THE CENTRALIZER OF TENSOR PRODUCTS OF BANACH SPACES (A FUNCTION SPACE REPRESENTATION)

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Let $X, Y$ be real Banach spaces, $X \hat{\otimes} Y$ their usual $\varepsilon$-tensor product. We represent $Z(X \hat{\otimes} Y)$, the centralizer of $X \hat{\otimes} Y$, as a space of real-valued functions on a suitable compact Hausdorff space. As a corollary we obtain Wicks-Beaumont's result: $Z(X \hat{\otimes} Y)$ is the closure with respect to the strong operator topology of $Z(X) \otimes Z(Y)$. In addition it is shown that $Z(X \hat{\otimes} Y)$ is in fact the uniform closure of $Z(X) \otimes Z(Y)$ provided the norm topology and the strong operator topology coincide on the centralizers of $X$ and $Y$.

1. Introduction. Let $X$ be a real Banach space. By $Z(X)$, the centralizer of $X$, we denote the set of $M$-bounded operators on $X$, i.e., the collection of those continuous linear operators $T: X \to X$ for which there is a $\lambda \in R$ such that $T_x$ is contained in every open ball which contains $\pm \lambda x$ (for $x \in X$); cf [2], [3], [4], [5], [8]. $Z(X)$ is, as a Banach algebra, isometrically isomorphic to the space $C(K_X)$ of continuous real-valued functions on a suitable compact Hausdorff space $K_X: C(K_X) \cong Z(X)$ ([2], 4.8).

For example, if $L$ is a locally compact Hausdorff space and $X = C_0 L: = \{f|f: L \to R, f$ continuous, $f$ vanishes at infinity$\}$, provided with the supremum norm, then it is easy to see that $Z(X)$ is identical with the space of all multiplication operators $M_h, f \mapsto h f$ (all $f \in C_0 L), h$ a bounded continuous function. Therefore $Z(X)$ is isometrically isomorphic with $C^b L: = \{h|h: L \to R, h$ continuous and bounded$\}$ so that $K_X = \beta L$ the Stone-Čech compactification of $L$ (up to homeomorphism).

Centralizers of Banach spaces play an important role in a great number of papers (cf. for example the references in [2]). We will investigate the centralizer of tensor products. In particular we are interested in the relation between the centralizer of the tensor product and the centralizers of the factors. Let $X$ and $Y$ be real Banach spaces, $X \otimes Y$ their algebraic tensor product. For $\sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ we define

$$\left\| \sum_{i=1}^n x_i \otimes y_i \right\| = \sup \left\{ \sum_{i=1}^n f(x_i) \tilde{f}(y_i) \mid f \in X', \|f\| \leq 1, \tilde{f} \in Y', \|\tilde{f}\| \leq 1 \right\}$$

$$= \sup \left\{ \left\| \sum_{i=1}^n f(x_i) y_i \right\| \mid f \in X', \|f\| \leq 1 \right\}$$

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we will use the same symbol $\| \|$ to denote the norm in all tensor products of Banach spaces which will appear in this paper — this is justified because we will not consider any other tensor product norms. $X \hat{\otimes} Y$ means the completion of $X \otimes Y$ provided with this norm.

It is not hard to see that, for $T \in Z(X)$ and $S \in Z(Y)$ we have $S \otimes T \in Z(X \hat{\otimes} Y)$ ([8], p. 564; note that Wickstead uses another but equivalent definition of $\Lambda f$-boundedness and that he writes $\hat{\otimes}$ instead of $\otimes$). Therefore $Z(X) \otimes Z(Y)$ may be thought of as a subspace of $Z(X \hat{\otimes} Y)$. We note that the tensor product norm of the operators in $Z(X) \otimes Z(Y)$ is exactly their operator norm. Wickstead proves ([8], Th. 3) that $Z(X) \otimes Z(Y)$ is the strong closure of $Z(X \hat{\otimes} Y)$. In general the strong closure may not be replaced by the uniform closure in this theorem. There are, however, important classes of Banach spaces for which $Z(X) \otimes Z(Y)$ is the uniform closure of $Z(X \otimes Z(Y)$. We will prove in §4 that this is the case if the strong operator topology and the norm topology are equivalent on the centralizers of $X$ and $Y$.

We will proceed as follows: In §2 we will state without proof those results of the function module representation theory introduced in [5] which we will need in the sequel. We will show that $X \hat{\otimes} Y$ has a function module representation which is related to the function module representations of $X$ and $Y$ in a natural way, a theorem which will be of fundamental importance for the following considerations. Section 3 contains a discussion of those Banach spaces $X$ for which the norm topology and the strong operator topology on $Z(X)$ are equivalent. In §4 we will show that $Z(X \hat{\otimes} Y)$ is isometrically isomorphic to a space of real-valued bounded (not necessarily continuous) functions on a suitable compact Hausdorff space. Finally, we investigate some consequences of this representation theorem. For example, we derive Wickstead's result as a corollary.

Note. In the first version of this paper Wickstead's theorem was used at a crucial point in the proof of Theorem 4.2. We are grateful to the referee for suggesting that we give an independent proof using the theory of function modules.

2. A function module representation of $X \hat{\otimes} Y$.

DEFINITION 2.1 ([5]). Let $K$ be a compact Hausdorff space, $(W_k)_{k \in K}$ a family of Banach spaces indexed by the points of $K$. A closed subspace $W$ of
\[ \prod_{k \in K} W_k = \left\{ (w(k))_{k \in K} \mid (w(k))_{k \in K} \in \prod_{k \in K} W_k, \quad \| (w(k))_{k \in K} \| = \sup_{k \in K} \| w(k) \| < \infty \right\} \]

is called a function module in \( \prod_{k \in K} W_k \) if the following conditions are satisfied:

(a) \( hw \in W \) for \( h \in CK, w \in W \) \((hw)(k) := h(k)w(k) \) for \( k \in K \)

(b) \( k \mapsto \| w(k) \| \) is upper semi-continuous on \( K \) for \( w \in W \)

(c) \( W_k = \{ w(k) \mid w \in W \} \) for \( k \in K \).

**Note.** By [5], p. 621, \( \{ w(k) \mid w \in W \} \) is closed for each \( k \in K \) if \( W \) is a closed subspace of \( \prod_{k \in K} W_k \) and (a) and (b) are satisfied.

**Proposition 2.2.** Let \( W \) be as in the preceding definition. For \( h \in CK \), the multiplication operator \( M_h : W \to W, w \mapsto hw \), is well-defined by 2.1(a). We claim that \( M_h \in Z(W) \). More generally, if \( \alpha : K \to \mathbb{R} \) is a bounded function such that \( M_\alpha(W) \subset W \), then \( M_\alpha \in Z(W) \). In addition, \( M_\alpha \) is contained in the strong operator closure of \( \{ M_h \mid h \in CK \} \).

**Proof.** It is easy to see that \( M_\alpha : W \to W \) is linear and continuous with \( \| M_\alpha \| \leq \| \alpha \| := \sup \{ |\alpha(k)| \mid k \in K \} \) (\( \alpha : K \to \mathbb{R} \) a bounded function such that \( M_\alpha(W) \subset W \)). \( M_\alpha \) obviously satisfies the condition for \( M \)-bounded operators with \( \lambda = \| \alpha \| \).

Let \( w_1, \ldots, w_n \in W, \varepsilon > 0 \) be arbitrarily given. For every \( k \in K \), \( \alpha(k)w_i - \alpha w_i \) is in \( W \) and vanishes at \( k \), so that, by 2.1(b), there is an open neighborhood \( U_k \) of \( k \) such that \( \|(\alpha(k)w_i - \alpha w_i)(1)\| \leq \varepsilon \) for 1 in \( U_k \) (all \( i \in \{1, \ldots, n\} \)). Let \( U_{k_1}, \ldots, U_{k_r} \) be a finite covering of \( K \). Then \( \| hw_i - \alpha w_i \| \leq \varepsilon \) for \( i = 1, \ldots, n \), where \( h := \sum_{j=1}^r \alpha(k_j)h_j \) and \( h_1, \ldots, h_r \) is a suitable partition of unity subordinate to \( U_{k_1}, \ldots, U_{k_r} \). This proves that \( M_\alpha \) is in the strong closure of \( \{ M_h \mid h \in CK \} \).

**Theorem 2.3.** Let \( X \) be a real Banach space, \( K_X \) a compact Hausdorff space such that \( Z(X) \cong CK_X \) (note that \( K_X \) is uniquely determined up to homeomorphism). \( X \) can be identified with a function module in \( \prod_{k \in K_X} X_k \) ((\( X_k \) \( k \in K_X \) a family of Banach spaces, the component spaces) such that the operators in \( Z(X) \) correspond to multiplication operators associated with the elements of \( CK_X \). More precisely, there is a linear isometry \( \omega : X \to \prod_{k \in K_X} X_k \) such that

(i) \( \omega(X) \) is a function module in \( \prod_{k \in K_X} X_k \).

(ii) for \( T \in Z(X), x \in X \) we have \( \omega(Tx) = \tilde{T}\omega(x) \), where \( \tilde{T} \in CK_X \) corresponds to \( T \) according to the isometry \( Z(X) \cong CK_X \).

In addition we have

(iii) \( \{ k \mid X_k \neq 0 \} \) is dense in \( K_X \).
Proof. (i) and (ii) are proved in [5] (Theorem 6 and Theorem 3; note that the maximal $M$-structure of $X$ is just $Z(X)$ by [2], 4.8). (iii) can be verified as follows: If $\tilde{T} \in \mathcal{C}K_X$ is an arbitrary function with corresponding operator $T \in Z(X)$, then we have $\|\tilde{T}\| = \|T\| = \sup \{|\tilde{T}\omega(x)| \mid \|x\| = 1\} = \sup \{|\tilde{T}(k)| \mid X_k \neq 0\}$. This implies that $\{k \mid X_k \neq 0\}$ is dense in $K_X$.

Theorem 2.4. Let $X$ (resp. $Y$) be a function module in $\prod_{k \in K} X_k$ (resp. $\prod_{k \in L} Y_k$), where $K$ and $L$ are compact Hausdorff spaces. For $\sum_{i=1}^m x_i \otimes y_i \in X \otimes Y$ let $\sum_{i=1}^m x_i \otimes y_i$ be the element

$$
\left(\sum_{i=1}^m x_i(k) \otimes y_i(1)\right)_{(k,1) \in K \times L}
$$

of $\prod_{k \in K} X_k \otimes_{\mathcal{F}} Y_1$. Then

(i) $\|\sum_{i=1}^m x_i \otimes y_i\| = \|\sum_{i=1}^m x_i \otimes y_i\|$ for $\sum_{i=1}^m x_i \otimes y_i \in X \otimes Y$ so that $X \otimes \hat{Y}$ can be identified with a closed subspace of $\prod_{k \in K} X_k \otimes_{\mathcal{F}} Y_1$; further, it is not necessary to distinguish between $x \otimes y$ and $x \otimes y$.

(ii) $X \otimes Y$ is a function module in $\prod_{k \in K} X_k \otimes_{\mathcal{F}} Y_1$.

Proof. (i) We will use the fact that the extreme points of the unit ball $S^{+}_{1'}$ (resp. $S^{+}_{1''}$) of $X'$(resp. $Y'$) are contained in the set of functionals of the form $x \mapsto f(x(k))$(resp. $y \mapsto f(y(1))$) where $k \in K$, $f \in (X_k)'$, $\|f\| \leq 1$ (resp. $1 \in L$, $\tilde{f} \in (Y_1)'$, $\|\tilde{f}\| \leq 1$); [6].

$$
\left\|\sum_{i=1}^m x_i \otimes y_i\right\| = \sup \left\{\|\sum F(x_i)\tilde{F}(y_i)\| \mid F \in X', \|F\| \leq 1, \tilde{F} \in Y', \|\tilde{F}\| \leq 1\right\}
= \sup \left\{\|\sum F(x_i)\tilde{F}(y_i)\| \mid F \in \text{ex} S^{+}_{1'}, \tilde{F} \in \text{ex} S^{+}_{1''}\right\}
= \sup \left\{\|\sum f(x_i(k))\tilde{f}(y_i(1))\| \mid k \in K, f \in (X_k)', \|f\| \leq 1, 1 \in L, \tilde{f} \in (Y_1)', \|\tilde{f}\| \leq 1\right\}
= \sup \left\{\|\sum x_i(k) \otimes y_i(1)\| \mid k \in K, 1 \in L\right\}
= \|\sum x_i \otimes y_i\|.
$$

Similarly one can prove that $\|\sum_{i=1}^m x_i(k) \otimes y_i(1)\| = \sup_{1 \in L} \|\sum_{i=1}^m x_i(k) \otimes y_i(1)\|$ for $k \in K$ (where the norms are calculated in $X_1 \otimes Y$ and $X_k \otimes Y_1$, respectively).

(ii) We only have to show that

(a) $h(\sum x_i \otimes y_i) \in X \otimes Y$ for $h \in C(K \times L)$, $\sum x_i \otimes y_i \in X \otimes Y$.
(b) $(h, 1) \mapsto || \sum x_i(k) \otimes y_i(1) ||$ is upper semi-continuous for $\sum x_i \otimes y_i \in X \otimes Y$
(c) $X \otimes Y$ is dense in $X \otimes Y$.

(a), (b), and (c) easily imply that $(X \otimes Y)^{-} = X \otimes Y$ is a function module (cf. the note at the end of 2.1).

(a) Let $h \in C(K \times L)$, $\sum x_i \otimes y_i \in X \otimes Y$. For $\varepsilon > 0$ there are $h_i, \ldots,$
\[ h_n \in \mathcal{CK}, g_1, \ldots, g_n \in \mathcal{CL} \text{ such that } \| \sum_{i=1}^{n} h_i \otimes g_i - h \| \leq \varepsilon. \] We thus have
\[
\| h \sum x_i \otimes y_i - (\sum h_i \otimes g_i)(\sum x_i \otimes y_i) \| \\
= \| h \sum x_i \otimes y_i - \sum_{i,j} h_j x_i \otimes g_j y_i \| \leq \varepsilon \| \sum x_i \otimes y_i \|.
\]

Since \( \sum_{i,j} h_j x_i \otimes g_j y_i \in X \otimes Y \) this implies that \( h \sum x_i \otimes y_i \in X \otimes Y \).

(b) Let \( a \in \mathbb{R}, (k_0, l_0) \in K \times L, \sum_{i=1}^{r} x_i \otimes y_i \in X \otimes Y, \| \sum_{i} (k_0) \otimes y_i(l_0) \| < a \). We have to show that there are neighbourhoods \( U \) of \( k_0, V \) of \( l_0 \) such that \( \| \sum x_i(k) \otimes y_i(1) \| < a \) for \( k \in U, 1 \in V \).

At first we will prove that there is a neighborhood \( \tilde{V} \) of \( l_0 \) such that \( \| \sum x_i(k_0) \otimes y_i(1) \| < a - 2\eta \) for \( 1 \in \tilde{V} \) (where \( \eta > 0 \) is a number such that \( \| \sum x_i(k_0) \otimes y_i(1) \| < a - 3\eta \)). To this end we choose an \((\eta/R)\)-net \( f_1, \ldots, f_N \) in the dual unit ball of the linear hull of \( x_i(k_0), \ldots, x_r(k_0)(R : = \sum \| x_i \| \| y_i \| + 1) \). It follows that, for \( f \in (X_{k_0})' \), \( \| f \| \leq 1 \), there is an \( f_j \in \{ f_1, \ldots, f_N \} \) such that \( \| \sum f_j(x_i(k_0)) y_i(1) \| = \left\| f_j - f \right\| R \) (all \( 1 \in L \)), i.e.,
\[
\| \sum x_i(k_0) \otimes y_i(1) \| = \sup \| \sum f(x_i(k_0)) y_i(1) \| \| f \in (X_{k_0})', \| f \| \leq 1 \|
\leq \sup \| \sum f_j(x_i(k_0)) y_i(1) \| \| j = 1, \ldots, N \| + \eta
\]

(All \( 1 \in L \)).

For \( j \in \{1, \ldots, N\}, \sum f_j(x_i(k_0)) y_i \) belongs to \( Y \) and \( \| \sum f_j(x_i(k_0)) y_i(1) \| \leq \| \sum x_i(k_0) \otimes y_i(1) \| < a - 3\eta \) so that by 2.1(b) there is a neighbourhood \( \tilde{V} \) of \( l_0 \) with \( \| \sum f_j(x_i(k_0)) y_i(1) \| < a - 3\eta \) for \( 1 \in \tilde{V} \) and \( j \in \{1, \ldots, N\} \).

For \( 1 \in \tilde{V} \) we thus have \( \| \sum x_i(k_0) \otimes y_i(1) \| < a - 2\eta \).

We now choose a function \( g \in \mathcal{CL} \) such that \( \| g \| = 1, g(1) = 1 \) in a suitable neighborhood \( \tilde{V} \) of \( l_0 \) contained in \( \tilde{V} \) and \( g |_{\tilde{V}} = 0 \). We then have (cf. the proof of (i)) \( \| \sum x_i(k_0) \otimes g y_i \| = \sup_{1 \in L} \| \sum x_i(k_0) \otimes g(1) y_i(1) \| \leq a - 2\eta \). Similarly to the first step of this proof we select an \((\eta/R)\)-net \( \tilde{f}_1, \ldots, \tilde{f}_M \) in the dual unit ball of the linear hull of \( g y_1, \ldots, g y_M \) (it follows that \( \| \sum x_i(k) \otimes g y_i \| \leq \sup \{ \| \sum \tilde{f}_j(g y_i)x_i(k) \| \| j = 1, \ldots, M \} + \eta \) for \( k \in K \)). For \( j \in \{1, \ldots, M\} \) we have \( \sum \tilde{f}_j(g y_i)x_i \in X \) and \( \| \sum \tilde{f}_j(g y_i)x_i(k_0) \| < a - \eta \). Therefore there is a neighborhood \( U \) of \( k_0 \) such that \( \| \sum \tilde{f}_j(g y_i)x_i(k) \| < a - \eta \) for \( k \in U, j = 1, \ldots, M \). This yields
\[
\sup_{1 \in \tilde{V}} \| \sum x_i(k) \otimes y_i(1) \| \leq \sup_{1 \in L} \| \sum x_i(k) \otimes (g y_i)(1) \|
\leq \| \sum x_i(k) \otimes g y_i \|
\leq \sup \{ \| \sum \tilde{f}_j(g y_i)x_i(k) \| \| j = 1, \ldots, M \} + \eta
< a \text{ for } k \in U.
\]

(c) This is obvious.
Remark. For the rest of this paper we will assume that $X$ and $Y$ are real Banach spaces which are identified with function modules in $\mathbb{P}^\infty_{k\in K_X} X_k$ resp. $\mathbb{P}^\infty_{k\in K_Y} Y_k$ as described in 2.3. With this identification, $X \otimes \tau Y$ is a function module in $\mathbb{P}^\infty_{k,1} X_k \otimes_k Y_1$ by 2.4.

Another way of representing the centralizer as a space of real-valued continuous functions is the Dauns-Hofmann type theorem of Alfsen-Effros ([2], 4.9). The relationship between this and the function module approach (2.3(ii)) is shown by the following proposition.

**Proposition 2.5.** Let $X, K_X, (X_k)_{k\in K_X}$ be as above, $K^*_X: = \{k \mid k \in K_X, X_k \neq 0\}$.

(i) Every $h_0 \in C^0(K^*_X)$ has a unique continuous extension to $K_X$ (so that $K_X = \beta K^*_X$).

(ii) Let $E_X$ be the set of extreme points in the unit ball of $X'$. By [6] we have $E_X = \bigcup_{k\in K_X^*} E_{X_k}$. Let $\pi: E_X \rightarrow K^*_X$ be defined by $\pi(p) := k$ for $p \in E_{X_k}$. Then, for every bounded structurally continuous mapping $g: E_X \rightarrow \mathbb{R}$ there is a function $h \in C^0(K^*_X)$ such that $g = h \circ \pi$. Conversely, for $h \in C^0(K^*_X)$, $h \circ \pi$ is structurally continuous.

**Proof.** (i) Let $h_0 \in C^0(K^*_X)$ be given. We define $h: K_X \rightarrow \mathbb{R}$ by $h(k) := h_0(k)$ for $k \in K^*_X$ and $h(k) = 0$ for $k \in K_X \setminus K^*_X$. Let $x \in X$ be given and $\varepsilon > 0$. $h$ is continuous on the closed set $D := \{k \mid \|x(k)\| \geq \varepsilon\} \subset K^*_X$ so that we may choose a continuous function $h_D: K_X \rightarrow \mathbb{R}$ such that $h_D = h|_D$, $\|h\| = \|h_D\|$. We then have $h_Dx \in X$ and $\|h_Dx - hx\| \leq 2\varepsilon\|x\|$ so that we may conclude that $hx \in X^- = X$. 2.2 and 2.3(ii) imply that there is a function $h' \in CK_X$ such that $M_k = M_{k'}$. $h'$ is obviously a continuous extension of $h$ which is uniquely determined by 2.3 (iii).

(ii) Let $g: E_X \rightarrow \mathbb{R}$ be a bounded structurally continuous function. By [2], 4.9, there is a $T \in Z(X)$ such that $p \circ T = g(p)p$ for every $p \in E_X$. Let $\tilde{T} \in CK_X$ be that function which corresponds to $T$. We then have $\tilde{T}(k)p = g(p)p$ for $p$ in $E_{X_k}$ so that $\tilde{T} \circ \pi = g$. Conversely, let $\tilde{T} \in CK_X$ be given. For $p \in E_{X_k}$ we have $p \circ T = \tilde{T}(k)p = (\tilde{T} \circ \pi)(p)p$. By [2], 4.9 this implies that $\tilde{T} \circ \pi$ is structurally continuous.

3. Centralizer-norming systems. In view of the following considerations we want to single out those Banach spaces for which, in a sense, the centralizer is “not too great”.

**Definition 3.1.** Let $X$ be a real Banach space. A finite family $x_1, \ldots, x_n$ in $X$ is called a centralizer-norming system (abbreviated: *ens*) if there is a number $r > 0$ such that max $\{\|Tx_i\| \mid i = 1, \ldots, n\} \geq r$.
r\|T\| for every $T \in Z(X)$. Obviously $X$ has a cns iff the norm topology and the strong operator topology coincide on $Z(X)$.

**Examples.** (1) Let $X$ be a Banach space for which $Z(X)$ is finite-dimensional (those spaces play an important role in the applications of $M$-structure to theorems of the Banach-Stone type; cf. [3], [4]). It is clear that $X$ has a cns (in fact, $X$ has a cns consisting of a single element).

We note that, for example, spaces which are smooth or strictly convex have one-dimensional centralizer and that $Z(X)$ is finite-dimensional for every reflexive space $X$ ([4]).

(2) If $L$ is a locally compact Hausdorff space, then $C_0L$ has a cns iff $L$ is compact. In this case we may choose $n = 1$ and $x_1 = 1$ (= the constant function assuming the value 1 at every point).

(3) Let $A$ be a $C^*$-algebra with unit $e$, $X$ the self-adjoint part of $A$. Then $\{e\}$ is a cns in $X$ since $Z(X)$ is just the space of multiplication operators corresponding to the self-adjoint elements in the center of $A$ ([2], Cor. 6.17).

(4) One might suggest that for Banach spaces $X$ having a cns it is always possible to find a cns consisting of a single element. We will use the Borsuk-Ulam theorem from algebraic topology to prove that $\inf \{n | n \in N, \text{there exists a cns in } X \text{ consisting of } n \text{ elements} \}$ may be an arbitrarily large number:

For $m \in N$ let $S^m$ be the $m$-dimensional sphere (i.e., the surface of the unit ball in the $(m + 1)$-dimensional Hilbert space), $X: = \{f \in C(S^m), f(-x) = -f(x) \text{ for all } x \in S^m\}$. $X$ is just the space $C_\Sigma(S^m)$, where $\Sigma: S^m \rightarrow S^m$ is the homeomorphism $x \mapsto -x$; cf. [7], Chapter 3, p. 71). A routine computation shows that $T \in Z(X)$ iff there is a continuous function $h: S^m \rightarrow R$ such that $h(x) = h(-x)$ for all $x \in S^m$ and $Tf = hf$ for $f \in X$. Therefore a family $f_1, \cdots, f_n$ in $X$ is a cns iff $\max \{||f_i(x)|| | i = 1, \cdots, n\} > 0$ for all $x \in S^m$. $X$ obviously has a cns consisting of $m + 1$-elements (for example, $f_i(x): = \text{the } i\text{th component of } x, x \in S^m, i = 1, \cdots, m + 1, \text{defines a family of functions with this property}$). On the other hand, if $g_1, \cdots, g_m$ are arbitrary functions in $X$, there is an $x_0 \in S^m$ such that $g_1(x_0) = \cdots = g_m(x_0) = 0$, i.e., $g_1, \cdots, g_m$ cannot be a cns ([1], p. 485).

We will need the fact that there is a characterization of centralizer-norming systems in terms of the function module representation 2.3:

**Lemma 3.2.** Let $X$ be a real Banach space, $X$ represented as a function module in $\prod_{k \in K} X_k$ as described in §2.

A finite family $x_1, \cdots, x_n$ in $X$ is a cns iff $\inf_k \max_i ||x_i(k)|| > 0$. 
Proof. Suppose that $x_1, \ldots, x_n$ is a cns in $X$, i.e., there is a number $r > 0$ such that $\max_i ||T x_i|| \geq r ||T||$ for $T \in Z(X)$. We claim that $\max_i ||x_i(k)|| \geq r$ for $k \in K_x$. Assume that there is a $k_0 \in K_x$ such that $||x_i(k_0)|| < r$ for $i = 1, \ldots, n$. Since $X$ is a function module, there is a neighborhood $U$ of $k_0$ such that $||x_i(k)|| \leq r' < r$ for $k \in U$ and $i = 1, \ldots, n$. But then, for a suitable function $h \in CK_x$ (which corresponds to $M_h \in Z(X)$) we get $\max_i ||M_h x_i|| = \max_i ||h x_i|| \leq r' ||h|| < r ||M_h||$, a contradiction.

The reverse conclusion is obvious.

In §4 we will also need a related definition, which by 3.2 is a local version of Definition 3.1.

**Definition 3.3.** $(X, K_x$ as in 3.2). Let $k_0$ be a point of $K_x$. A finite family $x_1, \ldots, x_n$ is called a local centralizer-norming system (local cns) at $k_0$, if there are a number $r > 0$ and a neighborhood $U$ of $k_0$ such that $\max_i ||x_i(k)|| \geq r$ for $k \in U$.

A simple compactness argument guarantees that $X$ has a cns iff every point in $K_x$ has a local cns.

**Example.** Let $L$ be a locally compact Hausdorff space, $X := C_0 L$. A point $k$ in $K_x = \beta L$ has a local cns iff $k \in L$. However, every point $k$ in $K_x$ has a local cns provided $X_k \neq 0$. There are known to the author only very complicated examples of Banach spaces not having this property. We will say that $X$ has the local cns property if every $k$ with $X_k \neq 0$ has a local cns.

4. The structure of $Z(X \hat{\otimes}_T Y)$. Let $X, K_x, (X_k)_{k \in K_x}, Y, K_y, (Y_l)_{l \in K_y}$ be as in §2.

**Definition 4.1.** $M(K_x \times K_y) := \{\alpha | \alpha : K_x \times K_y \to R$ a bounded function, $\alpha(k, 1) = 0$ whenever $X_k \hat{\otimes} Y_1 = 0, M_a(X \hat{\otimes} Y) \subseteq X \hat{\otimes} Y\}$. It is clear that $M(K_x \times K_y)$ is Banach algebra (with $||\alpha|| := \sup \{||\alpha(k, 1)|| | k \in K_x, 1 \in K_y\}$).

**Theorem 4.2.** (i) The mapping $\alpha \mapsto M_a$ is an isometric algebra isomorphism from $M(K_x \times K_y)$ onto $Z(X \hat{\otimes}_T Y)$ so that we may identify these two spaces.

(ii) Let $T$ be an operator in $Z(X \hat{\otimes}_T Y)$. Then $T \in (Z(X) \otimes Z(Y))^{-}$ iff there is an $\alpha \in C(K_x \times K_y)$ such that $T = M_a$. It follows that $(Z(X) \otimes Z(Y))^{-} = C(K_x \times K_y)$.

**Proof.** (i) The mapping is well-defined by 2.2. For $(k, 1) \in$
$K_X \times K_Y$ such that $X_k \boxtimes Y_l \neq 0$, $\varepsilon > 0$, there exist $x \in X$ and $y \in Y$ such that $\|x(k) \otimes y(1)\| = \|x(k)\| \cdot \|y(1)\| \geq 1 - \varepsilon$, $\|x\| \leq 1$, $\|y\| \leq 1$. This follows at once from 2.1(a), (b). Because of this fact we have $\|M_\alpha\| = \|\alpha\|$ for $\alpha \in M(K_X \times K_Y)$. The mapping $\alpha \mapsto M_\alpha$ is obviously an algebra homomorphism, and it remains to show that it is onto.

Let $T$ be an $M$-bounded operator on $X \boxtimes Y$. By [2], 4.8, every element of $E_x \otimes E_y$ is an eigenvector for $T'$. It can be shown that this is also true for every $p \otimes q$, where $(p, q) \in E_x \times E_y$. The proof of this fact depends on elementary properties of tensor products and weak*-topologies. We refer the reader to [8], p. 506. Therefore there is a function $a: E_x \times E_y \to \mathbb{R}$ such that $a(p \otimes q) = \alpha(p, q)$ for $(p, q) \in E_x \times E_y$. We claim that $a$ is separately continuous. Let $\alpha \in M(K_X \times K_Y)$. The following theorem asserts local continuity if there are local centralizer-norming systems:—

COROLLARY 4.3 (Wickstead). $Z(X) \otimes Z(Y)$ is the closure with respect to the strong operator topology of $Z(X) \otimes Z(Y)$.

Proof. This is a consequence of 4.2 and 2.2.

Because of 4.2 it is clear that in order to describe the relations between $Z(X) \otimes Z(Y)$ and $Z(X \boxtimes Y)$ when considering the norm topology we have to investigate the continuity properties of the functions $\alpha \in M(K_X \times K_Y)$. The following theorem asserts local continuity if there are local centralizer-norming systems:—
Theorem 4.4. Let \( k_0 \in K_x, l_0 \in K_Y \). If \( k_0 \) has a local eNSs \( x, \ldots, x_n \) in \( X \) and \( l_0 \) has a local eNSs \( y, \ldots, y_m \) in \( Y \), then all \( \alpha \in M(K_x \times K_Y) \) are continuous at \( (k_0, l_0) \).

**Proof.** Let \( U(\text{resp.} V) \) be a neighborhood of \( k_0(\text{resp.} l_0) \) such that \( \max \{||x_i(k)|| | i = 1, \ldots, n\} \geq r \) for \( k \in U(\text{resp.} \max \{||y_j(1)|| | j = 1, \ldots, m\} \geq \tilde{r} \) for \( 1 \in V \) where \( r \in \mathbb{R}, r > 0(\text{resp.} \tilde{r} \in \mathbb{R}, \tilde{r} > 0) \) is a suitable chosen number.

Now let \( \alpha \) be a function in \( M(K_x \times K_Y) \), \( \varepsilon > 0 \) arbitrary. For \( i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\} \) the function \( z_{ij} = \alpha(x_i \otimes y_j) - \alpha(k_0, l_0)(x_i \otimes y_j) \) is in \( X \otimes Y \) and vanishes at \( (k_0, l_0) \). Since the norm of the elements of \( X \otimes Y \) is upper semi-continuous (2.4(ii)) there are neighborhoods \( U' \) of \( k_0 \) on \( V' \) of \( l_0 \) such that

\[
||z_{ij}(k, 1)|| = ||\alpha(k, 1) - \alpha(k_0, l_0)|| \leq ||x_i(k)|| ||y_j(1)|| \leq \varepsilon r r'
\]

for \( k \in U', 1 \in V', i = 1, \ldots, n, j = 1, \ldots, m \). It follows that \( |||\alpha(k, 1) - \alpha(k_0, l_0)|| \leq \varepsilon \) for \( (k, 1) \in (U \cap U') \times (V \cap V') \).

Theorem 4.5. Let \( X \) and \( Y \) be real Banach spaces such that the norm topology and the strong operator topology are equivalent on \( Z(X) \) and \( Z(Y) \) (i.e., \( X \) and \( Y \) have a eNSs). We will identify \( Z(X) \otimes Z(Y) \) with a subspace of \( Z(X \otimes Y) \). Then the following assertions are valid:

(i) \( (Z(X) \otimes Z(Y)) = Z(X \otimes Y) \)

(ii) \( Z(X) \otimes Z(Y) = Z(X \otimes Y) \)

(iii) \( K_x \otimes K_y = K_x \times K_y \) (up to homeomorphism)

(iv) \( X \otimes Y \) has a eNSs

(more precisely: if \( x, \ldots, x_n \) is a eNSs in \( X \) and \( y, \ldots, y_m \) is a eNSs in \( Y \), then \( \{x_i \otimes y_j | i = 1, \ldots, n, j = 1, \ldots, m\} \) is a eNSs in \( X \otimes Y \).

**Proof.** (i) This is a consequence of 4.2(ii) and 4.4.

(ii) This follows from (i) since the norm of the operators in \( Z(X) \otimes Z(Y) \) is their tensor product norm.

(iii) \( C(K_x \otimes K_y) \equiv Z(X \otimes Y) \equiv Z(X \otimes Z(Y) \equiv C(K_x) \otimes C(K_y) \equiv C(K_x \times K_y) \). It follows that \( K_x \otimes K_y \) up to homeomorphism.

(iv) It is clear that \( \inf \{\max_i ||x_i(k) \otimes y_j(1)|| | (k, 1) \in K_x \times K_y \} > 0 \). As in 3.2 it follows that \( \{x_i \otimes y_j | i = 1, \ldots, n, j = 1, \ldots, m\} \) is a eNSs in \( X \otimes Y \).

Finally, we want to point out that for Banach spaces which are not too pathological the difference between \( Z(X \otimes Y) \) and \( Z(X) \otimes Z(Y) \) is just the difference between \( \beta(K_x^* \times K_y^*) \) and \( \beta K_x^* \times \beta K_y^* \)——
THEOREM 4.6. Let $X$ and $Y$ be Banach spaces having the local ene property. Then $K_{X\hat{\otimes}_s Y} = \beta(K^*_X \times K^*_Y)$.

Proof. By 4.2 and 4.4, $C(K_{X\hat{\otimes}_s Y}) \cong Z(X \hat{\otimes} Y) \cong C^h(K^*_X \times K^*_Y) \cong C(\beta(K^*_X \times K^*_Y))$. The Banach-Stone theorem implies that $K_{X\hat{\otimes}_s Y} = \beta(K^*_X \times K^*_Y)$.

REFERENCES


Received September 15, 1977 and in revised form May 5, 1978.

I. MATHEMATISCHES INSTITUT
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