)} \leqq e^{\gamma(\log \tilde{r})} , \end{aligned}$$

establishing (3.55) and hence (4). It is clear that the lower order of f is infinite because the lower order of $N(t)$ is infinite. This finishes the proof of Theorem 2.

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