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GENERALIZATIONS OF THE ROBERTSON FUNCTIONS

EDWARD JEAN MOULIS, JR.

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We study a class of analytic functions which unifies a number of classes previously studied, including functions with boundary rotation at most $k\pi$, functions convex of order ρ and the Robertson functions, i.e., functions f for which zf' is α -spirallike. We obtain representation theorems for this general class, and using a simple variational formula, also obtain sharp bounds on the modulus of the second coefficient of the series expansion of these functions. Using a univalence criterion due to Ahlfors, we determine a condition on the parameters k , α , and ρ which will ensure that a function in this class is univalent. This result improves previously published results for various subclasses and is sharp for the class of functions f for which zf' is α -spirallike of order ρ .

1. Let $P_\alpha^k(\rho)$ denote the class of regular functions $p(z)$ in $E = \{z: |z| < 1\}$ such that $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} \{e^{i\alpha} p(z) - \rho \cos \alpha\}}{1 - \rho} \right| d\theta \leq k\pi \cos \alpha,$$

$k \geq 2$, $0 \leq \rho < 1$, α real, $|\alpha| < \pi/2$, $z = re^{i\theta}$, $0 \leq r < 1$.

Let $V_\alpha^k(\rho)$ denote the class of functions regular in E with $f(0) = f'(0) - 1 = 0$ and

$$1 + \frac{zf''(z)}{f'(z)} \in P_\alpha^k(\rho),$$

k , α , and ρ as above. $V_0^k(0)$ is the class of functions with bounded boundary rotation. $V_\alpha^k(0)$ is a generalization of this class which has been studied recently ([7] and [13]). Padmanabhan and Parvatham [9] have studied properties of $V_0^k(\rho)$. In this paper we study properties of $V_\alpha^k(\rho)$ which unlike $V_0^k(\rho)$ contains functions whose boundary rotation is not necessarily bounded. A function f belongs to $V_\alpha^k(\rho)$ if and only if

$$\operatorname{Re} \left\{ e^{i\alpha} \left[\frac{1 + zf''(z)}{f'(z)} \right] \right\} > \rho \cos \alpha,$$

ρ and α as above. When $\rho = 0$, we obtain the class of functions $f(z)$ for which $zf'(z)$ is α -spirallike, which has been studied by M.S. Robertson [10], Libera and Ziegler [6], Bajpai and Mehrook [2], and Kulshrestha [5]. The case when $k = 2$ but ρ and α are not zero has been studied by Chichra [4] who denoted the class F_α^ρ . This

class also has been studied by Sizuk [12], who has called $zf'(z)$ α -spiral-shaped of order ρ . The class $V_0^k(\rho)$ is the class of functions which are convex of order ρ , introduced by M. S. Robertson in 1936.

LEMMA 1. If $p(z) \in P_\alpha^k(\rho)$, then

$$(1.1) \quad e^{i\alpha}p(z) = \frac{\cos \alpha}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\rho)ze^{i\theta}}{2 - ze^{i\theta}} d\psi(\theta) + i \sin \alpha,$$

where $\psi(\theta)$ is a function with bounded variation in $[0, 2\pi]$ satisfying

$$(1.2) \quad \int_0^{2\pi} d\psi(\theta) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\psi(\theta)| \leq k\pi.$$

Proof. Let

$$g(z) = \frac{e^{i\alpha}p(z) - \rho \cos \alpha - i \sin \alpha}{(1 - \rho) \cos \alpha},$$

and let

$$u(z) = \operatorname{Re} \{g(z)\} = \operatorname{Re} \left\{ \frac{\rho(z) - \rho \cos \alpha}{(1 - \rho) \cos \alpha} \right\}.$$

Then since $p(z) \in P_\alpha^k(\rho)$, $\int_0^{2\pi} |u(re^{i\theta})| d\theta \leq k\pi$, and according to a representation theorem due to Paatero [8],

$$\frac{e^{i\alpha}p(z) - \rho \cos \alpha - i \sin \alpha}{(1 - \rho) \cos \alpha} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{i\theta}}{1 - ze^{i\theta}} d\psi(\theta),$$

where $\psi(\theta)$ has bounded variation and satisfies condition (1.2) above. The conclusion of the lemma follows.

Now let $f(z) \in V_\alpha^k(\rho)$. By a theorem due to Padmanabhan and Parvatham [9], the integral in (1.1)

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 + (1 - 2\rho)ze^{i\theta}}{1 - ze^{i\theta}} d\psi(\theta) = 1 + zf_0''(z)/f_0'(z),$$

for some f_0 in $V_0^k(\rho)$. So

$$e^{i\alpha} \left[1 + \frac{zf''(z)}{f'(z)} \right] = \cos \alpha \left[1 + \frac{zf_0''(z)}{f_0'(z)} \right] + i \sin \alpha.$$

$$\frac{f''(z)}{f'(z)} = e^{i\alpha} \cos \alpha \left[\frac{1}{z} + \frac{f_0''(z)}{f_0'(z)} \right] + i \frac{e^{-i\alpha} \sin \alpha - 1}{z}.$$

Integrating, we obtain

LEMMA 2. $f(z)$ is in $V_\alpha^k(\rho)$ if and only if there is a function $f_0(z)$ in $V_0^k(\rho)$ such that

$$f'(z) = [f_0'(z)]^{e^{-i\alpha} \cos \alpha} .$$

The function $f_0(z)$ in $V_0^k(\rho)$ has associated with it a function $g_0(z)$ in $V_0^k(0)$. ([9], Lemma 2.)

LEMMA 3. $f(z)$ is in $V_\alpha^k(\rho)$ if and only if there is a function $g_0(z)$ in $V_0^k(0)$ such that

$$f'(z) = [g_0'(z)]^{(1-\rho)e^{-i\alpha} \cos \alpha} .$$

LEMMA 4. $f(z)$ is in $V_\alpha^k(\rho)$ if and only if there exists a function $g(z)$ in $V_\alpha^k(0)$ such that

$$f'(z) = [g'(z)]^{(1-\rho)} .$$

Proof. The function $[g_0'(z)]^{e^{-i\alpha} \cos \alpha}$ determines a function $g'_\alpha(z)$, where $g_\alpha(z)$ is in $V_\alpha^k(0)$ [7].

From Paatero's representation theorem for functions with bounded variation [8], we obtain the following representation.

THEOREM 1. $f(z)$ is in $V_\alpha^k(\rho)$ if and only if there exists a function $\psi(\theta)$ with bounded variation on $[0, 2\pi]$ satisfying condition (1.2) and

$$f'(z) = \exp \left\{ \frac{-(1-\rho)e^{-i\alpha} \cos \alpha}{\pi} \int_0^{2\pi} \log(1 - ze^{i\theta}) d\psi(\theta) \right\} .$$

THEOREM 2. $f(z)$ is in $V_\alpha^k(\rho)$ if and only if

(A) there exist starlike functions S_1, S_2 such that

$$f'(z) = \left\{ \frac{\left[\frac{S_1(z)}{z} \right]^{(k+2)/4}}{\left[\frac{S_2(z)}{z} \right]^{(k-2)/4}} \right\}^{(1-\rho)e^{-i\alpha} \cos \alpha}$$

(B) there exist α -spiral functions T_1, T_2 such that

$$f'(z) = \left\{ \frac{\left[\frac{T_1(z)}{z} \right]^{(k+2)/4 \cdot 1-\rho}}{\left[\frac{T_2(z)}{z} \right]^{(k-2)/4}} \right\} .$$

(C) there exist functions L_1, L_2 in $V_0^2(0)$ such that

$$f'(z) = \left\{ \frac{[L_1'(z)]^{(k+2)/4}}{[L_2'(z)]^{(k-2)/4}} \right\}^{(1-\rho)e^{-i\alpha} \cos \alpha} .$$

(D) there exist functions H_1, H_2 in $V_0^2(\rho)$ such that

$$f'(z) = \left\{ \frac{[H_1'(z)]^{(k+2)/4}}{[H_2'(z)]^{(k-2)/4}} \right\}^{e^{-i\alpha} \cos \alpha} .$$

Proof. (A) follows from Lemma 3 and Brannan’s representation for functions with bounded boundary rotation [3]. (B) follows from (A) since $s(z)$ is starlike if and only if $T(z) = z[s(z)/z]^{e^{-i\alpha} \cos \alpha}$ is α -spirallike. (C) follows from (A) because of the fact that $H(z)$ is convex if and only if $zH'(z) = S(z)$ is starlike. (D) follows trivially from (C).

2. Properties of functions in $V_\alpha^k(\rho)$.

COROLLARY 1. Suppose $f(z) = z + a_2z^2 + \dots$ is in $V_\alpha^k(\rho)$. Then $|a_2| \leq k(1 - \rho) \cos \alpha/2$, and this bound is sharp.

Proof. It is well known that if g_0 is in $V_0^k(0)$, then $|g_0''(0)| \leq k$, so the result follows directly from Lemma 3. This bound is sharp for the function $f(z)$ in $V_\alpha^k(\rho)$ defined by

$$f'(z) = \left\{ \frac{[(1 - z)^{(k-2)/2}]}{[(1 + z)^{(k+2)/2}]} \right\}^{(1-\rho)e^{-i\alpha} \cos \alpha}$$

LEMMA 5. If $f(z)$ is in $V_\alpha^k(\rho)$, then $F(z)$ defined by

$$F'(z) = \frac{f'\left(\frac{z + a}{1 + \bar{a}z}\right)}{f'(a)(1 + \bar{a}z)^{2(1-\rho)e^{-i\alpha} \cos \alpha}} , \quad F(0) = 0, \quad |a| < 1, \quad |z| < 1 ,$$

is also in $V_\alpha^k(\rho)$.

Proof. By Lemma 2, for $f(z)$ in $V_\alpha^k(\rho)$, there exists $f_0(z)$ in $V_0^k(\rho)$ such that $f'(z) = [f_0'(z)]^{e^{-i\alpha} \cos \alpha}$. By Lemma 3 in [9],

$$\frac{f_0'\left(\frac{z + a}{1 + \bar{a}z}\right)}{f_0'(a)(1 + \bar{a}z)^{2(1-\rho)}} \text{ is the derivative of}$$

a function in $V_0^k(\rho)$. Hence

$$\left[\frac{f_0'\left(\frac{z + a}{1 + \bar{a}z}\right)}{f_0'(a)(1 + \bar{a}z)^{2(1-\rho)}} \right]^{e^{-i\alpha} \cos \alpha} = \frac{f'\left(\frac{z + a}{1 + \bar{a}z}\right)}{f'(a)(1 + \bar{a}z)^{2(1-\rho)e^{-i\alpha} \cos \alpha}}$$

is the derivative of a function in $V_\alpha^k(\rho)$.

THEOREM 3. *If $f(z)$ is in $V_\alpha^k(\rho)$ and $0 < k(1 - \rho) \cos \alpha \leq 1$, then $f(z)$ is univalent in $|z| < 1$.*

Proof. By the previous lemma, if $f(z)$ is in $V_\alpha^k(\rho)$, then $F(z)$ defined by

$$F'(z) = \frac{f'\left(\frac{z+a}{1+\bar{a}z}\right)}{f'(a)(1+\bar{a}z)^{2(1-\rho)e^{-i\alpha}\cos\alpha}}, \quad F(0) = 0,$$

is in $V_\alpha^k(\rho)$ also, with $|a| < 1$ and $|z| < 1$. Then

$$\begin{aligned} F'''(z) = & \left[(1+az)^{2(1-\rho)e^{-i\alpha}\cos\alpha} f''\left(\frac{z+a}{1+\bar{a}z}\right) \cdot \frac{1-|a|^2}{(1+\bar{a}z)^2} \right. \\ & \left. - 2(1-\rho)e^{-i\alpha}\cos\alpha(1+\bar{a}z)^{2(1-\rho)e^{-i\alpha}\cos\alpha-1} \bar{a}f'\left(\frac{z+a}{1+\bar{a}z}\right) \right] \\ & \times [f'(a)(1+\bar{a}z)^{4(1-\rho)e^{-i\alpha}\cos\alpha}]^{-1}, \end{aligned}$$

$$F'''(0) = \frac{f''(a)}{f'(a)}(1-|a|^2) - 2(1-\rho)e^{-i\alpha}\cos\alpha \bar{a}.$$

Replacing a by z , using Corollary 1 of Theorem 2, and multiplying through by $|z|$, we have

$$\begin{aligned} & \left| \frac{zf''(z)}{f'(z)}(1-|z|^2) - 2(1-\rho)e^{-i\alpha}\cos\alpha|z|^2 \right| \\ & \leq k(1-\rho)\cos\alpha|z| < k(1-\rho)\cos\alpha. \end{aligned}$$

Ahlfors' univalence criterion [1], with $c = 2(1-\rho)e^{-i\alpha}\cos\alpha$, shows that f is univalent in E when $0 < k(1-\rho)\cos\alpha \leq 1$.

COROLLARY 1. *If $f(z)$ is in $V_\alpha^k(0)$, f is univalent in E whenever*

$$(2.1) \quad 0 < \cos \alpha \leq 1/k.$$

This simplifies and improves bounds previously published for this class [7].

COROLLARY 2. *If $f(z)$ is in $V_0^k(\rho)$, then f is univalent in E for*

$$(2.2) \quad \rho \geq \frac{k-1}{k}.$$

Previously, it was shown in [9] that f is univalent for $\rho \geq (k+1)/(k+2)$.

COROLLARY 3. *If $f(z)$ is in $V_\alpha^2(\rho)$, then $f(z)$ is univalent in E when $0 < \cos \alpha \leq 1/2(1 - \rho)$. f need not be univalent if $\cos \alpha > 1/[2(1 - \rho)]$.*

Chichra [4] has shown that for each α , $1/[2(1 - \rho)] < \cos \alpha < 1$, there exists a function $f(z)$ in $F_\alpha^\rho = V_\alpha^2(\rho)$ such that $f(z)$ is not univalent in E . Hence the problem of univalence in $V_\alpha^2(\rho)$ is solved.

3. We may use the same function f as in [4] to study conditions on k , α , and ρ which will allow functions in $V_\alpha^k(\rho)$ to be non-univalent. Let

$$(3.1) \quad g(z) = \frac{1}{\mu} [(1-z)^{-\mu} - 1],$$

and note

$$g'(z) = \frac{1}{(1-z)} \mu + 1.$$

$g'(z)$ has the form given in Theorem 2C, with $L_1'(z) = (1-z)^{-1}$ and $L_2'(z) = 1$ and

$$(3.2) \quad \mu + 1 = e^{-i\alpha} \cos \alpha (1 - \rho)(k + 2)/4.$$

Hence $g(z)$ is in $V_\alpha^k(\rho)$ and, from an earlier result due to Royster [11], will not be univalent in $|z| < 1$ when $|\mu + 1| > 1$ and $|\mu - 1| > 1$. The first condition requires that

$$(3.3) \quad \cos \alpha (1 - \rho)(k + 2)/4 > 1,$$

while the second condition simplifies to

$$(3.4) \quad \cos^2 \alpha (1 - \rho)(k + 2) \left[\frac{(1 - \rho)(k + 2)}{16} - 1 \right] > -3.$$

We may use these conditions to analyze the nonunivalence of functions in subclasses of $V_\alpha^k(\rho)$ which have been previously studied. When $\rho = 0$, the conditions defined by (2.1), (3.3) and (3.4) appear in Fig. 1. All functions in $V_\alpha^k(0)$ with k and α corresponding to points in region 1 are univalent, by (2.1). In region 3, $(k+2) \cos \alpha / 4 > 1$ and condition (3.4) is satisfied for all $k > 6$ when $0 < \cos \alpha < \sqrt{3}/2$; for $\sqrt{3}/2 \leq \cos \alpha < 1$, (3.4) is equivalent to $k > 6 - 4[4 \cos^2 \alpha - 3]^{1/2} / \cos \alpha$. When $g(z)$ defined by (3.1) is chosen so as to correspond with points in region 3, it will not be univalent. When

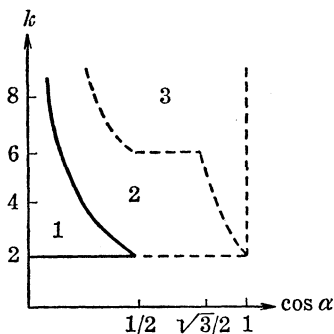


FIGURE 1

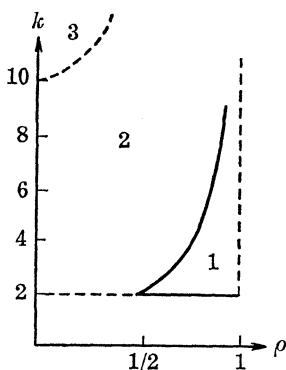


FIGURE 2

k and α correspond to points in region 2, it is an open question whether such f in $V_\alpha^k(0)$ will be univalent.

Fig. 2 is the corresponding diagram for univalence in the class $V_0^k(\rho)$. Region 1 depicts inequality (2.2), and all functions g defined by (3.1) with k, ρ satisfying (3.2) for $\alpha = 0$ are univalent in $|z| < 1$. Conditions (3.3) and (3.4) require that $\rho < (k - 10)/(k + 2)$, and for these values of ρ and k (in region 3), $g(z)$ will not be univalent. Region 2 shows those values of k and ρ for which the univalence of functions in $V_0^k(\rho)$ is an open question. We note that when $k = 2$, the equation (3.1) defines the function used by Chichra in showing that there exist functions f in $F_\alpha^\rho = V_\alpha^2(\rho)$ where f is not univalent in $|z| < 1$, for $1/2(1 - \rho) < \cos \alpha < 1$.

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