

# Pacific Journal of Mathematics

**DEDEKIND'S PROBLEM: MONOTONE BOOLEAN FUNCTIONS  
ON THE LATTICE OF DIVISORS OF AN INTEGER**

PAUL HESS

# DEDEKIND'S PROBLEM: MONOTONE BOOLEAN FUNCTIONS ON THE LATTICE OF DIVISORS OF AN INTEGER

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This paper is concerned with the combinatorial problem of counting the number of distinct collections of divisors of an integer  $N$  having the property that no divisor in a collection is a multiple of any other. It is shown that if  $N$  factors into primes  $N = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$  the number of distinct collections of divisors with the stated property does not exceed  $(\sum_{i=1}^n a_i - n + 3)^M$ , where  $M$  is the maximum coefficient in the expansion of the polynomial

$$(1+x+x^2+\cdots+x^{a_1})(1+x+x^2+\cdots+x^{a_2})\cdots(1+x+x^2+\cdots+x^{a_n}).$$

In the special case where  $N$  is squarefree the problem is equivalent to that of counting the number of "Sperner families" on  $n$  letters, for which G. Hansel obtained the upper bound  $3^{M_n}$ , where  $M_n$  is the binomial coefficient  $\binom{n}{[n/2]}$ ; the result in this paper is then a generalization of Hansel's theorem to the non-squarefree case.

The problem has also been formulated as that of counting the number of families consisting of incomparable subsets of a set of  $n$  objects (the objects of course corresponding to the primes in the number-theoretic formulation), with the variation that each object may appear in a set with a specifically limited number of repetitions (these limits corresponding to the prime exponents).

NOTATION. Given  $n$  letters  $x_1, x_2, \dots, x_n$ , and  $n$  positive integers  $a_1, a_2, \dots, a_n$ , consider the lattice consisting of all terms  $(x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n})$  in the polynomial  $\prod_{i=1}^n (\sum_{k=0}^{a_i} x_i^k)$ , with the partial ordering defined  $(x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}) \subseteq (x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n})$  if  $j_i \leq k_i$  for all  $i$ . A single term  $X = (x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n})$  in this lattice will be referred to as a "set", the empty set  $\phi$  denoting the term with all exponents  $j_1, j_2, \dots, j_n$  equal to zero. If  $X = (x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n})$ , the notation  $(X, x_k^c)$  will indicate the set  $(x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k+c} \cdots x_n^{j_n})$ , and the exponent sum  $j_1 + j_2 + \cdots + j_n$  will be written  $|X|$ .

A monotone Boolean function is defined to be a function taking the values 0 or 1 on each set of this lattice with the property that  $f(X) \leq f(Y)$  if  $X \subseteq Y$ . The problem of counting the number of monotone Boolean functions on this lattice is then equivalent to the problem concerning collections of divisors of  $N$  stated at the begin-

ning.

(1) The lattice defined above can be partitioned into chains, constructed inductively:

If  $n = 1$ , the chain covering consists of the single chain  $\phi \subseteq (x_1) \subseteq (x_1^2) \subseteq \dots \subseteq (x_1^{a_1})$ .

If  $n > 1$ , assume the chain covering has already been constructed on the  $n - 1$  letters  $x_1, \dots, x_{n-1}$ . Each chain  $C: X_1 \subseteq X_2 \subseteq \dots \subseteq X_r$  of the covering on  $n - 1$  letters gives rise to the chains

$$\begin{aligned} X_1 \subseteq X_2 \subseteq \dots \subseteq X_r \subseteq (X_r, x_n) \subseteq (X_r, x_n^2) \subseteq \dots \subseteq (X_r, x_n^{a_n}) \\ (X_1, x_n) \subseteq (X_1, x_n^2) \subseteq \dots \subseteq (X_1, x_n^{a_n}) \subseteq (X_2, x_n^{a_n}) \subseteq \dots \subseteq (X_{r-1}, x_n^{a_n}) \\ (X_2, x_n) \subseteq (X_2, x_n^2) \subseteq \dots \subseteq (X_{r-1}, x_n^{a_n-1}) \end{aligned}$$

⋮

terminating in

$$(X_{r-1}, x_n) \subseteq \dots \subseteq (X_{r-1}, x_n^{a_n - (r-2)}) \text{ if } 2 \leq r \leq a_n$$

or in

$$(X_{a_n}, x_n) \subseteq (X_{a_n+1}, x_n) \subseteq \dots (X_{r-1}, x_n) \text{ if } r > a_n .$$

If  $r = 1$ , the chain  $C$  gives rise only to the chain

$$X_1 \subseteq (X_1, x_n) \subseteq \dots \subseteq (X_1, x_n^{a_n}) .$$

EXAMPLES. If  $n = 1, a_1 = 2$ , the covering consists of the single chain  $\phi \subseteq (x_1) \subseteq (x_1^2)$ .

If  $n = 2, a_1 = 2, a_2 = 4$ , the covering consists of the three chains

$$\begin{aligned} \phi \subseteq (x_1) \subseteq (x_1^2) \subseteq (x_1^2 x_2) \subseteq (x_1^2 x_2^2) \subseteq (x_1^2 x_2^3) \subseteq (x_1^2 x_2^4) \\ (x_2) \subseteq (x_2^2) \subseteq (x_2^3) \subseteq (x_2^4) \subseteq (x_1 x_2^4) \\ (x_1 x_2) \subseteq (x_1 x_2^2) \subseteq (x_1 x_2^3) . \end{aligned}$$

An easy induction on  $n$  suffices to show that each chain contains a set  $X$  for which the exponent sum

$$|X| = \begin{cases} \sum_{i=1}^n a_i / 2 & \text{if } \sum_{i=1}^n a_i \text{ is even} \\ \left( \sum_{i=1}^n a_i + 1 \right) / 2 & \text{if } \sum_{i=1}^n a_i \text{ is odd} \end{cases}$$

and that all sets in the lattice appear once and only once in the covering. It follows that the number of chains in the covering is given by  $M$ , the maximum coefficient in the expansion of the polynomial  $\prod_{i=1}^n (\sum_{k=0}^{a_i} x_i^k)$ . (The coefficient of  $x^j$  in this polynomial is the number of sets in the lattice with exponent sum  $j$ .)

A theorem of Dilworth [2], states that a partially ordered set with  $k$  but not  $k + 1$  incomparable elements can be covered by  $k$

chains. The chain covering defined above is the covering whose existence is guaranteed by Dilworth's theorem.

*The set function  $\sigma$ .* If three sets  $X \subseteq Y \subseteq Z$  appear in succession within a chain, we define  $\sigma(X)$  to be the set  $X + (Z - Y)$ .  $\sigma(X)$  is undefined if  $X$  is not at least three places from the end of its chain.

EXAMPLES.  $\phi \subseteq (x_1) \subseteq (x_1^2); \sigma(\phi) = (x_1)$

$$(x_1^2 x_2^3) \subseteq (x_1^2 x_2^4) \subseteq (x_1^2 x_2^4 x_3); \sigma(x_1^2 x_2^3) = (x_1^2 x_2^3 x_3).$$

If  $X \subseteq Y \subseteq Z$  are three sets in succession within a chain in the covering, it is easy to see that if  $\sigma(X) = Y$ , then all the letters in  $Z$  are also letters in  $Y$ . This situation will be abbreviated " $\sigma(X) = \text{next}$ ", and we note that the length  $l$  of the longest possible sequence in a chain of the form  $\dots X_{i+1} \subseteq X_{i+2} \subseteq \dots \subseteq X_{i+l} \dots$  where  $X_{i+1} \neq \phi$  and all  $X$  in the sequence are composed of the same letters, is  $\sum_{i=1}^n a_i - n + 1$ .

Within the chain covering (1), define an ordering of the chains as follows: If  $n = 1$ ,  $C_1$  is the single chain  $\phi \subseteq (x_1) \subseteq (x_1^2) \subseteq \dots \subseteq (x_1^{a_1})$ , and inductively if  $n > 1$ , and  $C'_1, C'_2, \dots, C'_k$  are the ordered chains in the covering for the  $n - 1$  letters  $x_1, \dots, x_{n-1}$ , and if  $C'_j$  gives rise to the chains  $C_{j_1}, C_{j_2}, \dots, C_{j_{l_j}}$  in the covering on  $n$  letters in the sequence in which they appear in the definition (1), then let  $C_{11}, C_{12}, \dots, C_{1_{l_1}}; C_{21}, C_{22}, \dots, C_{2_{l_2}}; \dots; C_{k1}, C_{k2}, \dots, C_{k_{l_k}}$  be the ordering of the chains  $C_1, C_2, \dots, C_M$  in the  $n$ -letter covering. (In other words, simply order the chains as they appear in the inductive definition). An easy induction on  $n$  then establishes the following property of the function  $\sigma$ : (2) If  $\sigma(X)$  is defined and " $\neq \text{next}$ ", and  $X$  appears in chain  $C_i$ ,  $\sigma(X)$  in chain  $C_j$ , then  $j > i$ .

*Proof of (2).* Induction on  $n$ . The statement is true for  $n = 1$  vacuously. Consider the chain on  $n - 1$  letters  $X_1 \subseteq X_2 \subseteq \dots \subseteq X_r$  giving rise to the chains on  $n$  letters

$$\begin{aligned} X_1 \subseteq X_2 \subseteq \dots \subseteq X_r \subseteq (X_r, x_n) \subseteq \dots \subseteq (X_r, x_n^{a_n}) \\ (X_1, x_n) \subseteq (X_1, x_n^2) \subseteq \dots \subseteq (X_1, x_n^{a_n}) \subseteq (X_2, x_n^{a_n}) \subseteq \dots \subseteq (X_{n-1}, x_n^{a_n}) \\ \vdots \\ (X_{j-1}, x_n) \subseteq \dots \subseteq (X_{j-1}, x_n^{a_n - (j-2)}) \subseteq (X_j, x_n^{a_n - (j-2)}) \subseteq \dots \subseteq (X_{r-1}, x_n^{a_n - (j-2)}). \end{aligned}$$

In the first chain above, if  $\sigma(X_k)$  is defined and " $\neq \text{next}$ ",  $k \leq r - 2$ , so that  $\sigma(X_k)$  is in a later  $n - 1$  chain by induction, therefore in a later  $n$ -chain.  $\sigma(X_r)$  " $= \text{next}$ " and the same holds for  $\sigma(X_r, x_n)$ ,  $\sigma(X_r, x_n^2)$ , etc.  $\sigma(X_{r-1}) = (X_{r-1}, x_n)$  which is in a later  $n$ -chain. In

subsequent chains,  $\sigma(X_{j-1}, x_n^{a_n-(j-1)}) = (X_j, x_n^{a_n-(j-1)})$  which appears in the chain immediately following.  $\sigma(X_i, x_n^{a_n-(j-2)})$ , where  $i \geq j-1$ , if defined and " $\neq$  next", is the set  $(\sigma(X_i), x_n^{a_n-(j-2)})$  where  $\sigma(X_i)$  " $\neq$  next". By induction,  $\sigma(X_i)$  is in a later  $n-1$  chain so that  $(\sigma(X_i), x_n^{a_n-(j-2)})$  is in a later  $n$ -chain, which completes the proof of the assertion.

(3) If  $C$  is a chain in the covering and  $f$  is a monotone Boolean function already defined on all sets  $\sigma(W)$ , where  $W$  is any set in the chain  $C$  for which  $\sigma(W)$  is defined and " $\neq$  next", then the number of possible definitions for  $f$  on the chain  $C$  does not exceed  $\sum_{i=1}^n a_i - n + 3$ .

*Proof of (3).* Let the chain  $C$  consist of  $l$  sets  $W_1 \subseteq W_2 \subseteq \dots \subseteq W_l$ . Suppose  $\sigma(W)$  is undefined or " $=$  next" for all  $W$  in the chain  $C$ . Then if  $l \geq 3$ ,  $W_2 \neq \phi$  and  $W_2 \dots W_l$  are sets consisting of the same letters. Then the number of ways of defining a monotone Boolean function on the chain is at most  $l + 1 \leq \sum_{i=1}^n a_i - n + 3$ . Otherwise, let  $W_m$  be the  $W$  farthest to the right in the chain for which  $f(\sigma(W)) = 0$ , and  $W_k$  the  $W$  farthest to the left for which  $f(\sigma(W)) = 1$ . Either  $m$  or  $k$  exists. If  $k$  does not exist, then  $m$  does. In this case  $f(\sigma(W_m)) = 0$  and since  $W_m \subseteq \sigma((W_m))$ ,  $f$  is undetermined only on the portion of the chain  $W_{m+1}, W_{m+2}, \dots, W_{m+l}$ . But  $\sigma$  is undefined or " $=$  next" on these sets, so that  $W_{m+2} \dots W_l$  are sets consisting of the same letters (or  $W_{m+1} \dots W_l$  is shorter than 3 sets in length). Thus  $f$  is undetermined on at most  $\sum_{i=1}^n a_i - n + 2$  sets and the number of ways of defining  $f$  is at most  $\sum_{i=1}^n a_i - n + 3$  (either 0 throughout the chain, or  $\sum_{i=1}^n a_i - n + 2$  choices for the position of the 1 farthest to the left). A similar argument takes care of the case where  $m$  does not exist and  $k$  does. If  $m$  and  $k$  both exist, first suppose  $m < k$ . Then we have  $f = 0$  on the sets  $W_m, W_{m-1}, \dots$ , down to  $W_1$ , and  $f = 1$  on the sets  $W_{k+2}, W_{k+3}, \dots$  up to  $W_l$ . In this case  $W_{m+2} \dots W_{k-1} W_k W_{k+1}$  are all sets consisting of the same letters, so that the length of the segment on which  $f$  is undetermined,  $(k+1) - (m+1) + 1$ , is at most  $\sum_{i=1}^n a_i - n + 2$ , and as before the number of possible definitions of  $f$  on the chain is at most  $\sum_{i=1}^n a_i - n + 3$ . The final possibility is  $m \geq k$ , but by definition of  $m$  and  $k$ ,  $m \neq k$  and obviously  $m$  cannot exceed  $k+1$ . The situation is then:  $W_1 \subseteq \dots \subseteq W_k \subseteq W_m \subseteq W_{m+1} \subseteq \dots \subseteq W_l$ ,  $m = k+1$ ,  $f(\sigma(W_k)) = 1$  and  $f(\sigma(W_m)) = 0$  so that  $f = 1$  on the sets  $W_{m+1} \dots W_l$ ,  $f = 0$  on the sets  $W, \dots, W_k, W_m$ , and  $f$  is completely predetermined on the chain in this case.

**Conclusion.**  $(\sum_{i=1}^n a_i - n + 3)^M$ , where  $M$  is the maximal coeffi-

cient in the expansion of  $(1 + x + \cdots + x^{a_1})(1 + x + \cdots + x^{a_2}) \cdots (1 + x + \cdots + x^{a_n})$  is an upper bound on the number of monotone Boolean functions on the lattice of divisors of  $N = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ .

*Proof.* Let  $C_1, C_2, \dots, C_M$  be the ordered chains in the covering. On the last chain, the function  $\sigma$  is undefined or “=next” throughout. (Otherwise, according to (2), for  $X$  in the chain  $C_M$ ,  $\sigma(X)$  would appear in a later chain which is impossible.) It then follows from (3) that the number of ways of defining  $f$  on  $C_M$  does not exceed  $\sum_{i=1}^n a_i - n + 3$ . On chain  $C_{M-1}$ , if  $X$  is a set in this chain for which  $\sigma(X)$  is defined and “ $\neq$  next”, then according to (2)  $\sigma(X)$  appears in the chain  $C_M$ . Thus  $f(\sigma(X))$  is already defined for all such  $X$  in the chain  $C_{M-1}$ , and from (3) there are at most  $\sum_{i=1}^n a_i - n + 3$  possible definitions of  $f$  on  $C_{M-1}$ . Continuing in this way to the first chain  $C_1$  gives the upper bound stated.

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