CONGRUENT SECTIONS OF A CONVEX BODY

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It is shown that if all the 3-dimensional sections of a convex body $K$, of dimension at least 4, through a fixed inner point are congruent, then $K$ is a euclidean ball. A dual result concerning projections is also proved.

1. Introduction. W. Süss [8] showed that if all the plane sections of a 3-dimensional convex body passing through a fixed inner point are congruent, then the body is a euclidean ball. P. Mani [5] generalized this result to the case of congruent 2$\times$-dimensional sections of a (2$n$ + 1)-dimensional convex body. Both of these results are deduced immediately from topological proofs that a nonspherical 2$n$-dimensional body cannot be completely turned in dimension 2$n$ + 1, and the assumption that the sections fit together to form a convex body is only used to prove continuity. However, every centrally symmetric 3-dimensional body can be completely turned in 4-dimensional euclidean space $E^4$, so in this case a proof using properties of convex bodies is required; the present paper provides one. Our main results are:

THEOREM 1. Let $K$ be a convex body of dimension at least 4, let $p$ be an inner point of $K$, and suppose that all 3-dimensional sections of $K$ passing through $p$ are congruent. Then $K$ is a euclidean ball with center $p$.

THEOREM 2. Let $K$ be a convex body of dimension at least 4, and suppose all the 3-dimensional orthogonal projections of $K$ are congruent. Then $K$ is a euclidean ball.

A result which follows directly from the work of Mani is the following:

THEOREM 3. Let $n \geq 1$, let $K$ be a convex body of dimension at least 2$n$ + 1 and let $p$ be an inner point of $K$. Suppose all the 2$\times$-dimensional sections of $K$ passing through $p$ are affinely equivalent. Then $K$ is an ellipsoid.

2. Complete turnings of 3-dimensional bodies. When $A$ is a $d$-dimensional convex body, a field of bodies congruent to $A$ is a continuous function $A(u)$ defined for $u$ in the unit sphere $S^d$, where $A(u)$ is a congruent copy of $A$ lying in a hyperplane of $E^{d+1}$ perpendicular to $u$; here $A(u)$ is meant to be continuous in the Hausdorff
metric. If additionally $A(u) = A(-u)$ for each $u$, we say $A(u)$ is a complete turning of $A$ in $E^{d+1}$. Clearly if all the $d$-dimensional sections of a $(d+1)$-dimensional convex body through a fixed inner point are congruent, they give rise to a complete turning of some $d$-dimensional body in $E^{d+1}$. We make use of the methods of Mani [5] and H. Hadwiger [4] to determine which 3-dimensional convex bodies can be completely turned in $E^4$. When $v$ is a fixed unit vector in $E^4$ and for $u = (t_1, t_2, t_3, t_4) \in S^3$ we define $p_1(u) = (-t_2, t_1, -t_4, t_3)$, $p_2(u) = (t_2, -t_1, -t_4, t_3)$, $p_3(u) = (-t_4, -t_3, t_2, t_1)$, then let $\Psi_u$ be the orthogonal transformation such that $\Psi_u(v) = u$ and $\Psi_u(p_i(v)) = p_i(u)$ for $i = 1, 2, 3$. Notice that $\Psi_{-u} = -\Psi_u$.

**Lemma 2.1.** Let $A$ be a 3-dimensional convex body whose symmetry group is finite, and suppose $A$ can be completely turned in $E^4$. Then $A$ is centrally symmetric.

**Proof.** Let $A(u)$ be a complete turning of $A$ in $E^4$. We may assume that each $A(u)$ has its centroid at the origin $o$, and that $A = A(v)$ for some unit vector $v$. Let $\Psi_u$ be defined as above. Since $A(u)$ is a field of bodies congruent to $A$, the proof of Proposition 2 in [5] shows the existence of orthogonal transformations $\Phi_u$ depending continuously on $u$ with $\Phi_u(A) = A(u)$. The restriction $\Phi^{-1}_u \Phi_u |_A$ is a continuously varying symmetry of $A$, and by connectedness it must be a constant $\Theta$.

The map $\Psi^{-1}_u \Phi_u$ preserves the linear span of $A$, so consider $\Psi^{-1}_u \Phi_u(v)$ for a fixed $v \in A$. The mapping $u \mapsto \Psi^{-1}_u \Phi_u(v)$ maps $S^3$ continuously into a copy of $E^3$, so by the Borsuk-Ulam theorem (see [7], p. 266) it maps some pair of antipodal points into coincidence. Thus for some $u$ we have

$$\Psi^{-1}_u \Phi_u(v) = \Psi^{-1}_u \Phi_u(v)$$

and since $\Psi_{-u} = -\Psi_u$ this yields

$$-\Phi_{-u}(v) = \Phi_u(v)$$

and so $-v = \Phi_{-u} \Phi_u(v) = \Theta(v)$. It follows that $\Theta$ is a central reflection, and $A$ is centrally symmetric.

**Lemma 2.2.** Let $A$ be a 3-dimensional convex body whose symmetry group is infinite, and suppose $A$ can be completely turned in $E^4$. Then $A$ is centrally symmetric.

**Proof.** Let $A(u)$ be a complete turning of $A$. We may assume that each $A(u)$ has its centroid at the origin, and that $A = A(v)$ where $v$ is a unit vector. Let $\Psi_u$ be the map defined above. Since
A has an infinite symmetry group, it has an axis of revolution; let such an axis be parallel to the unit vector \( w \).

Suppose that \( A \) is not centrally symmetric, so that \( A \) has only one axis of revolution, and for some \( \lambda > 0 \) the two sections

\[ \{ x \in A : x \cdot w = \pm \lambda \} \]

are discs of different radii. Any symmetry of \( A \) maps the axis onto itself, and maps \( \lambda w \) onto \( \lambda w \) also.

It follows that for each \( u \in S^3 \), there is a unit vector \( w(u) \) in the linear span of \( A(u) \) such that \( \Phi(w) = w(u) \) for every orthogonal transformation \( \Phi \) with \( \Phi(A) = A(u) \). Hence \( w(u) \) is a continuous function of \( u \) and \( w(-u) = w(u) \). The mapping \( u \mapsto \varphi_u^{-1}(w(u)) \) is a continuous map of \( S^3 \) into a copy of \( E^3 \), so by the Borsuk-Ulam theorem, for some \( u \) we have

\[ \varphi_u^{-1}(w(u)) = \varphi_u^{-1}(w(-u)) = -\varphi_u^{-1}(w(u)) \]

so that \( w(u) = -w(u) \) which is impossible. We conclude that \( A \) is centrally symmetric.

**REMARKS.** Lemmas 2.1 and 2.2 show that any 3-dimensional convex body which can be completely turned in \( E^4 \) is centrally symmetric. Conversely, the map \( \varphi_u \) allows every 3-dimensional centrally symmetric convex body to be completely turned in \( E^4 \).

3. Congruent central sections of a convex body. Throughout this section \( K \) will be a fixed 4-dimensional convex body in \( E^4 \) having the origin as center of symmetry, and such that all the 3-dimensional central sections of \( K \) are congruent. We assume \( K \) is not a euclidean ball, and seek a contradiction. For nonzero \( u \) and \( v \) the hyperplane \( \{ x \in E^4 : x \cdot u = 0 \} \) is denoted \( H(u) \), the orthogonal projection on \( H(u) \) is denoted \( \pi_u \) and \( \Phi_{u,v} \) is some orthogonal transformation which maps \( H(u) \cap K \) onto \( H(v) \cap K \); clearly the choice of \( \Phi_{u,v} \) may not be unique.

**LEMMA 3.1.** Let \( v \in S^3 \). Then the section \( H(v) \cap K \) is not a body of revolution.

**Proof.** Suppose the lemma is false. Then since \( H(v) \cap K \) is not a euclidean ball, it has just one axis of rotation \( l \). Consider a plane \( A \) with \( l \subset A \subset H(v) \). For any \( u^* \in X = S^3 \cap A^\perp \), there is a neighborhood of \( u^* \) in \( X \) in which \( \Phi_{u,v} \) can be chosen as a continuous function of \( u \). Let \( X_0 \) be a compact simple arc of \( X \) containing \( v \) in its interior. By compactness \( X_0 \) can be dissected into a finite collection of interior-disjoint arcs, on each of which \( \Phi_{u,v} \) is chosen continuously; if this
gives rise to two choices $\Phi'_u, v$ and $\Phi''_u, v$ of $\Phi_{u,v}$ at a common end $u$ of two such arcs, then $\Phi''_u, \Phi''_u^{-1}$ preserves $H(v) \cap K$, so by composing $\Phi''_u, v$ with a suitable orthogonal transformation we can suppose $\Phi''_u, v = \Phi'_u, v$. Hence we can choose $\Phi_{u,v}$ continuously for $u \in X_0$.

We claim $\Phi_{u,v}(\Lambda)$ contains $I$ for every $u \in X$. Suppose this is false, and let $x \in I \cap \partial K$. Then as $u$ varies on $X$, a nontrivial arc on a sphere is described by $\Phi_{u,v}(x)$, so $H(v) \cap \partial K$ contains a maximal spherical cap $A$ with pole $x$ and at constant distance from $o$. Let $y$ and $z$ be the points of $A$ on the perimeter of $A$. Then for each $u \in X_0$, the points $\Phi_{u,v}(y)$ and $\Phi_{u,v}(z)$ lie within $cl A$ and $||\Phi_{u,v}(y) - \Phi_{u,v}(z)|| = ||y - z||$, so let $\Lambda$, $\Phi_{u,v}(y)$ and $\Phi_{u,v}(z)$ are coplanar. This contradiction shows that $\Phi_{u,v}(\Lambda)$ contains $I$ for each $u \in X_0$.

By composing $\Phi_{u,v}$ with a suitable continuously varying orthogonal transformation that acts as a symmetry on $H(v) \cap K$ we can suppose $\Phi_{u,v}(\Lambda) = A$ for each $u \in X_0$ and $\Phi_{u,v}$ is the identity so $\Phi_{u,v}(u) = v$. Since the symmetry group of $\Lambda \cap K$ is finite, $\Phi_{u,v}|\Lambda$ is the identity for all $u \in X_0$. Thus $l$ is the axis of $H(u) \cap K$ for all $u \in X_0$, and hence (by letting $X_0$ tend to $X$) for all $u \in X$. Then for any $s \in l \cap \partial K$, the length $||s||$ is equal to the radius of the central section of $H(v) \cap K$ perpendicular to $l$. It follows that $l \cap K$ is a euclidean ball and so $K$ is a euclidean ball contrary to hypothesis. This proves the lemma.

REMARKS. From Lemma 3.1 it follows that each $H(u) \cap K$ has only a finite symmetry group. It follows from the proof of Proposition 2 in [5] that for fixed $v \in S^3$ we can choose $\Phi_{u,v}$ as a continuous function of $u \in S^3$. We can further suppose $\Phi_{u,v}$ is the identity so $\Phi_{u,v}(u) = v$. When $u$ and $v$ are not unit vectors, we define $\Phi_{u,v} = \Phi_{u',v'}$, where $u' = ||u||^{-1}u$, $v' = ||v||^{-1}v$.

LEMMA 3.2. $K$ is smooth.

Proof. Let $K^*$ be the polar reciprocal of $K$ relative to the origin. Then $\Phi_{u,v}(\pi_u K^*) = \pi_v K^*$ for each $u, v \in S^3$. To prove $K^*$ is smooth, it will suffice to show $K^*$ is strictly convex. In the ensuing argument, faces are meant to be exposed faces.

Suppose first that $K^*$ has a 2-face $F$, and let $F$ be the face of $K^*$ in the direction of $w \in S^3$. Fix a unit vector $v$ perpendicular to $w$ and the affine hull $aff F$. Then $\pi_w F$ is a 2-face of $\pi_u K^*$ for every $u$ perpendicular to $w$ and close to $v$, and by continuity $\Phi_{u,v}(\pi_u F) = \pi_w F$. However, if $u$ is chosen perpendicular to $w$ but not perpendicular to $aff F$, then $\pi_w F$ has smaller area than $\pi_u F$. This contradiction shows that $K^*$ has no 2-faces.

Next suppose that $K^*$ has 3-faces, and consider any 3-face $G$,
having an outer unit normal $m$ say at its centroid. If $u$ is any unit
vector perpendicular to $m$ then $\pi_u G$ is a 2-face of $\pi_u K^*$. Conversely,
suppose $J$ is a 2-face of a projection $\pi_u K^*$. Then there is a face
$G'$ of $K^*$ such that $\pi_u G' = J$. We necessarily have $\dim G' \geq \dim J$, and
since $K^*$ has no 2-faces, $G'$ must be a 3-face. Hence $w$ is
perpendicular to the normal of $K^*$ at the centroid of $G'$. Since the
facets of $K^*$ form a countable set, $\pi_w (K^*)$ can only have a 2-face
when $w$ lies in a certain countable union of hyperplanes. This is
impossible since all the 3-dimensional orthogonal projections of $K^*$
are congruent. We conclude that $K^*$ has no 3-faces.

Finally suppose $K^*$ has an edge $L$, with ends $x$ and $x + \lambda t$
where $\lambda > 0$ and $t$ is a unit vector. Let $L$ be the face of $K^*$
in the direction of the unit vector $p$, let $\Theta$ be the plane through $o$
orthogonal to $p$ and $t$, and let $v$ be a unit vector in $\Theta$. For each
$u \in \Theta \cap S^3$ the line segment $L(u) = \phi_{u,v}(\pi_u L)$ is an edge of $\pi_v K^*$ and
has length $\lambda$; we claim that $L(u)$ is the same edge for every $u \in 
\Theta \cap S^3$. Suppose this is false; then by continuity the region $\cup \{L(u):
 u \in \Theta \cap S^3\}$ contains on open neighborhood $N$ in the relative boundary
of $\pi_v K^*$. Choose $u \in \Theta \cap S^3$ such that $L(u)$ intersects $N$. For every
unit vector $w$ orthogonal to $p$ and close to $u$, the segment $L(w) = 
\phi_{u,v}(\pi_w L)$ is an edge of $\pi_v K^*$ that intersects $N$, so $L(w) = L(u')$ for
some $u' \in \Theta \cap S^3$. Hence $L(w)$ has length $\lambda$. But we can choose
$w$ not to be orthogonal to $t$, in which case $L(w)$ is shorter than $L$.
This contradiction shows that $L(u)$ is the same edge for all $u \in \Theta \cap S^3$.

It follows that $\Phi_{u,v}(\pi_u x) = \pi_v(x)$ and $\Phi_{u,v}(\pi_u (x + \lambda t)) = \pi_v(x + \lambda t)$
for all $u \in \Theta \cap S^3$, and since $\pi_u$ and $\pi_v$ fix $t$ we find that $\Phi_{u,v}(t) = t$.
Further $\pi_u (p) = \pi_v (p) = p$ so $\Phi_{u,v}(p) = p$, and it follows that $\Phi_{u,v}$ fixes all
points of $\Theta \perp$ for $u \in \Theta \cap S^3$. Hence all sections of $K$ parallel to $\Theta$ are
circular and have centers on $\Theta \perp$. It follows that $K$ has 3-dimensional
central sections which are bodies of revolution, contrary to Lemma
3.1. We conclude that $K^*$ is strictly convex, so $K$ is smooth.

DEFINITION. An open neighborhood $A$ on the relative boundary
of a section $H(v) \cap K$ is said to be contoured if the intersection of
$A$ with every sphere with center $o$ is empty or a circular arc.

LEMMA 3.3. Let $x$ be a boundary point of $K$ at which the unit
outward normal $n$ is not a multiple of $x$, let $v$ be a unit vector
perpendicular to $x$, and suppose relbd $H(v) \cap K$ contains no contoured
neighborhoods. Then $\Phi_{u,v}$ is a differentiable function of $u$ for $u$
close to $v$.

Proof. Choose a neighborhood $A$ of $x$ in the boundary of $K$
such that at no point of $A$ is the normal direction to $K$ parallel to
the radius vector. We show $A$ contains a neighborhood $B \subset relbd H(v) \cap K$ so that at no point of $B$ is the outward normal to $H(v) \cap K$ parallel to the radius vector. Suppose this is false so by continuity of the normal directions, the normal to $H(v) \cap K$ at each point of $H(v) \cap A$ is parallel to the radius vector. Hence $H(v) \cap A$ is a subset of a 3-sphere $S$ with center $o$. For $u \in S^3$ we have $\Phi_u(H(v) \cap A) \subset S$, and the regions $\Phi_u(H(v) \cap A)$ cover a neighborhood of $x$ in $bdK$. Thus $x$ is parallel to $n$ contrary to hypothesis. We deduce the existence of $B$ as required.

It now follows from the Implicit Function theorem that each set $C(\alpha) = \{y \in B: ||y|| = \alpha\}$ is a union of simple continuously differentiable arcs if it is nonempty. We may suppose $B$ is chosen so that each $C(\alpha)$ is connected. Consider two curves $C(\alpha)$ and $C(\beta)$ with $\alpha \neq \beta$, and let $a_0 \in C(\alpha)$ and $b_0 \in C(\beta)$ be two points for which $a_0 - b_0$ is not perpendicular to the tangent line of $C(\beta)$ at $b_0$. We can continuously differentiably select $f_{\alpha, \beta}(\lambda, a) \in C(\beta)$ with $||f_{\alpha, \beta}(\lambda, a) - a|| = \lambda$ for $a \in C(\alpha)$ close to $a_0$ and $\lambda$ close to $||a_0 - b_0||$, such that $f_{\alpha, \beta}(||a_0 - b_0||, a_0) = b_0$.

Let us suppose there exist open neighborhoods $M, N$ in $B$ such that for each $\alpha \neq \beta$, each $a_0 \in C(\alpha) \cap M$ and each $b_0 \in C(\beta) \cap N$ with $a_0 - b_0$ not perpendicular to the tangent line of $C(\beta)$ at $b_0$, we have

\[(*) \quad D_2||f_{\alpha, \beta}(\lambda, a) - f_{\alpha, \beta}(\mu, a)|| = 0\]

for all $\lambda$ and $\mu$ close to $||a_0 - b_0||$ and $a$ on $C(\alpha)$ close to $a_0$. Additionally we may suppose that each $C(\alpha)$ intersects $M$ and $N$ in (connected, but possibly empty) arcs.

Consider $a_0 \in M$ with $||a_0|| = \alpha$. Suppose $N$ contains a neighborhood $P$ such that each $b \in P$ satisfies $||b|| \neq \alpha$ and $b - a_0$ is perpendicular to the tangent line of $C(\beta)$ at $b$. We can suppose the intersection of $P$ with each $C(\beta)$ is connected, so that each $C(\beta)$ which intersects $P$ is at constant distance from $a_0$; thus each such $C(\beta)$ is a circular arc, being in the intersection of two spheres. Hence $P$ is a contoured neighborhood contrary to hypothesis. Thus for the given $a_0$, for a dense set of $b_0$ in $N$ we have $a_0 - b_0$ not perpendicular to the tangent line of $C(\beta)$ at $b_0$ and $D_2||f_{a, \beta}(\lambda, a) - f_{a, \beta}(\mu, a)|| = 0$ for all $a$ on $C(\alpha)$ close to $a_0$ and $\lambda, \mu$ close to $||a_0 - b_0||$ where $\beta = ||b_0||$. Consider such a $b_0$, which we can suppose chosen so that $a_0 - b_0$ is not perpendicular to the tangent line of $C(\alpha)$ at $a_0$, let $\lambda_0 = ||a_0 - b_0||$, and suppose $D_2||f_{a, \beta}(\lambda, a) - f_{a, \beta}(\mu, a)|| = 0$ for all $\lambda$ and $\mu$ in an interval $J$ with center $\lambda_0$ and all $a$ in an arc $F$ of $C(\alpha)$ surrounding $a_0$.

Then $||f_{a, \beta}(\lambda, a) - f_{a, \beta}(\mu, a)||$ is a function only of $\lambda$ and $\mu$ for $\lambda, \mu \in J$, $a \in F$. For fixed $\lambda, \mu \in J$, the triangles $\{a, f_{a, \beta}(\lambda, a), f_{a, \beta}(\mu, a)\}$ are then all congruent for $a \in F$. Letting $\mu$ tend to $\lambda$, the angle
between the tangent line to \( C(\beta) \) at \( f_{\alpha,\beta}(\lambda, a) \) and the vector \( f_{\alpha,\beta}(\lambda, a) - a \) is a function of \( \lambda \) only, say \( \rho(\lambda) \), for \( \lambda \in J \) and \( a \in F \). We can suppose \( F \) and \( J \) are so short that \( f_{\beta,\alpha}(\mu, f_{\alpha,\beta}(\lambda, a)) \) is defined for \( \lambda, \mu \in J, a \in F \).

Consider \( a \) and \( a_0 \) in the interior of \( F \), let \( b_i = f_{\alpha,\beta}(\lambda_0, a_i) \) and let \( g_i(\lambda) = f_{\beta,\alpha}(\lambda, b_i) \in C(\alpha) \) for \( i = 1, 2 \). We can choose an open interval \( J' \) with \( \lambda_0 \in J' \subset J \) which is so short that \( g_i(\lambda) \in F \) for all \( \lambda \in J', i = 1, 2 \). Then \( f_{\alpha,\beta}(\lambda, g_i(\lambda)) = b_i \); choose unit vectors \( t_i \) parallel to the tangent lines of \( C(\beta) \) at \( b_i \) so that \( (g_i(\lambda) - b_i) \cdot t_i = \lambda \cos \rho(\lambda) \).

There is an orthogonal transformation \( \Psi \) in \( H(v) \) with \( \Psi(b_1) = b_2 \), \( \Psi(t_i) = t_2 \) and \( \Psi(a_i) = a_2 \). The continuously varying points \( g_i(\lambda) \) satisfy:

\[
\|g_i(\lambda)\| = \|\Psi g_i(\lambda)\| = \alpha \\
\|g_i(\lambda) - b_i\| = \|\Psi g_i(\lambda) - b_i\| = \lambda \\
(g_2(\lambda) - b_2) \cdot t_2 = (\Psi g_i(\lambda) - b_2) \cdot t_2 = \lambda \cos \rho(\lambda)
\]

and these conditions ensure \( \Psi g_i(\lambda) = g_2(\lambda) \) for all \( \lambda \in J' \). Thus \( \Psi \) maps \( a_1 \) onto \( a_2 \) and maps a neighborhood of \( a_1 \) in \( C(\alpha) \) onto a neighborhood of \( a_2 \) in \( C(\alpha) \). If \( F \) contains in its interior a point of 2-fold differentiability of \( C(\alpha) \), then \( F \) has constant curvature, and since it lies on a sphere it must be an arc of a circle.

Since relbd \( H(v) \cap K \) is twice differentiable almost everywhere, \( C(\alpha) \cap M \) has a point of two-fold differentiability for a dense set of \( \alpha \). If \( C(\alpha) \cap M \) is twice differentiable somewhere, the above arguments show it contains a circular arc; choose a maximal such arc \( C \). Then the above arguments apply taking \( a_0 \) as an end of \( C \), and this contradicts the maximality of \( C \) unless \( C = C(\alpha) \cap M \). We conclude that \( C(\alpha) \cap M \) is a circular arc for a dense set of \( \alpha \); by taking limits \( M \) is contoured contrary to hypothesis.

It follows that our supposition (*) is false. Thus for a dense of \( (a_0, b_0) \) in \( B \times B \), for \( \alpha = \|a_0\| \) and \( \beta = \|b_0\| \) we find that the tangent line of \( C(\beta) \) at \( b_0 \) is not perpendicular to \( a_0 - b_0 \) and \( D_2[f_{\alpha,\beta}(\lambda, a) - f_{\alpha,\beta}(\mu, a)] \neq 0 \) for \( (\lambda, \mu, a) \) arbitrarily close to \( (\lambda_0, \lambda_0, a_0) \) in \( R \times R \times C(\alpha) \) where \( \lambda_0 = \|a_0 - b_0\| \).

We can therefore choose \( \lambda, \mu, \nu, a_0, b_0, c_0 \) with \( \|a_0\| = \alpha, \|b_0\| = \beta, b_0 = f_{\alpha,\beta}(\lambda, a_0), c_0 = f_{\alpha,\beta}(\mu, a_0), \nu = \|b_0 - c_0\| \), such that the tangent lines of \( C(\alpha) \) at \( b_0 \) and \( c_0 \) are not parallel to \( b_0 - a_0 \) and \( c_0 - a_0 \), respectively, \( D_2[f_{\alpha,\beta}(\lambda, a_0) - f_{\alpha,\beta}(\mu, a_0)] \neq 0 \), and by choosing \( \lambda, \mu \) and \( \nu \) small with \( b_0 - a_0 \) not too nearly parallel to the tangent line of \( C(\beta) \) at \( b_0 \) we can also ensure that \( a_0, b_0, c_0 \) is linearly independent.

We can write \( K = \{y: h(y) \leq 1\} \) where \( h \) is a positive-homogeneous continuously differentiable convex function. Regarding points of \( E^3 \) as column matrices, for points \( a, b, c, u \neq 0 \) define
\[
F \begin{bmatrix}
a \\
b \\
c \\
u
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2}||a||^2 \\
h(a) \\
\frac{1}{2}||b||^2 \\
h(b) \\
\frac{1}{2}||c||^2 \\
h(c) \\
\frac{1}{2}||a-b||^2 \\
\frac{1}{2}||b-c||^2 \\
\frac{1}{2}||c-a||^2 \\
u \cdot a \\
u \cdot b \\
u \cdot c
\end{bmatrix}
\]

so that
\[
DF \begin{bmatrix}
a \\
b \\
c \\
u
\end{bmatrix} = \begin{bmatrix}
a^T \\
\nabla h(a) \\
b^T \\
\nabla h(b) \\
c^T \\
\nabla h(c) \\
\n-a^T - b^T - a^T \\
\n-b^T - c^T - b^T \\
\n-c^T - a^T \\
u^T \\
u^T \\
u^T
\end{bmatrix}
\]

where \( \nabla h \) is the gradient of \( h \); notice that if \( y \) is a boundary point of \( K \) then \( \nabla h(y) \) is a nonzero multiple of the unit normal to \( K \) at \( y \). We will show that

\[
(1) \quad \text{rank } D_{abc} F = 12.
\]

To this end define \( m(x) \) to be the orthogonal projection of \( \nabla h(x)^T \) on \( H(v) \), and let

\[
Q' = \begin{bmatrix}
a_0^T \\
m^T(a_0) \\
b_0^T \\
m^T(b_0) \\
c_0^T \\
m^T(c_0) \\
a_0^T - b_0^T - a_0^T \\
b_0^T - c_0^T - b_0^T \\
a_0^T - c_0^T \\
c_0^T - a_0^T
\end{bmatrix}.
\]
We first prove $\text{rank } Q' = 9$.

Let $s^*, t^*, w^*$ be unit vectors parallel to the tangent lines of $C(\alpha)$ at $a_0$, of $C(\beta)$ at $b_0$, and of $C(\beta)$ at $c_0$ respectively.

Suppose that there are points $s, t, w \in H(v)$ such that

$$Q' \begin{bmatrix} s \\ t \\ w \end{bmatrix} = 0 .$$

Then $a_0 \cdot s = 0$ and $m(a_0) \cdot s = 0$ which ensures that $s$ is a multiple of $s^*$. Similarly $t$ and $w$ are multiples of $t^*$ and $w^*$ respectively. By choice of $a_0$, $b_0$, and $c_0$ we have

$$(a_0 - b_0) \cdot t^* \neq 0 , \quad (a_0 - c_0) \cdot w^* \neq 0$$

and this ensures that the equations

(2) $$(a_0 - b_0) \cdot (\sigma s^* - \tau t^*) = 0$$

(3) $$(a_0 - c_0) \cdot (\sigma s^* - \omega w^*) = 0$$

have a one-dimensional space of solutions $(\sigma, \tau, \omega)$. We can choose numbers $\tau^*$ and $\omega^*$ such that

$$\tau^* t^* = D_2 f_{\alpha, \beta}(\lambda, a_0)$$

$$\omega^* w^* = D_2 f_{\alpha, \beta}(\mu, a_0) ;$$

if we take $\sigma^* = 1$ then $(\sigma^*, \tau^*, \omega^*)$ is a solution of (2) and (3) since $||f_{\alpha, \beta}(\lambda, a) - a|| = \lambda$ and $||f_{\alpha, \beta}(\mu, a) - a|| = \mu$ for $a$ on $C(\alpha)$ close to $a_0$. Also if $\chi$ is the projection on the 8th coordinate of $R^8$ we have

$$\chi Q' \begin{bmatrix} \sigma^* s^* \\ \tau^* t^* \\ \omega^* w^* \end{bmatrix} = \frac{1}{2} D_2 ||f_{\alpha, \beta}(\lambda, a_0) - f_{\alpha, \beta}(\mu, a_0)||^2 \neq 0 .$$

Thus

$$Q' \begin{bmatrix} \sigma s^* \\ \tau t^* \\ \omega w^* \end{bmatrix} = 0$$

implies $\sigma = \tau = \omega = 0$ ,

which shows that $\text{rank } Q' = 9$.

Suppose $p, q,$ and $r$ are vectors in $E^4$ for which

(4) $$D_{abc} F' \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ v \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = 0 .$$
By considering the last 3 components in (4) we find that $v \cdot p = v \cdot q = v \cdot r = 0$, so if coordinates are chosen such that $v$ is on the $x_i$ axis, we have $p = (p', 0)$, $q = (q', 0)$, $r = (r', 0)$. Also the 4th, 8th and 12th columns of $Q'$ are zero, (4) show that

$$Q' \begin{bmatrix} p \\ q \\ r \end{bmatrix} = 0$$

and since $\text{rank } Q' = 9$ it follows that $p' = q' = r' = 0$. Hence $p = q = r = 0$ which proves (1). Now it follows from the Implicit Function theorem that in a certain neighborhood of $(a_0, b_0, c_0)$, for each $u$ close to $v$ the equation

$$F' \begin{bmatrix} a \\ b \\ c \\ u \end{bmatrix} = F' \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ v \end{bmatrix}$$

has a unique solution $(a, b, c)$, and $a$, $b$, and $c$ are differentiable functions of $u$. Roughly, we can say that no tetrahedron close to $(o, a, b, c)$ with $o$ as a vertex and 3 vertices on $H(u) \cap bdK$ is congruent to $(o, a, b, c)$. It follows that $\Phi_{a,v}(a) = a_0$, $\Phi_{a,v}(b) = b_0$ and $\Phi_{a,v}(c) = c_0$. Thus $\Phi_{a,v}$ is a differentiable function of $u$ near $v$.

**Lemma 3.4.** Some 3-dimensional central section of $K$ has a contoured neighborhood on its relative boundary.

**Proof.** Suppose the lemma is false. Since $K$ is assumed not to be a euclidean ball, there is a point $x$ on the boundary of $K$ at which the unit outward normal vector $n$ is not parallel to $x$. Let $v$ be the unit vector perpendicular to $x$ and which is coplanar with $n$ and $x$ having $n \cdot v > 0$. Then $\Phi_{a,v}$ is a differentiable function of $u$ by Lemma 3.3 for $u$ close to $v$. For real $\theta$ let $u = u(\theta) = -\theta x + v$, let $y = y(\theta) = \Phi_{a,v}(x)$ and let $f = y'(0)$. We have $y(0) = x$ and $y(\theta) \cdot u(\theta) = 0$. Since $||y(\theta)||$ is constant we have $y \cdot y' = 0$ so $x \cdot f = 0$. Thus

$$(x + f\theta + o(\theta)) \cdot (-\theta x + v) = 0$$

whence

$$-\theta x \cdot x + x \cdot v - \theta^2 f \cdot x + \theta f \cdot v = o(\theta)$$

so that
\[-x \cdot x + f \cdot v = o(1)\]
as \(\theta \to 0\). It follows that \(f \cdot v = x \cdot x > 0\).

We can write \(n = \alpha x + \beta v\) where \(\beta = n \cdot v > 0\), and then

\[
\begin{align*}
n \cdot y - n \cdot x &= n \cdot (y - x) = (\alpha x + \beta v) \cdot (\theta f + o(\theta)) \\
&= \theta \alpha x \cdot f + \theta \beta v \cdot f + o(\theta) \\
&= \theta \beta v \cdot f + o(\theta)
\end{align*}
\]

which is positive for small positive \(\theta\). This is impossible since \(n \cdot x \geq n \cdot z\) for all \(z \in K\). We conclude that some \(H(w) \cap K\) has a contoured neighborhood on its relative boundary.

**Lemma 3.5.** No 3-dimensional central section of \(K\) has a contoured neighborhood on its relative boundary. Our assumption that \(K\) is not a euclidean ball is therefore untenable.

**Proof.** Suppose \(v\) is a unit vector and that \(relbd H(v) \cap K\) contains a contoured neighborhood \(A\). Define \(C(\alpha) = \{x \in A: \|x\| = \alpha\}\). First consider the possibility that all of the circular arcs \(C(\alpha)\) are parallel to a certain plane \(A\) through \(o\) in \(H(v)\). Let \(\Theta\) be a plane through \(o\) in \(H(v)\) which intersects \(A\) and which makes a positive angle \(\gamma\) with \(A\). Then \(\Theta \cap K\) is not circular, for then \(A\) would contain a spherical region which is impossible since \(A\) is contoured. The symmetry group of \(\Theta \cap K\) is therefore finite.

Suppose that \(\Phi_{u,v}(\Theta) = \Theta\) for every \(u \in \Theta^\perp \cap S^3\); then \(\Phi_{u,v}\) would be a continuously varying symmetry of \(\Theta \cap K\), and since \(\Phi_{u,v}\) is the identity we find \(\Phi_{u,v}\) is the identity for all \(u \in \Theta^\perp \cap S^3\). It follows that every section of \(K\) parallel to \(\Theta^\perp\) is circular with center on \(\Theta\). Hence some 3-dimensional central sections of \(K\) are bodies of revolution, contrary to Lemma 3.1.

Therefore there exists some \(u\) such that \(\Phi_{u,v}(\Theta) \neq \Theta\). Choose distinct numbers \(\alpha\) and \(\beta\) such that \(C(\alpha)\) and \(C(\beta)\) both intersect \(\Theta\). There is arc \(\Gamma\) of \(\Theta^\perp \cap S^3\) which has \(v\) as one end, such that \(\Phi_{u,v}(\Theta)\) intersects \(C(\alpha)\) and \(C(\beta)\) for every \(u \in \Gamma\) but \(\Phi_{u,v}(\Theta) \neq \Theta\) for some \(u \in \Gamma\). For all \(u \in \Gamma\) we have \(\Phi_{u,v}(C(\alpha) \cap \Theta) = C(\alpha) \cap \Phi_{u,v}(\Theta)\) and \(\Phi_{u,v}(C(\beta) \cap \Theta) = C(\beta) \cap \Phi_{u,v}(\Theta)\), so \(\Phi_{u,v}(\Theta)\) makes an angle \(\gamma\) with \(A\). Hence for every \(x\) in \(\Theta \cap bdK\), the arc \(\{\Phi_{u,v}(x): u \in \Gamma\}\) is a compact circular arc in \(H(v) \cap bdK\), is parallel to \(A\) and has its center on the the line \(l\) in \(H(v)\) through \(o\) perpendicular to \(A\). By taking various values of \(\gamma\), it follows that for any plane \(A'\) in \(H(v)\) parallel to \(A\) but distinct from \(A\), the closed curve \(A' \cap bdK\) is a union of compact circular arcs centered on \(l\). We can express \(A' \cap bdK\) as the union of a countable collection \(\mathcal{F}\) of interior-disjoint maximal compact circular arcs with centers on \(l\). The end-points of the arcs in \(\mathcal{F}\) form a
compact countable set $\mathcal{E}$. If $\mathcal{E}$ is nonempty, it follows from the Baire Category theorem that some point of $\mathcal{E}$ is isolated; such an isolated point is a common end-point of two members of $\mathcal{E}$, which cannot exist. We conclude that $\mathcal{E}$ is empty so that $A' \cap \text{bd} K$ is a circle with its center on $l$. It follows that $H(v) \cap K$ is a body of revolution contrary to Lemma 3.1.

We may therefore assume that not all of the arcs $C(\alpha)$ are parallel to one plane. We can then choose distinct numbers $\alpha$ and $\beta$ and a plane $\Lambda$ through $o$ in $H(v)$ such that $\Lambda$ intersects each of $C(\alpha)$ and $C(\beta)$ in two points, and $C(\alpha)$ is not in a plane parallel to the plane of $C(\beta)$. For no plane $\Lambda'$ through $o$ in $H(v)$ close to $\Lambda$ are the configurations $(o, \Lambda \cap C(\alpha), \Lambda \cap C(\beta))$ and $(o, \Lambda' \cap C(\alpha), \Lambda' \cap C(\beta))$ congruent, so it follows that $\Phi_{u, v}(\Lambda) = \Lambda$ for all $u \in \Lambda^2 \cap S^3$. Further, $\Lambda \cap K$ is not circular so $\Phi_{u, v, t, \Lambda}$ is the identity for all $u \in \Lambda^2 \cap S^3$. It follows as in the case considered above that $K$ has 3-dimensional central sections which are bodies of revolution contrary to Lemma 3.1.

Lemma 3.5 contradicts Lemma 3.4, so we conclude that $K$ is a euclidean ball.

We have now proved:

**Proposition.** If $K$ is a centrally symmetric 4-dimensional convex body and all the 3-dimensional central sections of $K$ are congruent, then $K$ is a euclidean ball.

4. Proof of the theorems.

**Proof of Theorem 1.** Let $d$ denote the dimension of $K$, and consider first the case when $d = 4$. For $u \in S^3$ let $A(u)$ be the section of $K$ through $p$ which is perpendicular to the direction $u$. Then $A(u)$ is a complete turning of some 3-dimensional body $A$ in $E^t$, so by Lemmas 2.1 and 2.2, $A$ is centrally symmetric. Hence $A(u)$ is centrally symmetric for each $u \in S^3$. Consider an orthogonal projection $K_0$ of $K$ on a 3-flat through $p$. Then every 2-dimensional section of $K_0$ through $p$ is a projection of a 3-dimensional section of $K$ through $p$. Thus all 2-dimensional sections of $K_0$ through $p$ are centrally symmetric, and it follows from a result of Rogers [6] that $K_0$ is centrally symmetric. Every 2-dimensional orthogonal projection of $K$ is a projection of some 3-dimensional projection, and so is centrally symmetric. It follows from another result of Rogers [6] that $K$ is centrally symmetric.

If $p$ is the center of $K$, it follows immediately from the Proposition above that $K$ is a euclidean ball with center $p$. Suppose therefore that the center of $K$ is $a \neq p$, and consider a 3-dimensional
orthogonal projection $\pi$ with $\pi(a) \neq \pi(p)$. As we have seen above, every 2-dimensional section of $\pi(K)$ through $\pi(p)$ is centrally symmetric, but $\pi(a)$ is the center of $\pi(K)$. It follows from the False Center theorem of Aitchison, Petty and Rogers [1] that $\pi(K)$ is an ellipsoid. Since $\pi(a) \neq \pi(p)$ for almost all projections $\pi$, by taking limits we find that every 3-dimensional projection of $K$ is an ellipsoid, so $K$ is an ellipsoid by the dual of a result of Busemann [2, p. 91]. The 3-dimensional central sections of $K$ are all similar, and it is easily shown that $K$ must therefore be a euclidean ball. Since the 3-dimensional sections of $K$ through $p$ are all congruent, $p$ must be the center of $K$.

In the case $d > 4$, it follows from the 4-dimensional case considered above that every 4-dimensional section of $K$ through $p$ is a euclidean ball with center $p$, so $K$ is a euclidean ball with center $p$.

**Proof of Theorem 2.** We may assume that the centroid of $K$ is $o$. Consider an orthogonal projection $K_o$ of $K$ on a 4-flat through $o$. The 3-dimensional orthogonal projections of $K_o$ are all orthogonal projections of $K$ and are therefore congruent. So the 3-dimensional orthogonal projections of $K_o$ give rise to a complete turning of some 3-dimensional convex body in 4 dimensions, and by Lemmas 2.1 and 2.2 they are all centrally symmetric. Hence $K_o$ is centrally symmetric. It follows that $K$ is centrally symmetric with center $o$, using a result of Rogers. Let $K^*$ be the polar reciprocal of $K$ about $o$. Then all the central 3-dimensional sections of $K^*$ are congruent so by Theorem 1, $K^*$ is a euclidean ball with center $o$. Hence $K$ is a euclidean ball.

**Proof of Theorem 3.** First consider the case when the dimension of $K$ is $2n + 1$. For each unit vector $u$ let $K(u)$ be the $2n$-dimensional section of $K$ through $p$ perpendicular to $u$, and let $F(u)$ be the $2n$-dimensional ellipsoid of least volume containing $K(u)$; the uniqueness of $F(u)$ was proved by Danzer, Laugwitz, and Lenz [3]. The affine transformation $\Phi_u$ which maps $F(u)$ onto a $2n$-dimensional euclidean unit ball $B(u)$ in the hyperplane of $F(u)$ by dilating its principal axes is a continuous function of $u$. Then all $\Phi_u K(u)$ for $u \in S^1$ are congruent, so $\Phi_u K(u)$ is a field of congruent $2n$-dimensional bodies in $E^{n+1}$. A result of Mani [5] shows that each $\Phi_u K(u)$ is a euclidean ball, so $K(u)$ is an ellipsoid. It follows from a theorem of Busemann [2, p. 91] that $K$ is an ellipsoid.

Now suppose the dimension of $K$ is greater than $2n + 1$. From the case already considered it follows that every $(2n + 1)$-dimensional section of $K$ through $p$ is an ellipsoid, and Busemann’s result then
shows that $K$ is an ellipsoid.

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Received June 12, 1978.

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