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JANET E. MILLS

An ω-semigroup is a semigroup whose idempotents form an ω-chain $e_0 > e_1 > e_2 > \cdots$. In this paper we characterize the semilattice of idempotents of a 0-simple inverse semigroup whose nonzero $\mathscr{D}$-classes form ω-semigroups.

A semilattice E is an interlaced union of ω-chains $C_\alpha = \{e_{\alpha,0} > e_{\alpha,1} > \cdots\}$, $\alpha \in A$, if $E = \bigcup_{\alpha\in A} C_\alpha$ and if $\alpha, \beta \in A$, $i \geqq 0$, then there exists a unique $j \geqq 0$ such that

$$e_{\beta,j} < e_{\alpha,i} \quad \text{but} \quad e_{\beta,j} \not< e_{\alpha,i+1}\,.$$

It will be shown that Y is the semilattice of a 0-simple inverse semigroup whose nonzero $\mathscr{D}$-classes form ω-semigroups if and only if Y is an interlaced union of ω-chains, with zero adjoined. One such 0-simple inverse semigroup with semilattice Y will be explicitly displayed.

In the semigroups under consideration, every nonzero $\mathscr{D}$-class is an ω-semigroup, that is, a bisimple ω-semigroup. Since bisimple ω-semigroups were described completely by N. R. Reilly, [8], our semigroups are unions of well-known semigroups; it is the manner in which the idempotents of these ω-semigroups relate to each other that is of interest here. This class of semigroups includes several which have already been explored, for example, simple ω-semigroups, [4] and [7], and certain simple inverse semigroups whose idempotents form the ordinal product of a ω-chain and a semilattice with identity, [6]. Bisimple ω-semigroups occur in abundance within most regular semigroups (see [1]), so it is natural to consider, as a first step, those semigroups whose $\mathscr{D}$-classes are all ω-semigroups.

1. Preliminaries. Let S be an inverse semigroup. For an element a of S, a^{-1} denotes the unique element of S for which $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. For any subset D of S, E_D is the set of idempotents of S contained in D. Equivalences $\mathscr{D}$ and $\mathscr{J}$ denote the usual Green's relations.

For inverse semigroups, the property of being 0-simple is easily seen to be equivalent to the condition: if e and f are nonzero idempotents then there exists an idempotent g such that $g \leqq f$ and $g\mathscr{D}e$, where $\leqq$ is the usual partial order on idempotents.

Let e and f be idempotents with $e\mathscr{D}f$. Then there exists a in

S such that $aa^{-1} = e$ and $a^{-1}a = f$. Furthermore, the mapping $\sigma_a: x \to a^{-1}xa$ is an isomorphism of E_Se onto E_Sf, [3].

The following result is crucial to our development of the structure of the semigroups under consideration.

LEMMA 1.1. *Let S be an inverse semigroup in which every nonzero $\mathscr{D}$-class is an ω-semigroup. Then S is 0-simple if and only if for any two distinct nonzero $\mathscr{D}$-classes D, D', if $g, h \in E_D$ with $g < h$, then there exists $d \in E_{D'}$ such that $d < h$ but $d \nless g$.*

Proof. Let S be 0-simple and D, D' be two distinct nonzero $\mathscr{D}$-classes with $g < h$, g, h in E_D. By 0-simplicity, there exists $e \in E_{D'}$ such that $e < g$. Since $E_{D'}$ is inversely well-ordered, e can be picked to be the maximal idempotent of D' beneath g. Moreover, since there is an idempotent of D below e, there are only a finite number above e, so we let g' be the minimal such one. That is,

$$e < g' \leqq g < h .$$

Since $g' \mathscr{D} h$, there exists a in S with $aa^{-1} = h, a^{-1}a = g'$. Now $a^{-1}ea \mathscr{D} e$ and $a^{-1}ea < g' < g$. By maximality of e, it follows that $a^{-1}ea \leqq e < g'$. If $a^{-1}ea = e$, then σ_a, as defined above, acts in the following manner: $\sigma_a(h) = g'$, $\sigma_a(e) = e$ and $\sigma_a(g) = g''$ for some $g'' \mathscr{D} g$. Since $e < g < h$, then $e < g'' < g'$. But by minimality of g', this is impossible. Thus $a^{-1}ea < e < g'$.

Since $\sigma_{a^{-1}}$ is also an isomorphism, $a^{-1}ea < e < g'$ implies

$$a(a^{-1}ea)a^{-1} < aea^{-1} < ag'a^{-1} .$$

That is, $e < aea^{-1} < h$. Consequently $d = aea^{-1}$ *is* $\mathscr{D}$-related to e and d satisfies the condition that $d < h$. Furthermore, since e is the maximal idempotent of D' below g, $d \nless g$.

The converse follows directly from the remark preceding Lemma 1.1.

An ideal I is called prime if $ab \in I$ implies $a \in I$ or $b \in I$.

LEMMA 1.2. *If S is a 0-simple inverse semigroup whose nonzero $\mathscr{D}$-classes are ω-semigroups then 0 is a prime ideal, and $S \backslash 0$ is a simple inverse semigroup whose $\mathscr{D}$-classes are ω-semigroups.*

Proof. Let e and f be nonzero idempotents of S with $ef = 0$. Then e and f must be in distinct $\mathscr{D}$-classes, since each $\mathscr{D}$-class is closed. By 0-simplicity, there exists an idempotent g such that $g \leqq e$ and $g \mathscr{D} f$. Since f and g are in an ω-semigroup, either $g \leqq f$ or

$f < g$. But if $f < g$, then $f \leqq e$ and $ef \neq 0$. Hence $g \leqq f$ and $g \leqq e$. But this implies that $g \leqq ef = 0$. But $g \neq 0$, and thus $ef \neq 0$. Therefore, 0 is a prime ideal of E_S, and thus of S.

2. The idempotent structure. In light of Lemma 1.2, we now restrict ourselves to simple inverse semigroups whose $\mathscr{D}$-classes are ω-semigroups. In such a semigroup, we now show that the semilattice of idempotents is an interlaced union of ω-chains.

LEMMA 2.1. *Let S be a simple inverse semigroup whose $\mathscr{D}$-classes are ω-semigroups D_α, $\alpha \in A$, and $E_{D_\alpha} = \{e_{\alpha,0} > e_{\alpha,1} > \cdots\}$. The following properties hold in E_S.*

(i) *If $e_{\alpha,i} \leqq e_{\beta,j}$ then $i \geqq j$.*

(ii) *For $\alpha, \beta \in A$, $i, j \geqq 0$, and for all n such that $-j \leqq n < +\infty$,*

$$e_{\alpha,i} < e_{\beta,j} \longleftrightarrow e_{\alpha,i+n} < e_{\beta,j+n} .$$

(iii) *If $e_{\alpha,i}e_{\beta,j} = e_{\gamma,k}$ then $e_{\alpha,i+n}e_{\beta,j+n} = e_{\gamma,k+n}$, for all $n \geqq -\min\{i, j\}$.*

(iv) *For $\alpha \in A$, if $aa^{-1} = e_{\alpha,i}$, $a^{-1}a = e_{\alpha,j}$ then $\sigma_a: Ee_{\alpha,i} \to Ee_{\alpha,j}$ defined by $x\sigma_a = a^{-1}xa$, is an isomorphism such that if $e_{\beta,k} \leqq e_{\alpha,i}$, then*

$$(1) \qquad e_{\beta,k}\sigma_a = e_{\beta,k+(j-i)} .$$

Proof. (i) Let $e_{\alpha,i} < e_{\beta,j}$. Consider the set

$$M = \{k \mid e_{\alpha,k} < e_{\beta,0}, e_{\alpha,k} \nless e_{\beta,j}\} .$$

Then if k is in M, $k < i$ since $e_{\alpha,i} < e_{\beta,j}$. On the other hand, by Lemma 1.1, for all $p < j$, there exists p' such that $e_{\alpha,p'} < e_{\beta,p}$, $e_{\alpha,p'} \nless e_{\beta,p+1}$; each p' is in M and they are all distinct. Consequently $j - 1 \leqq |M| < i$, so $i > j$.

We know from [3] that σ_a is an isomorphism and thus preserves $\mathscr{D}$-classes. Therefore, if $e_{\beta,k} \leqq e_{\alpha,i}$, then $e_{\beta,k}\sigma_a = e_{\beta,m}$ for some m. In addition it is clear that for $e_{\alpha,k} \leqq e_{\alpha,i}$, $e_{\alpha,k}\sigma_a = e_{\alpha,k+(j-i)}$, since there must be a one-to-one correspondence between the sets $\{e_{\alpha,k} < \cdots < e_{\alpha,i}\}$ and $\{e_{\alpha,k}\sigma_a < \cdots < e_{\alpha,j}\}$. The proof of (1) for arbitrary β will be made after (ii) and (iii) are proved.

(ii) Let $e_{\alpha,i} < e_{\beta,j}$. It will first be shown that $e_{\alpha,i+1} < e_{\beta,j+1}$. Either $e_{\alpha,i} < e_{\beta,j+1}$ and thus $e_{\alpha,i+1} < e_{\alpha,i} < e_{\beta,j+1}$, or $e_{\alpha,i} \nless e_{\beta,j+1}$. We may assume the latter. By simplicity, there exists $e_{\beta,k} < e_{\alpha,i}$, so let $r = \min\{k \mid e_{\beta,k} < e_{\alpha,i}\}$. That is, using 1.1

$$e_{\beta,r} < e_{\alpha,i} < e_{\beta,j} \quad \text{and} \quad e_{\beta,r} \nless e_{\alpha,i+1} .$$

Let $aa^{-1} = e_{\beta,j}$ and $a^{-1}a = e_{\beta,j+1}$. Then

$$a^{-1}e_{\beta,r}a < a^{-1}e_{\alpha,i}a < a^{-1}e_{\beta,j}a ,$$

where the strict inequalities hold since σ_a is an isomorphism. That is,

$$(2) \qquad e_{\beta,r+1} < a^{-1}e_{\alpha,i}a < e_{\beta,j+1}$$

since $e_{\beta,r}\sigma_a = e_{\beta,r+1}$ as we have seen earlier. Now $a^{-1}e_{\alpha,i}a\mathscr{D}e_{\alpha,i}$ and thus $a^{-1}e_{\alpha,i}a < e_{\alpha,i}$ since $e_{\alpha,i} \not< e_{\beta,j+1}$. If $a^{-1}e_{\alpha,i}a < e_{\alpha,i+1}$ then by 1.1, there exists p such that $e_{\beta,p} < e_{\alpha,i+1}$, $e_{\beta,p} \not< a^{-1}e_{\alpha,i}a$. By definition of r, $p \geqq r$ and in fact $p > r$ since $e_{\beta,r} \not< e_{\alpha,i+1}$. But then by (2) $e_{\beta,p} \leqq e_{\beta,r+1} < a^{-1}e_{\alpha,i}a$, contrary to the assumption. Hence $a^{-1}e_{\alpha,i}a = e_{\alpha,i+1}$ and thus $e_{\alpha,i+1} < e_{\beta,j+1}$.

That $e_{\alpha,i+n} < e_{\beta,j+n}$ for all $n \geqq 0$ follows by induction.

Now consider the case $n = -1$. Let $j > 0$. Then $i > j > 0$ by (i). Either $e_{\alpha,i}$ is the maximal idempotent of D_α less than $e_{\beta,j}$, or $e_{\alpha,i} < e_{\alpha,i-1} < e_{\beta,j} < e_{\beta,j-1}$. Thus we may assume that the former holds. By 1.1, there exists m such that $e_{\alpha,m} < e_{\beta,j-1}$, $e_{\alpha,m} \not< e_{\beta,j}$. Since $e_{\alpha,i} < e_{\beta,j}$, it follows that $m \leqq i-1$. Hence $e_{\alpha,i-1} \leqq e_{\alpha,m} < e_{\beta,j-1}$. The proof for n such that $-j \leqq n \leqq -1$ is by induction.

(iii) The proof of (iii) is made using repeated applications of (ii).

To see that (1) holds for arbitrary β, let σ_a be defined as in (iv). Then, as we have stated, for $e_{\beta,k} < e_{\alpha,i}$, $a^{-1}e_{\beta,k}a = e_{\beta,p}$ for some p. By (ii), $e_{\beta,k} < e_{\alpha,i}$ if and only if $e_{\beta,k+(j-i)} \leqq e_{\alpha,i+(j-i)} = e_{\alpha,j}$. Since σ_a is one-to-one and preserves $\mathscr{D}$-classes, $e_{\beta,k}\sigma_a = e_{\beta,k+(j-i)}$.

THEOREM 2.2. *If S is a simple inverse semigroup whose $\mathscr{D}$-classes are ω-semigroups, then E_S is an interlaced union of ω-chains.*

Proof. We know that E_S is a union of ω-chains $E_{D_\alpha} = \{e_{\alpha,0} > e_{\alpha,1} > \cdots\}$, $\alpha \in A$, where D_α is a $\mathscr{D}$-class. Let $\alpha, \beta \in A$, $i \geqq 0$. We must find a unique $j \geqq 0$ such that $e_{\beta,j} < e_{\alpha,i}$, $e_{\beta,j} \not< e_{\alpha,i+1}$. Consider the set

$$K = \{j \mid e_{\beta,j} < e_{\alpha,i}\} .$$

By Lemma 1.1, K is nonempty, and thus K must have a least element, call it m. Then $e_{\beta,m} < e_{\alpha,i}$. If $e_{\beta,m} < e_{\alpha,i+1}$, then by Lemma 2.1 (ii), $e_{\beta,m-1} < e_{\alpha,(i+1)-1}$. That is, $e_{\beta,m-1} < e_{\alpha,i}$. By minimality of m, this is impossible. Thus $e_{\alpha,m} \not< e_{\alpha,i+1}$.

Since $e_{\alpha,i}\mathscr{D}e_{\alpha,i+1}$, there exists $a \in S$ such that $aa^{-1} = e_{\alpha,i}$, $a^{-1}a = e_{\alpha,i+1}$ and σ_a defined by $e_{\gamma,k}\sigma_a = e_{\gamma,k+1}$ is an isomorphism of $Ee_{\alpha,i}$ onto $E_{\alpha,i+1}$, by Lemma 2.1(iv). Now $e_{\beta,m} < e_{\alpha,i}$ so $e_{\beta,m}\sigma_a = e_{\beta,m+1} < e_{\alpha,i+1}$. Hence $e_{\beta,k} < e_{\alpha,i+1}$ for all $k > m$. From this and minimality of m, it follows that $e_{\beta,m}$ is the unique idempotent in D_β such that $e_{\beta,m} < e_{\alpha,i}$ and $e_{\beta,m} \not< e_{\alpha,i+1}$. Therefore, E_S is an interlaced union of ω-chains E_{D_α}, $\alpha \in A$.

3. An interlaced union of ω-chains.

Given an interlaced union

of ω-chains, we now construct a simple inverse semigroup associated with it.

Let E be an interlaced union of ω-chains $e_{\alpha,0} > e_{\alpha,1} > \cdots, \alpha \in A$. Recall that this means that for all $\alpha, \beta \in A, i \geqq 0$, there exists a unique $j \geqq 0$ such that $e_{\beta,j} < e_{\alpha,i}, e_{\beta,j} \nless e_{\alpha,i+1}$.

LEMMA 3.1. *For E as described, the following hold.*
(i) *If $e_{\alpha,i} \leqq e_{\beta,j}$ then $i \geqq j$.*
(ii) *If $e_{\alpha,i} \leqq e_{\beta,j}$ then $e_{\alpha,i+n} \leqq e_{\beta,j+n}$ for all $n \geqq -j$.*
(iii) *If $e_{\alpha,i}e_{\beta,j} = e_{\gamma,k}$ then $e_{\alpha,i+n}e_{\beta,j+n} = e_{\gamma,k+n}$ for all $n \geqq 0$.*

Proof. First we prove (ii) for all $n \geqq -\min\{i, j\}$. Assume that $e_{\alpha,i} \leqq e_{\beta,j}$. Let $n \geqq 0$ and assume $e_{\alpha,i+n} \leqq e_{\beta,j+n}$. If $e_{\alpha,i+n} \leqq e_{\beta,j+n+1}$, then $e_{\alpha,i+n+1} < e_{\alpha,i+n} \leqq e_{\beta,j+n+1}$ and the result holds. If $e_{\alpha,i+n} \nless e_{\beta,j+n+1}$ then $e_{\alpha,i+n}$ is the unique element below $e_{\beta,j+n}$ which is not below $e_{\beta,j+n+1}$. Consider $e_{\alpha,i+n+1}$. We know $e_{\alpha,i+n+1} < e_{\beta,j+n}$ since $e_{\alpha,i+n+1} < e_{\alpha,i+n}$; therefore, by uniqueness of $i+n$, we have $e_{\alpha,i+n+1} \leqq e_{\beta,j+n+1}$. By induction, (ii) holds for all $n \geqq 0$.

Now let $n > -\min\{i, j\}$ and let $e_{\alpha,i-n} \leqq e_{\beta,j-n}$. Either $e_{\alpha,i-n-1} \leqq e_{\beta,j-n} < e_{\beta,j-n-1}$, or else $e_{\alpha,i-n-1} \nless e_{\beta,j-n}$. There exists a unique $k \geqq 0$ such that $e_{\alpha,k} < e_{\beta,j-n-1}$ and $e_{\alpha,k} \nless e_{\beta,j-n}$. If $e_{\alpha,i-n-1} \nless e_{\beta,j-n}$ then it must be that $k \leqq i-n-1$ and $e_{\alpha,i-n-1} \leqq e_{\alpha,k} < e_{\beta,j-n-1}$. Consequently, for all n such that $-\min\{i, j\} \leqq n < +\infty$, (ii) holds.

(i) Let $e_{\alpha,i} \leqq e_{\beta,j}$ and assume $i < j$. Then by the above paragraph, $e_{\alpha,i-i} \leqq e_{\beta,j-i}$. That is, $e_{\alpha,0} \leqq e_{\beta,j-i} < e_{\beta,0}$. Since E is an interlaced union of ω-chains, there exists $k \geqq 0$ such that $e_{\alpha,k} < e_{\beta,0}$ and $e_{\alpha,k} \nless e_{\beta,1}$. But $j - i \geqq 1$ and $e_{\alpha,k} \leqq e_{\beta,0} \leqq e_{\beta,j-i} \leqq e_{\beta,1}$. This is impossible. Therefore $i \geqq j$. This also shows that (ii) is true for all $n \geqq -j = -\min\{i, j\}$.

(iii) Let $e_{\alpha,i}e_{\beta,j} = e_{\gamma,k}$. Then $e_{\gamma,k} \leqq e_{\alpha,i}$ and $e_{\gamma,k} \leqq e_{\beta,j}$, so that by (ii), $e_{\gamma,k+1} \leqq e_{\alpha,i+1}, e_{\gamma,k+1} \leqq e_{\beta,j+1}$. That is,

$$e_{\gamma,k+1} \leqq e_{\alpha,i+1}e_{\beta,j+1} < e_{\alpha,i}e_{\beta,j} = e_{\gamma,k} .$$

Let $e_{\alpha,i+1}e_{\beta,j+1} = e_{\delta,p}$. Then $e_{\delta,p} \leqq e_{\alpha,i+1}, e_{\delta,p} \leqq e_{\beta,j+1}$, so by (ii), $e_{\delta,p-1} \leqq e_{\alpha,i}, e_{\delta,p-1} \leqq e_{\beta,j}$. That is, $e_{\delta,p-1} \leqq e_{\alpha,i}e_{\beta,j} = e_{\gamma,k}$. Consequently, $e_{\gamma,k+1} \leqq e_{\delta,p} < e_{\delta,p-1} \leqq e_{\gamma,k}$. But then by uniqueness in the definition of E, both $e_{\delta,p}$ and $e_{\delta,p-1}$ can not be strictly between $e_{\gamma,k+1}$ and $e_{\gamma,k}$. Thus $e_{\delta,p} = e_{\gamma,k+1}$ and $e_{\gamma,k+1} = e_{\alpha,i+1}e_{\beta,j+1}$. By induction, (iii) holds for all $n \geqq 0$.

THEOREM 3.2 *Let E be an interlaced union of ω-chains $\{e_{\alpha,0} > e_{\alpha,1} > \cdots\}, \alpha \in A$. For $\alpha \in A$, $m, n \geqq 0$, let $\tau_{(m,\alpha,n)}$ be the mapping from $Ee_{\alpha,m}$ onto $Ee_{\alpha,n}$ defined by*

$$e_{\beta,j}\tau_{(m,\alpha,n)} = e_{\beta,j+(n-m)} .$$

Then $W = \{\tau_{(m,\alpha,n)} \mid \alpha \in A,\ m, n \geqq 0\}$, under composition, is a simple inverse semigroup whose $\mathscr{D}$-classes are ω-semigroups, and $E_W \cong E$.

Proof. By Theorem 3.2 of [5], to see that W is a simple inverse semigroup, it suffices to show that W is a subtransitive inverse subsemigroup of T_E, the set of isomorphisms of principal ideals of E. Using (ii) and (iii) of 3.1, it is not difficult to show that $\tau_{(m,\alpha,n)}$ is an isomorphism of $Ee_{\alpha,m}$ onto $Ee_{\alpha,n}$, and thus W is contained in T_E.

To see that W is closed, let $\tau_{(m,\alpha,n)}, \tau_{(i,\beta,j)}$ be in W. Certainly $\tau_{(m,\alpha,n)}\tau_{(i,\beta,j)}$ is an isomorphism from one subset of E to another. We need to show its domain is $Ee_{\delta,p}$ and its range is $Ee_{\delta,q}$ for some $\delta \in A$, $p, q \geqq 0$.

Now, $e_{\gamma,k} \in$ domain of $\tau_{(m,\alpha,n)}\tau_{(i,\beta,j)}$ if and only if

$$e_{\gamma,k} \leqq e_{\alpha,m} \quad \text{and} \quad e_{\gamma,k+(n-m)} \leqq e_{\beta,i}\ ,$$

which by Lemma 3.1 (ii) is equivalent to

$$e_{\gamma,k} \leqq e_{\alpha,m} \quad \text{and} \quad e_{\gamma,k} \leqq e_{\beta,i-(n-m)}\ .$$

This is equivalent to

$$e_{\gamma,k} \leqq e_{\alpha,m}e_{\beta,i-(n-m)}\ .$$

Thus the domain of $\tau_{(m,\alpha,n)}\tau_{(i,\beta,j)}$ is $Ee_{\alpha,m}e_{\beta,i-(n-m)}$.

Now, $e_{\delta,s}$ is in the range of $\tau_{(m,\alpha,n)}\tau_{(i,\beta,j)}$ if and only if

$$e_{\delta,s} \leqq e_{\beta,j} \quad \text{and} \quad e_{\delta,s-(j-i)} \leqq e_{\alpha,n}\ ,$$

which is equivalent to

$$e_{\delta,s} \leqq e_{\beta,j} \quad \text{and} \quad e_{\delta,s} \leqq e_{\alpha,n+(j-i)}\ .$$

This in turn is equivalent to

$$e_{\delta,s} \leqq e_{\alpha,n+(i-j)}e_{\beta,j}\ .$$

Therefore, the range of $\tau_{(m,\alpha,n)}\tau_{(i,\beta,j)}$ is $Ee_{\alpha,n+(j-i)}e_{\beta,j}$.

If $(n-m)+(j-i) \geqq 0$, and $e_{\alpha,m}e_{\beta,i-(n-m)} = e_{\delta,p}$ for some $\delta \in A$, $p \geqq 0$, then by Lemma 3.1 (iii),

$$e_{\alpha,m+(n-m)+(j-i)}e_{\beta,i-(n-m)+(n-m)+(j-i)} = e_{\delta,p+(n-m)+(j-i)}\ .$$

That is,

$$e_{\alpha,n+(j-i)}e_{\beta,j} = e_{\delta,p+(n-m)+(j-i)} = e_{\delta,q}\ ,$$

for some $q \geqq 0$, and $\tau_{(m,\alpha,n)}\tau_{(i,\beta,j)} = \tau_{(p,\delta,q)}$. If $(n-m)+(j-i) \leqq 0$, a similar argument works for $e_{\alpha,n+(i-j)}e_{\beta,j}$. Thus W is closed and is a subsemigroup of T_E. It is clearly an inverse semigroup since $\tau_{(n,\alpha,m)} = \tau^{-1}_{(m,\alpha,n)}$.

In order that W be subtransitive, it must satisfy the condition: for e, f in E, there exists $\theta \in W$ such that domain of $\theta = Ee$, range of $\theta \subseteqq Ef$. For $e_{\alpha,i}$, $e_{\beta,j}$ in E, there exists $k \geqq 0$ such that $e_{\alpha,k} \leqq e_{\beta,j}$, since E is interlaced. Thus $\theta = \tau_{(i,\alpha,k)}$ satisfies the necessary condition.

Since idempotents of W are of the form $\tau_{(i,\alpha,i)}$, E_W is an interlaced union of ω-chains, isomorphic to E under the map: $e_{\alpha,i} \rightarrow \tau_{(i,\alpha,i)}$. By Lemma 1.2 of [5], it is clear the $\tau_{(i,\alpha,i)} \mathscr{D} \tau_{(j,\beta,j)}$ if and only if $\alpha = \beta$, so the $\mathscr{D}$-classes of W are ω-semigroups.

THEOREM 3.3. *A semilattice E is the semilattice of idempotents of a 0-simple inverse semigroup whose nonzero $\mathscr{D}$-classes are ω-semigroups if and only if E is an interlaced union of ω-chains with 0 adjoined.*

Proof. This follows immediately from Corollary 1.2, Theorem 2.2 and Theorem 3.2.

4. An application.

The simplest example of an interlaced union of ω-chains is that of an ω-chain itself. The inverse semigroups corresponding are simple ω-semigroups, the structure of which was determined by Kochin [4] and Munn [7]. The following result demonstrates the strength of the condition imposed on an interlaced union of ω-chains.

THEOREM 4.1. *If S is a simple inverse semigroup with exactly two $\mathscr{D}$-classes, each of which is an ω-semigroup, then S is itself an ω-semigroup.*

Proof. Let $\{e_0 > e_1 > \cdots\}$ and $\{f_0 > f_1 > \cdots\}$ be the idempotents of the two $\mathscr{D}$-classes. Since E_S must be an interlaced union of ω-chains by Theorem 2.2, there exists unique $i \geqq 0$, $j \geqq 0$ such that

$$e_i < f_0,\ e_i \not< f_1\ , \quad \text{and} \quad f_j < e_0,\ f_j \not< e_1\ .$$

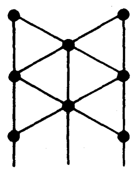
Now $e_0 f_0 \in E_S$ so $e_0 f_0 = e_k$ or f_k for some k. Without loss of generality we may assume $e_0 f_0 = e_k$. Then $e_k < f_0$. But $e_i < f_0$ implies that $e_i = e_i e_0 \leqq e_0 f_0 = e_k$, so $i \geqq k$. But if $e_i < e_k$, then $e_k \not< f_1$ since $e_i \not< f_1$. Thus by uniqueness, $k = i$ and $e_0 f_0 = e_i$. Now $f_j < e_0$ so $f_j < e_0 f_0 = e_i$. Since $f_j \not< e_1$, it follows that $i = 0$. Hence $e_0 f_0 = e_0$, i.e., $e_0 \leqq f_0$. By Lemma 3.1(ii), $e_n \leqq f_n$ for all n.

We need to show that $f_1 < e_0$. Since $e_1 < f_1$ and $e_0 < f_0$, then

$$e_1 \leqq e_0 f_1 < e_0 f_0 = e_0\ .$$

If $e_1 = e_0 f_1$ then $f_j < e_0$ and $f_j \not< e_1$ implies that $f_j = f_j f_0 < e_0 f_1 = e_1$. But this is impossible, so $e_1 < e_0 f_1 < e_0$. Thus $e_0 f_1 = f_j$, by uniqueness, and $e_1 < f_j < e_0$. By property (i) of Lemma 3.1, $j \leqq 1$, so $j = 1$, and $e_1 < f_1 < e_0 < f_0$. By property (ii), this means that E_S is an ω-chain.

To see that Theorem 4.1 does not hold for more than two $\mathscr{D}$-classes, consider the following semilattice E.



This semilattice E is the interlaced union of three ω-chains, each chain being a column, but E is not an ω-chain itself. For more than three $\mathscr{D}$-classes, one may add to E ω-chains each of whose elements is put between two elements of one of the columns in the semilattice E.

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James Madison University
Harrisonburg, VA 22807

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