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THE CARRIER SPACE OF A REFLEXIVE OPERATOR ALGEBRA

ALAN HOPENWASSER AND DAVID LARSON

Many properties of nest algebras are actually valid for reflexive operator algebras with a commutative subspace lattice. In this paper we collect a number of such results related to the carrier space of the algebra. Included among these results are a generalization of Ringrose's criterion, a description of the partial correspondence between lattice homomorphisms of the carrier space and projections in the lattice, the construction of isometric representations of certain quotient algebras, and a direct sum decomposition of the commutant of the core modulo the intersection of the spectral ideals.

Let $\mathscr{A} = \operatorname{Alg}\mathscr{L}$, where $\mathscr{L}$ is a commutative subspace lattice and let $\mathscr{I}$ be the intersection of all the spectral ideals in $\mathscr{A}$. (See §1 for definitions.) In §1 we generalize Ringrose's criterion to the commutative subspace lattice case: $A \in \mathscr{I}$ if, and only if, for each $\varepsilon > 0$ there is a finite family $\{E_i\}$ of mutually orthogonal intervals from $\mathscr{L}$ such that $\sum E_i = 1$ and $\|E_iAE_i\| < \varepsilon$, $i = 1, \cdots, n$. We also prove that $\mathscr{I}$ is the closed linear span of commutators of the form $AL - LA$, where $A \in \mathscr{A}$ and $L \in \mathscr{L}$. In §2 we describe the partial correspondence between certain projections in $\mathscr{L}$ and certain lattice homomorphisms in the carrier space X. A necessary (but not sufficient) condition for an operator A to be in the radical of $\mathscr{A}$ is given in §3. In §4 we exhibit isometric representations as algebras of operators acting on Hilbert space of each quotient algebra $\mathscr{A}/\mathscr{A}_\phi$ and of the quotient $\mathscr{A}/\mathscr{I}$. In the nest algebra case this was done by Lance in [5]. Finally, in §5 we generalize somewhat a theorem from [6] which identifies the $\mathscr{I}$-commutant of the core of $\mathscr{A}$ as the direct sum of the diagonal of $\mathscr{A}$ and $\mathscr{I}$.

1. Let $\mathscr{L}$ be a commutative subspace lattice acting on a separable Hilbert space $\mathscr{H}$, that is to say, $\mathscr{L}$ is a lattice of commuting, orthogonal projections on $\mathscr{H}$ which contains 0 and 1 and is closed in the strong operator topology. Let $\mathscr{A} = \operatorname{Alg}\mathscr{L}$, the algebra of all operators leaving invariant each projection in $\mathscr{L}$. Then $\mathscr{A}$ is a reflexive operator algebra whose lattice of invariant subspaces is just $\mathscr{L}$ [1]. Define the *carrier space*, X, of $\mathscr{L}$ to be the set of all lattice homomorphisms of $\mathscr{L}$ onto the trivial lattice $\{0, 1\}$. If the carrier space is given the topology in which a net, ϕ_ν, converges to ϕ if, and only if, $\phi_\nu(L) \to \phi(L)$ for each $L \in \mathscr{L}$, then it becomes a

compact, Hausdorff topological space.

A projection E in $\mathscr{A}$ is said to be an *interval* if $E = L - M$ for projections $L, M \in \mathscr{L}$ with $M < L$. If $\phi \in X$, we say that E is a *test interval* for ϕ if $\phi(\mathscr{L}) = 1$ and $\phi(M) = 0$. (It is easy to check that this is well-defined.) Let $\mathscr{F}_\phi = \{E | E$ is a test interval for $\phi\}$. $\mathscr{F}_\phi$ is a family of intervals which satisfies the finite intersection property and is maximal with respect to this property. Any family of intervals satisfying these conditions is called a *basic family*; there is a one-to-one correspondence between elements of the carrier space of $\mathscr{L}$ and basic families of intervals from $\mathscr{L}$ [3].

For each ϕ in X, define a continuous semi-norm N_ϕ on $\mathscr{A}$ by $N_\phi(T) = \inf\{\|ETE\| \mid E \in \mathscr{F}_\phi\}$, for each $T \in \mathscr{A}$. This, in turn, permits the definition of the *spectral ideal*, $\mathscr{A}_\phi$, associated with ϕ: $\mathscr{A}_\phi = \{T \in \mathscr{A} \mid N_\phi(T) = 0\}$. The spectral ideals are closed two-sided ideals in $\mathscr{A}$, as is the intersection, $\mathscr{I}$, of all the spectral ideals. Proofs of these facts can be found in [3], as well as the fact that $\mathscr{I}$ is contained in the radical, $\mathscr{R}$, of the algebra $\mathscr{A}$. It is known that $\mathscr{I} = \mathscr{R}$ if $\mathscr{A}$ is a nest algebra [7] and in a number of other cases, and we conjecture that equality always holds. Should it occur that $\mathscr{I}$ need not equal $\mathscr{R}$, it now appears clear that the role played by $\mathscr{I}$ in the structure of $\mathscr{A}$ is at least as important as the role played by the radical. As evidence in favor of the conjecture, we prove below that the Ringrose criterion for membership in the radical of a nest algebra ([7], Theorem 5.4) is a criterion for membership in $\mathscr{I}$ in the general case.

Each semi-norm N_ϕ on $\mathscr{A}$ can be identified with the quotient norm on $\mathscr{A}/\mathscr{A}_\phi$. Denote this quotient algebra by $\mathscr{D}_\phi$ and the canonical quotient map by q_ϕ. Later, in §4, we shall exhibit an isometric representation of $\mathscr{D}_\phi$ as an algebra of operators acting on Hilbert space.

PROPOSITION 1. *For each* $T \in \mathscr{A}$, $N_\phi(T) = \|q_\phi(T)\|$.

Proof. Let $T \in \mathscr{A}$. If E is a test interval for ϕ, then $ETE - T \in \mathscr{A}_\phi$. Hence $\|q_\phi(T)\| \leqq \|T + (ETE - T)\| = \|ETE\|$. Thus, $\|q_\phi(T)\| \leqq \inf\{\|ETE\| \mid E \in \mathscr{F}_\phi\} = N_\phi(T)$. For the opposite inequality, let $S \in \mathscr{A}_\phi$. Let ε be an arbitrary positive number and choose a test interval E for ϕ such that $\|ESE\| < \varepsilon$. Then

$$\begin{aligned} \|T + S\| &\geqq \|E(T + S)E\| = \|ETE + ESE\| \\ &\geqq \|ETE\| - \|ESE\| \\ &\geqq N_\phi(T) - \varepsilon . \end{aligned}$$

Since ε is arbitrary, $\|T + S\| \geqq N_\phi(T)$ for all $S \in \mathscr{A}_\phi$. Hence $\|q_\phi(T)\| \geqq$

$N_\phi(T)$.

A similar result holds for the quotient algebra $\mathscr{A}/\mathscr{I}$, where $\mathscr{I} = \bigcap_{\phi \in X} \mathscr{A}_\phi$. If $\mathscr{D} = \mathscr{A}/\mathscr{I}$ and if $q: \mathscr{A} \to \mathscr{D}$ is the canonical quotient map, then $\sup N_\phi(\cdot) = \|q(\cdot)\|$. Further, for each $T \in \mathscr{A}$, the supremum is attained. In the case of a nest algebra this is just Lemma 1.6 of [6] expressed in terms of the carrier space of $\mathscr{L}$. The ingredients needed for this result also yield the validity of the Ringrose criterion as a description of the elements of $\mathscr{I}$; a consequence of this criterion is the identification of $\mathscr{I}$ as the closed linear span of all commutators $AL - LA$, with $A \in \mathscr{A}$ and $L \in \mathscr{L}$. The first preliminary result which we need is the assertion that each such commutator lies in $\mathscr{I}$.

LEMMA 2. *If $A \in \mathscr{A}$ and $L \in \mathscr{L}$ then $AL - LA \in \mathscr{I}$.*

Proof. Let $\phi \in X$. If $\phi(L) = 1$ then $L \in \mathscr{F}_\phi$ and $N_\phi(AL - LA) \leqq \|L(AL - LA)L\| = 0$. If $\phi(L) = 0$ then $1 - L \in \mathscr{F}_\phi$ and $N_\phi(AL - LA) \leqq \|(1 - L)(AL - LA)(1 - L)\| = 0$. Thus, $AL - LA \in \mathscr{A}_\phi$, for all $\phi \in x$; i.e., $AL - LA \in \mathscr{I}$.

LEMMA 3. *If $\phi \in X$ and $A \in \mathscr{A}$ then $N_\phi(A) \leqq \|q(A)\|$.*

Proof. Since $\mathscr{I} \subseteqq \mathscr{A}_\phi$, we have $\|q_\phi(A)\| \leqq \|q(A)\|$, and the lemma follows from Proposition 1.

LEMMA 4. *Let $A \in \mathscr{A}$. Then there exists an element $\phi \in X$ such that $N_\psi(A) \leqq N_\phi(A)$, for all $\psi \in X$.*

Proof. In the remark on p. 379 of [3] it is shown that the mapping $\psi \to N_\psi(A)$ of X into $\boldsymbol{R}$ is upper semi-continuous. Since X is a compact Hausdorff space, this mapping achieves its supremum at some point ϕ in X.

DEFINITION. A projection in $\mathscr{A}$ is said to be *simple* if it is 0 or it is a finite sum of intervals from $\mathscr{L}$.

PROPOSITION 5. *The set of simple projections is a complemented lattice.*

Proof. We must show that if E and F are simple, then $E \wedge F = EF$, $E^\perp = I - E$, and $E \vee F = E + F - EF$ are simple.

First suppose that E and F are intervals from $\mathscr{L}$, say $E = L - M$ and $F = N - P$, where $L, M, N, P \in \mathscr{L}$ and $M \leqq L$ and $P \leqq N$. Then

$$E \wedge F = L \wedge N - ((L \wedge P) \vee (N \wedge M)) .$$

Since $\mathscr{L}$ is a lattice, $L \wedge N$ and $(L \wedge P) \vee (N \wedge M)$ both lie in $\mathscr{L}$; further $(L \wedge P) \vee (N \wedge M) \leqq L \wedge N$, so $E \wedge F$ is an interval. Now suppose that $E = \bigvee_{i=1}^{n} E_i$ and $F = \bigvee_{j=1}^{m} F_j$, where $\{E_i\}$ and $\{F_j\}$ are each families of mutually orthogonal intervals from $\mathscr{L}$. Then $E \wedge F = \bigvee_{i,j} (E_i \wedge F_j)$ and $\{E_i \wedge F_j\}$ is a family of mutually orthogonal intervals from $\mathscr{L}$; thus $E \wedge F$ is simple.

Next observe that if E is an interval then $E^{\perp}$ is the sum of two orthogonal intervals and hence is simple. If E is merely simple, then $E = \bigvee_{i=1}^{n} E_i$, where $\{E_i\}$ is a mutually orthogonal family of intervals from $\mathscr{L}$. Since $E^{\perp} = \bigwedge_{i=1}^{n} E_i^{\perp}$ and each $E_i^{\perp}$ is simple, the paragraph above implies that $E^{\perp}$ is simple.

It remains only to show that if E and F are simple, then $E \vee F$ is simple. When E and F are orthogonal, this is obvious; for the general case use the facts that $F \wedge E^{\perp}$ is simple and $E \vee F = E \vee (F \wedge E^{\perp})$.

For the following lemma, and for §2, recall that a subset $\mathscr{K}_1 \subseteq \mathscr{L}$ is said to be an *ideal* if it satisfies the two properties:

(a) $K_1, K_2 \in \mathscr{K}_1, \Rightarrow K_1 \vee K_2 \in \mathscr{K}_1$

(b) $K \in \mathscr{K}_1, L \in \mathscr{L}, L \leqq K \Rightarrow L \in \mathscr{K}_1$.

Similarly, a subset $\mathscr{K}_2$ is said to be a *co-ideal* if it satisfies the dual properties:

(a′) $K_1, K_2 \in \mathscr{K}_2 \Rightarrow K_1 \wedge K_2 \in \mathscr{K}_2$

(b′) $K \in \mathscr{K}_2, L \in \mathscr{L}, K \leqq L \Rightarrow L \in \mathscr{K}_2$.

An ideal is *prime* if its complement is a co-ideal.

LEMMA 6. *Every interval from $\mathscr{L}$ is a test interval for some lattice homomorphism in X.*

Proof. Let E be an interval from $\mathscr{L}$. Then there exist projections L and M in $\mathscr{L}$ with $M < L$ such that $E = L - M$. Let $\mathscr{J}_0 = \{R \in \mathscr{L} \mid R \geqq L\}$ and $\mathscr{K}_0 = \{S \in \mathscr{L} \mid S \leqq M\}$. It is routine to check that $\mathscr{K}_0$ is an ideal in $\mathscr{L}$ and that $\mathscr{J}_0$ is a co-ideal; it is obvious that $\mathscr{K}_0$ and $\mathscr{J}_0$ are disjoint. By a result of Stone, there exists an ideal $\mathscr{K} \supseteq \mathscr{K}_0$ and a co-ideal $\mathscr{J} \supseteq \mathscr{J}_0$ such that $\mathscr{K} \cap \mathscr{J} = \varnothing$ and $\mathscr{K} \cup \mathscr{J} = \mathscr{L}$. (See [4], page 80.) Since $\mathscr{K}$ is a prime ideal, there is a lattice homomorphism $\phi \in X$ such that $\mathscr{K} = \ker \phi$ ([2], p. 28). Since $\phi(L) = 1$ and $\phi(M) = 0$, E is a test interval for ϕ.

LEMMA 7. *Let E be an interval from $\mathscr{L}$ and let $U = \{\phi \in X \mid E$ is a test interval for $\phi\}$. Then U is a nonempty open and closed subset of X.*

Proof. The preceding proposition says that U is nonempty. To see that U is closed, suppose $\phi_\nu \in U$ and $\phi_\nu \to \phi$ in X. If $E = L - M$, with $L, M \in \mathscr{L}$, $L < M$, then $\phi_\nu(L) = 1$ and $\phi_\nu(M) = 0$ for all ν; hence $\phi(L) = 1$ and $\phi(M) = 0$. Thus $\phi \in U$ and U is closed. To see that U is open, let $\phi_\nu \in X - U$ and $\phi_\nu \to \phi$. Since ϕ_ν is convergent, the two nets $\phi_\nu(L)$ and $\phi_\nu(M)$ are each eventually constant. For all ν, $\phi_\nu(L) = \phi_\nu(M)$, since E is not a test interval for ϕ_ν; consequently $\phi(L) = \phi(M)$ and $\phi \in X - U$. Thus $X - U$ is closed and U is open.

LEMMA 8. *If E is a simple projection then, for any $A \in \mathscr{A}$, $EAE^\perp$ and $E^\perp AE$ lie in $\mathscr{I}$.*

Proof. By Lemma 2, every projection in $\mathscr{L}$ commutes with every member of $\mathscr{A}$ modulo $\mathscr{I}$, hence finite linear combinations of such projections (and, in particular, simple projections) have this property. Therefore $EAE^\perp = E(EA) - (EA)E \in \mathscr{I}$ and $E^\perp AE = (AE)E - E(AE) \in \mathscr{I}$.

PROPOSITION 9. *Let q be the canonical quotient map of $\mathscr{A}$ onto $\mathscr{D} = \mathscr{A}/\mathscr{I}$. For each $A \in \mathscr{A}$, $\sup_{\phi \in X} N_\phi(A) = \|q(A)\|$. Further, the supremum is attained by some lattice homomorphism (which depends upon A).*

Proof. Lemma 3 asserts that $\sup N_\phi(A) \leqq \|q(A)\|$, while Lemma 4 asserts that the supremum is attained. Suppose that A is an element of $\mathscr{A}$ for which $\alpha = \sup N_\phi(A) < \|q(A)\|$. Let β be such that $\alpha < \beta < \|q(A)\|$. For each ϕ in X choose a test interval E_ϕ such that $\|E_\phi A E_\phi\| < \beta$. (This is possible since $\inf\{\|EAE\| \mid E \in \mathscr{F}_\phi\} = N_\phi(A) < \beta$.) Let $U_\phi = \{\psi \in X \mid E_\phi \in \mathscr{F}_\psi\}$. Since $\phi \in U_\phi$ for each ϕ, the family $\{U_\phi\}$ is an open cover for X. But X is compact, so there exist finitely many lattice homomorphisms $\phi_1, \cdots, \phi_n \in X$ such that $U_{\phi_1}, \cdots, U_{\phi_n}$ cover X.

Observe that $\bigvee_{i=1}^n E_{\phi_i} = 1$. Indeed, by Proposition 5, the complement of $\bigvee_{i=1}^n E_{\phi_i}$ is simple. If the complement is not 0, it must contain an interval projection F. Let ϕ_0 be a lattice homomorphism for which F is a test interval. Since $F \wedge E_{\phi_i} = 0$ for $i = 1, \cdots, n$, none of the E_{ϕ_i} are test intervals for ϕ_0. But this says that $\phi_0 \notin \bigcup_{i=1}^n U_{\phi_i} = X$, a contradiction.

Now define $F_1 = E_{\phi_1}$ and $F_k = E_{\phi_k} \wedge (F_1 \vee \cdots \vee F_{k-1})^\perp$, for $k = 2, \cdots, n$. Then $F_1, \cdots, F_n$ are mutually orthogonal, each F_i is a sub-projection of E_{ϕ_i}, and $\sum_{i=1}^n F_i = \bigvee_{i=1}^n F_i = \bigvee_{i=1}^n E_{\phi_i} = 1$. It follows that $\|q(F_i A F_i)\| \leqq \|q(E_{\phi_i} A E_{\phi_i})\| \leqq \|E_{\phi_i} A E_{\phi_i}\| < \beta$, for each i; by [6], Lemma 1.1, $\|q(\sum_{i=1}^n F_i A F_i)\| = \max_{i=1,\dots,n} \|q(F_i A F_i)\| < \beta < \|q(A)\|$. Since $A = \sum_{i,j} F_i A F_j$, we will obtain a contradiction if we show that $\|q(\sum_{i=1}^n F_iAF_i)\| = \|q(\sum_{i,j=1}^n F_iAF_j)\|$. But this equality follows from Lemma 8, the fact that each F_i is simple, and the observation that $F_i \perp F_j$ if $i \neq j$. This proves the proposition.

If $\mathscr{A}$ is a nest algebra, the following theorem is precisely the Ringrose criterion ([7], Theorem 5.4).

THEOREM 10. *Let $A \in \mathscr{A}$. Then $A \in \mathscr{I}$ if, and only if, given $\varepsilon > 0$ othere exists a finite family, $\{E_i\}_{i=1,\cdots,k}$, of mutually orthogonal intervals from $\mathscr{L}$ such that $\sum_{i=1}^k E_i = 1$ and $\|E_iAE_i\| < \varepsilon$, for all i.*

Proof. It is clear that any operator in $\mathscr{A}$ which satisfies this condition must lie in each $\mathscr{A}_\phi$, and hence in $\mathscr{I}$. For the converse, suppose $A \in \mathscr{I}$. Let $\varepsilon > 0$. For each $\phi \in X$ there is a test interval E_ϕ in $\mathscr{F}_\phi$ such that $\|E_\phi AE_\phi\| < \varepsilon$. Let $U_\phi = \{\psi \in X \mid E_\phi \in \mathscr{F}_\psi\}$. The family $\{U_\phi\}$ is an open cover for X; let $U_{\phi_1}, \cdots, U_{\phi_n}$ be a finite subcover. Just as in the proof of Proposition 9, $\bigvee_{i=1}^n E_{\phi_i} = 1$. Since the simple projections form a complemented lattice, there is a finite sequence, $\{E_i\}_{i=1,\cdots,k}$, of mutually orthogonal intervals such that $1 = \sum_{i=1}^k E_i$ and, for each i and j, either $E_i \leqq E_{\phi_j}$ or $E_i \perp E_{\phi_j}$. Consequently, $\|E_iTE_i\| < \varepsilon$ for all i and the theorem is proven.

If $A \in \mathscr{A}$ and $L \in \mathscr{L}$ then $LA - AL = LAL^\perp$; from this it is clear that the linear span of such commutators is an ideal in $\mathscr{A}$. As a corollory of Theorem 10 we can identify $\mathscr{I}$ as the closure of this ideal.

THEOREM 11. *The closure of the linear span of the set of commutators of the form $LA - AL$, with $A \in \mathscr{A}$ and $L \in \mathscr{L}$, is the ideal $\mathscr{I}$.*

Proof. Lemma 2 shows that the closure is contained in $\mathscr{I}$. Let $\mathscr{I}_0$ denote the linear span of commutators $LA - AL$, $A \in \mathscr{A}$, $L \in \mathscr{L}$. From Theorem 10 it is sufficient to prove that if $\{E_i\}_{i=1,\cdots,n}$ is a family of mutually orthogonal intervals such that $\sum_{i=1}^n E_i = 1$ and if $A \in \mathscr{I}$ then $A - \sum_{i=1}^n E_iAE_i \in \mathscr{I}_0$. The argument in Lemma 8 shows that, for each i, $E_iA - E_iAE_i = E_iAE_i^\perp$ is in $\mathscr{I}_0$. Hence $A - \sum_{i=1}^n E_iAE_i = \sum_{i=1}^n E_iA - \sum_{i=1}^n E_iAE_i = \sum_{i=1}^n (E_iA - E_iAE_i) \in \mathscr{I}_0$.

2. Although the carrier space appears in the literature on nest algebras [5, 7], it does so in disguise. The reason for the disguise is that the carrier space can be parameterized in a natural way by the projections in the nest $\mathscr{L}$. Each projection $L \neq 0, 1$ corresponds to two lattice homomorphisms: define ϕ_L^+ by the requirement that it

map L and each subprojection of L to 0 and all other projections to 1; define ϕ_L^- by requiring that it map each proper subprojection of L to 0 and all other projections to 1. The projections 0 and 1 each correspond to a single lattice homomorphism, ϕ_0^+ and ϕ_1^-, respectively. The lattice homomorphisms so obtained are all distinct except when L is an immediate predecessor to M, in which case $\phi_L^+ = \phi_M^-$. It is easy to see that each element of the carrier space arises in this fashion. Given $L \neq 1$, the family of all intervals of the form $M - L$, with $M > L$, is a "cofinal" subfamily of the basic family for ϕ_L^+ and is used in place of the basic family. A similar remark applies for homomorphisms of the form ϕ_L^-.

In the general case, in which $\mathscr{L}$ is a commutative subspace lattice, this correspondence partially breaks down. Not every projection in $\mathscr{L}$ gives rise to a lattice homomorphism and not every homomorphism is associated with a projection in $\mathscr{L}$. In this section we describe that portion of the correspondence which remains valid.

Let $L \in \mathscr{L}$, with $L \neq 1$. The set $[0, L] = \{M \in \mathscr{L} \mid M \leqq L\}$ is an ideal in the lattice $\mathscr{L}$; its complement, $[0, L]^c = \{M \in \mathscr{L} \mid M \nleqq L\}$ need not be a co-ideal. (Although it is true that if $N \in [0, L]^c$ and $M \geqq N$ then $M \in [0, L]^c$, it is not necessarily the case that $N_1, N_2 \in [0, L]^c$ implies $N_1 \wedge N_2 \in [0, L]^c$.) The mapping $\phi: \mathscr{L} \rightarrow \{0, 1\}$ which maps each projection in $[0, L]$ to 0 and each projection in $[0, L]^c$ to 1 is a lattice homomorphism if, and only if, $[0, L]^c$ is a co-ideal. Similarly, if $L \neq 0$, the set $[L, 1] = \{M \mid M \geqq L\}$ is automatically a co-ideal while its complement, $[L, 1]^c = \{M \mid M \ngeqq L\}$, need not be an ideal. The mapping $\phi: \mathscr{L} \rightarrow \{0, 1\}$ which takes the value 1 on $[L, 1]$ and 0 on $[L, 1]^c$ is a lattice homomorphism if, and only if, $[L, 1]^c$ is an ideal.

An interval E in a nest algebra can be written in the form $E = L - M$ for a unique choice of $L, M \in \mathscr{L}$ with $M < L$. While this is not necessarily true in the general commutative case, it is possible to define upper and lower endpoints for E.

DEFINITION. Let E be an interval from $\mathscr{L}$. Define the *upper endpoint* of E to be the projection $P = \bigwedge \{L \in \mathscr{L} \mid E \leqq L\}$ and define the *lower endpoint* of E to be the projection $Q = \bigvee \{L \in \mathscr{L} \mid E \perp L\}$.

In general it is not necessarily true that $Q \leqq P$. However, we do have $E = P \vee Q - Q = P - P \wedge Q = P(I - Q)$.

We can now single out a class of projections in $\mathscr{L}$, the *fundamental* projections, which give rise to lattice homomorphisms in X.

DEFINITION. A projection $L \neq 1$ in $\mathscr{L}$ is said to be *upper fun-*

damental if L is the lower endpoint of every interval of the form $M - L$, with $M \in \mathscr{L}$, $M > L$. A projection $L \neq 0$ is said to be *lower fundamental* if L is the upper endpoint of every interval of the form $L - M$, with $M \in \mathscr{L}$, $M < L$.

LEMMA 12. (a) *L is upper fundamental* $\Leftrightarrow [0, L]^c$ *is a co-ideal.*
(b) *L is lower fundamental* $\Leftrightarrow [L, 1]^c$ *is an ideal.*

Proof. (a) Assume L is upper fundamental. Suppose $[0, L]^c$ is not a co-ideal: then there exist $M_1, M_2 \in [0, L]^c$ such that $M_1 \wedge M_2 \leqq L$. Since $M_1 \nleqq L$, $L \vee M_1 - L \vee (M_1 \wedge M_2) = L \vee M_1 - L$ is a nonzero interval. Further, since $M_1 \wedge M_2 \leqq L$, $M_2 \perp (L \vee M_1 - L)$. Since L is the lower endpoint of $L \vee M_1 - L$, $M_2 \leqq L$, a contradiction. Thus $[0, L]^c$ is a co-ideal.

Now suppose $[0, L]^c$ is a co-ideal. Let $M \in \mathscr{L}$ with $M > L$. To prove that L is lower fundamental, we must show that if $N \perp (M - L)$ then $N \leqq L$. But $N \perp (M - L)$ implies that $N \wedge M = L \wedge N \leqq L$, i.e., $N \wedge M \notin [0, L]^c$. Since $M \in [0, L]^c$ and $[0, L]^c$ is a co-ideal, we cannot have $N \in [0, L]^c$. Thus $N \leqq L$ as desired.

(b) The proof is just the dual of the argument in (a).

For each $L \neq 1$, define a mapping $\phi_L^+ : \mathscr{L} \to \{0, 1\}$ by requiring ϕ_L^+ to take the value 0 on $[0, L]$ and the value 1 on $[0, L]^c$. For each $L \neq 0$, define $\phi_L^- : \mathscr{L} \to \{0, 1\}$ by requiring ϕ_L^- to take the value 1 on $[L, 1]$ and the value 0 on $[L, 1]^c$. We then have the following corollary:

COROLLARY 13. (a) $\phi_L^+ \in X \Leftrightarrow L$ *is upper fundamental.*
(b) $\phi_L^- \in X \Leftrightarrow L$ *is lower fundamental.*

In order to describe which lattice homomorphisms are associated with fundamental projections, we recapitulate the classification of points in X in [3]. Fix an element $\phi \in X$ and define

$$K_0(\phi) = \bigvee \{L \in \mathscr{L} \mid \phi(L) = 0\} = \bigvee \ker \phi$$
$$K_1(\phi) = \bigwedge \{L \in \mathscr{L} \mid \phi(L) = 1\} = \bigwedge \operatorname{coker} \phi .$$

When ϕ is understood, we write K_0 and K_1 in place of $K_0(\phi)$ and $K_1(\phi)$.

DEFINITION. If $\phi(K_0) = 0$, we say that ϕ is *upper adjacent* to K_0. If $\phi(K_1) = 1$, we say that ϕ is *lower adjacent* to K_1.

It is easy to check that if ϕ is upper adjacent to K_0, then $[0, K_0]^c$ is a co-ideal and K_0 is upper fundamental. Similarly, if ϕ is lower adjacent to K_1, then $[K_1, 1]^c$ is an ideal and K_1 is lower fundamental.

In general the four possibilities $K_0 = K_1$, $K_0 < K_1$, $K_0 > K_1$, and K_0 and K_1 not comparable will each occur. The four further possibilities arising from the fact that each of K_0, K_1 may be in either $\ker \phi$ or in $\operatorname{coker} \phi$ also occur. Of the 16 a priori combinations, precisely 7 can occur. (See [3] for proofs and examples.) These 7 differenf types of lattice homomorphisms are described as follows:

DEFINITION. ϕ is said to be *atomic* if $\phi(K_0) = 0$ and $\phi(K_1) = 1$.

If ϕ is atomic we may have either $K_0 < K_1$ or K_0 not comparable with K_1; if either of these two conditions hold, then ϕ must be atomic. If ϕ is atomic then ϕ is both upper adjacent to K_0 and lower adjacent to K_1. The atomic lattice homomorphisms are the only ones which arise from two distinct fundamental projections. When ϕ is atomic, $K_1 - K_0 \wedge K_1$ is a minimal projection in the basic family $\mathscr{F}_\phi$.

DEFINITION. ϕ is said to be *local* if $K_0 = K_1$. If ϕ is local we may have either $\phi(K_0) = \phi(K_1) = 0$ (in which case ϕ is upper adjacent to K_0) or $\phi(K_0) = \phi(K_1) = 1$ (in which case ϕ is lower adjacent to K_1). These two types and the atomic case in which $K_0 < K_1$ are the only types which occur when $\mathscr{L}$ is a nest.

DEFINITION. ϕ is said to be *semi-local* if $K_1 < K_0$ and $\phi(K_1) = \phi(K_0)$.

If ϕ is semi-local and the common value of ϕ is 0 then ϕ is upper adjacent to K_0; if the common value is 1 then ϕ is lower adjacent to K_1. While these two types cannot occur in nest algebras, they are reasonably similar to the nest algebra types.

DEFINITION. ϕ is said to be *diffuse* if $K_1 < K_0$, $\phi(K_1) = 0$, and $\phi(K_0) = 1$.

This is the one remaining type. The diffuse homomorphisms are the only ones which do not arise from a fundamental projection.

It might be tempting to suspect that $\cap \{\mathscr{A}_\phi \mid \phi \text{ is not diffuse}\}$, rather than $\mathscr{I}$, is equal to the radical. That this is not so can be seen by considering the case in which $\mathscr{A}$ is a maximal abelian von Neumann algebra with no atoms. It is not difficult to see that in this case every lattice homomorphism on $\mathscr{L} = \operatorname{Lat} \mathscr{A}$ is diffuse.

3. A sufficient condition for an operator A in $\mathscr{A}$ to belong to the radical of $\mathscr{A}$ is that $A \in \mathscr{I}$. In this section we give a necessary

condition for A to be in the radical. This condition will not, in general, be sufficient; the set of operators which satisfy this condition forms a closed, two-sided ideal which may properly contain $\mathscr{R}$.

DEFINITION. If E_1, E_2 are nonzero orthogonal intervals from $\mathscr{L}$ such that $E_1\mathscr{B}(\mathscr{H})E_2 \subseteqq \mathscr{A}$, we say E_1, E_2 is strictly ordered and write $E_1 \ll E_2$

LEMMA 14. *The relation* $\ll$ *is transitive.*

Proof. Assume $E \ll F$ and $F \ll G$. We must show that $E \perp G$ and $E\mathscr{B}(\mathscr{H})G \subseteqq \mathscr{A}$. Let P be the upper endpoint of G, viz the smallest projection in $\mathscr{L}$ which contains G. Then $G = P - N$, for some $N \in \mathscr{L}$ with $N \leqq P$. Since $F \ll G$, we have $F \leqq P$ and $F \perp G$, hence $F \leqq N$. Since $E \ll F$, we have that E is contained in the upper endpoint for F, which is a subprojection of N. But the fact that $N \perp G$ now implies $E \perp G$.

Now suppose that $A \in E\mathscr{B}(\mathscr{H})G$, so $A = EAG$. We must show that A leaves invariant each member of $\mathscr{L}$. Fix $L \in \mathscr{L}$. If $GL = 0$ then $AL = 0$ and A leaves L invariant. Assume $GL \neq 0$ and let x be a vector in GL. Let y be an arbitrary vector in E and let z be any nonzero vector in F. By the assumption that $E \ll F$ and $F \ll G$, there exist operators S, $T \in \mathscr{A}$ such that $Sz = y$ and $Tx = z$. Thus $y = STx$. Since L is invariant under $\mathscr{A}$ and $x \in L$, we have $y \in L$ for all $y \in E$, i.e., $E \leqq L$. But $A = EAG$, hence $AL = LAL$ and $A \in \mathscr{A}$.

DEFINITION. A mutually orthogonal family of intervals is said to be *strictly ordered* if it is linearly ordered by the relation $\ll$. The *length* of such a family is its cardinality.

DEFINITION. If $A \in \mathscr{A}$ and $\varepsilon > 0$ we define the *ε-order* of A to be the number $R_\varepsilon(A) = \sup\{n \mid \text{there exists a strictly ordered family } \mathscr{F} \text{ of length } n \text{ with } \|EAE\| \geqq \varepsilon \text{ for all } E \in \mathscr{F}\}$.

REMARK. It is clear that if $\varepsilon_2 > \varepsilon_1 > 0$ then $R_{\varepsilon_1}(A) \geqq R_{\varepsilon_2}(A)$.

LEMMA 15. *If* $A, B, C \in \mathscr{A}$ *and* $\varepsilon > 0$, *then* $R_\varepsilon(A + B) \leqq R_{\varepsilon/2}(A) + R_{\varepsilon/2}(B)$, *and* $R_\varepsilon(BAC) \leqq R_\delta(A)$, *where* $\delta = \varepsilon/\|B\|\,\|C\|$.

Proof. The first assertion follows from the observation that if $\|E(A + B)E\| \geqq \varepsilon$, then either $\|EAE\| \geqq \varepsilon/2$ or $\|EBE\| \geqq \varepsilon/2$. The second follows from the fact that for an interval E we have $\|E(BAC)E\| = \|EBEAECE\| \leqq \|B\|\,\|C\|\,\|EAE\|$, so that if $\|E(BAC)E\| \geqq \varepsilon$, then $\|EAE\| \geqq \varepsilon/\|B\|\,\|C\|$.

DEFINITION. Let $\mathscr{J} = \{A \in \mathscr{A} \mid R_\varepsilon(A) < \infty$ for every $\varepsilon > 0\}$.

LEMMA 16. *$\mathscr{J}$ is a closed 2-sided ideal in $\mathscr{A}$.*

Proof. By Lemma 15 it is sufficient to show that $\mathscr{J}$ is closed. Suppose $A_n \in \mathscr{J}$ and $A_n \to A$. Fix $\varepsilon > 0$ and choose n so that $\|A - A_n\| < \varepsilon/2$. If E is an interval then $\|EAE\| \geqq \varepsilon$ implies $\|EA_nE\| \geqq |\,\|EAE\| - \|E(A - A_n)E\|\,| \geqq \varepsilon/2$. So $N_\varepsilon(A) \leqq N_{\varepsilon/2}(A_n) < \infty$. Since ε is arbitrary, $A \in \mathscr{J}$.

REMARK. In some cases, $\mathscr{J}$ may have nonzero intersection with the diagonal $\mathscr{A} \cap \mathscr{A}^*$. For example, if E is an atom for the lattice $\mathscr{L}$ (for each projection $L \in \mathscr{L}$, either $L \geqq E$ or $L \perp E$) and if $A = EAE$ then $R_\varepsilon(A) \leqq 1$ for every ε.

LEMMA 17. *Let $A \in \mathscr{A}$ and suppose that $A = LAL^\perp$, for some $L \in \mathscr{L}$. If $\{E_1, E_2\}$ is a strictly ordered pair of intervals then either $E_1AE_1 = 0$ or $E_2AE_2 = 0$.*

Proof. We may suppose $E_2 \ll E_1$, so that $E_2\mathscr{B}(\mathscr{H})E_1 \subseteqq \mathscr{A}$. If $E_1AE_1 \neq 0$ we have $E_1L \neq 0$. Let x be a vector in E_1L. Since $E_2 \ll E_1$, E_2 is contained in any projection which is invariant under $\mathscr{A}$ and which contains x; in particular, $E_2 \leqq L$. But then $E_2AE_2 = 0$.

LEMMA 18. *Let $A_i \in \mathscr{A}$, $L_i \in \mathscr{L}$, $i = 1, \cdots, n$. Let $A = \sum_{i=1}^n L_iAL_i^\perp$. If $E_1, \cdots, E_k$ is a strictly ordered set with $E_jAE_j \neq 0$ for all j, then $k \leqq n$.*

Proof. This follows from Lemma 17.

REMARK. Lemma 18 shows already that $\mathscr{I} \subseteqq \mathscr{J}$. We shall prove below that $\mathscr{R} \subseteqq \mathscr{J}$.

LEMMA 19. *Let $A \in \mathscr{A}$ and assume that $R_\varepsilon(A) = \infty$ for some $\varepsilon > 0$. Let $E_1, \cdots, E_n$ be a finite set of mutually orthogonal simple projections with $\sum_{i=1}^n E_i = 1$. Then $R_\varepsilon(E_iAE_i) = \infty$ for at least one i.*

Proof. It suffices to prove the assertion for $n = 2$. Suppose E_1 and E_2 are simple with $E_1 \perp E_2$, $E_1 + E_2 = 1$. Let $B = E_1AE_1 + E_2AE_2$ and $C = E_1AE_2 + E_2AE_1$. We claim first that C can be expressed as a finite sum $C = \sum_{i=1}^m L_iC_iL_i^\perp$, for appropriate $C_i \in \mathscr{A}$, $L_i \in \mathscr{L}$. Indeed, C is a sum of terms of the form FAG, where F and G are orthogonal intervals. If $F = M - N$ with $M > N$, $M, N \in \mathscr{L}$ then $FAG = FAGF^\perp = MN^\perp AG(M^\perp + N) = M(N^\perp AG)M^\perp$.

Let k be an arbitrary positive integer and let $\{F_j\}$ be a strictly ordered family of length $k+m$ such that $||F_jAF_j|| \geqq \varepsilon$, for all j. Since $A = B + C$, Lemma 18 and the transitivity of $\ll$ imply that there is a strictly ordered subfamily $\{F_j^1\}$ of length k such that $||F_j^1BF_j^1|| \geqq \varepsilon$ for all j. Since $||F_j^1BF_j^1|| = \max_{i=1,2}||F_j^1E_iAE_iF_j^1||$ and k is arbitrary, either $R_\varepsilon(E_1AE_1) = \infty$ or $R_\varepsilon(E_2AE_2) = \infty$.

LEMMA 20. *Let A be an element of $\mathscr{A}$ for which $R_\varepsilon(A) = \infty$, for some $\varepsilon > 0$. Then there exist infinite sequences E_n, K_n of intervals such that $E_n \perp K_n$, $E_{n+1} < K_n$, $K_{n+1} < K_n$, $R_\varepsilon(E_nAE_n) \geqq n$, and $R_\varepsilon(K_nAK_n) = \infty$, for all n.*

Proof. Let $E_{11} \ll E_{12}$ be strictly ordered intervals with $||E_{1i}AE_{1i}|| \geqq \varepsilon$, for $i = 1, 2$. If $E_{11}AE_{11}$ has finite ε-order, let $E_1 = E_{11}$ and let K_1 be an interval in $E_1^\perp$ such that $R_\varepsilon(K_1AK_1) = \infty$. If $E_{11}AE_{11}$ has infinite ε-order, let $E_1 = E_{12}$ and $K_1 = E_{11}$. In either case, $E_1 \perp K_1$, $R_\varepsilon(E_1AE_1) \geqq 1$, and $R_\varepsilon(K_1AK_1) = \infty$.

Now assume inductively that intervals E_i, K_i, $i = 1, \cdots, n$ have been constructed such that $E_{i+1} < K_i$ and $K_{i+1} < K_i$, for $i = 1, \cdots, n-1$ and $E_i \perp K_i$, $R_\varepsilon(E_iAE_i) \geqq i$, $R_\varepsilon(K_iAK_i) = \infty$, for $i = 1, \cdots, n$. Since $R_\varepsilon(K_nAK_n) = \infty$, there exist strictly ordered intervals $E_{n+1,1} \ll E_{n+1,2} \ll \cdots \ll E_{n+1,2n+2}$ contained in K_n with $||E_{n+1,i}AE_{n+1,i}|| \geqq \varepsilon$ for all i. Let L be the upper endpoint of $E_{n+1,n+1}$ and let $F_1 = L \wedge K_n$, $F_2 = L^\perp \wedge K_n$. Then F_1 is an interval which contains each of $E_{n+1,1}, \cdots, E_{n+1,n+1}$ while F_2 is an interval containing $E_{n+1,n+2}, \cdots, E_{n+1,2n+2}$. Therefore, $R_\varepsilon(F_iAF_i) \geqq n+1$, for $i = 1, 2$. If F_1AF_1 has finite ε-order let $E_{n+1} = F_1$ and let K_{n+1} be an interval contained in $K_n - E_{n+1}$ for which $R_\varepsilon(K_{n+1}AK_{n+1}) = \infty$. If F_1AF_1 has infinite ε-order, let $E_{n+1} = F_2$ and $K_{n+1} = F_1$. In either case E_{n+1} and K_{n+1} are orthogonal intervals, both contained in K_n, with $R_\varepsilon(E_{n+1}AE_{n+1}) \geqq n+1$ and $R_\varepsilon(K_{n+1}AK_{n+1}) = \infty$. Induction completes the proof.

LEMMA 21. *Let $E_1 \gg E_2 \gg \cdots \gg E_n$ be a strictly ordered set of intervals and let A be an operator in $\mathscr{A}$ for which $||E_iAE_i|| > 1$, $i = 1, \cdots, n$. Then there exists a contraction S in $\mathscr{A}$ with support and range contained in $E = \sum E_i$ such that SA is nilpotent of index n and $||E(SA)^kE|| \geqq 1$, for $k = 1, \cdots, n-1$.*

Proof. For each i, let x_i be a unit vector in E_i such that $||E_iAE_ix_i|| \geqq 1$. Let $y_i = E_iAx_i$ and let $S = \sum_{i=1}^{n-1}||y_i||^{-2}y_i \otimes x_{i+1}$. (The operator $y \otimes x$ is defined by $y \otimes x(z) = \langle z, y\rangle x$, for all $z \in \mathscr{H}$.) Since the E_i are mutually orthogonal, $||S|| = \max_i ||y_i||^{-2}||y_i \otimes x_{i+1}|| = \max_i ||y_i||^{-1} \leqq 1$. The fact that the E_i are strictly ordered guarantees that $S \in \mathscr{A}$. Observe also that $(SB)^n = 0$, for all $B \in \mathscr{A}$. If $k = 1, \cdots,$

$n-1$, then $(SA)^k x_1 = x_{k+1} + z$, for some vector z orthogonal to x_{k+1}; hence $\|E(SA)^k E\| \geqq 1$. Thus S satisfies the requirements of the lemma.

PROPOSITION 22. *If A is in the radical of $\mathscr{A}$ then the ε-order of A is finite for all $\varepsilon > 0$; i.e., $\mathscr{R} \subseteq \mathscr{J}$.*

Proof. We must show that if $A \in \mathscr{A}$ and $R_\varepsilon(A) = \infty$ for some $\varepsilon > 0$, then $A \notin \mathscr{R}$. We may assume $\varepsilon > 1$ by multiplying by a scalar, if necessary. By Lemma 20 there exists a sequence (E_n) of mutually orthogonal intervals such that $R_\varepsilon(E_n A E_n) \geqq n$, for all n. For each n there exists a strictly ordered family $E_{n1} \ll E_{n2} \ll \cdots \ll E_{nn}$ contained in E_n such that $\|E_{ni} A E_{ni}\| \geqq \varepsilon > 1$, $i = 1, \cdots, n$. By Lemma 21 there exists a contraction S_n in $\mathscr{A}$ with $S_n = E_n S_n E_n$ such that $\|E_n(S_n A)^{n-1} E_n\| > 1$. Let $S = \sum_{n=1}^\infty S_n$, the sum converging strongly. We have, for each $n \geqq 2$,

$$\begin{aligned} \|(SA)^{n-1}\| &\geqq \|E_n(SA)^{n-1}E_n\| \\ &= \|E_n((E_n S E_n)A)^{n-1}E_n\| \\ &= \|E_n(S_n A)^{n-1}E_n\| > 1 . \end{aligned}$$

Hence SA is not quasinilpotent and $A \notin \mathscr{R}$.

4. If $\mathscr{A}$ is a nest algebra, then by a result of Lance [5], there is an isometric representation of each quotient algebra, $\mathscr{D}_\phi = \mathscr{A}/\mathscr{A}_\phi$, as an algebra of operators on a Hilbert space. Propositions 1 and 9 permit us to exhibit similar isometric representations of $\mathscr{D}_\phi$ and $\mathscr{D} = \mathscr{A}/\mathscr{I}$ as algebras of operators on a Hilbert space in the general case in which $\mathscr{A}$ is a reflexive operator algebra with commutative subspace lattice $\mathscr{L}$, acting on the Hilbert space $\mathscr{H}$.

THEOREM 23. *For each $\phi \in X$, there is an isometric representation of $\mathscr{D}_\phi$ as an algebra of operators on Hilbert space.*

Proof. If ϕ is atomic, let E be the minimal projection in $\mathscr{F}_\phi$. It is easy to see that $\mathscr{D}_\phi$ is isometrically isomorphic to $\mathscr{B}(E\mathscr{H})$. (See [3], p. 381.) Assume henceforth that ϕ is not atomic. Let $B(\mathscr{F}_\phi)$ denote the set of all bounded, complex valued functions defined on $\mathscr{F}_\phi$. With the usual pointwise algebraic operations, complex conjugation as an involution, and the supremum norm, $\| \quad \|_\infty$, $B(\mathscr{F}_\phi)$ forms an abelian unital C^*-algebra. If $f \in B(\mathscr{F}_\phi)$, let us say $\lim f = p$ provided that, for each $\varepsilon > 0$ there is an element $E \in \mathscr{F}_\phi$, such that if $F \in \mathscr{F}_\phi$ and $F \leqq E$ then $|f(F) - p| < \varepsilon$. The set of all f in $B(\mathscr{F}_\phi)$ for which $\lim f$ exists is a unital C^*-subalgebra of $B(\mathscr{F}_\phi)$; lim is a pure state on this sub-algebra. Let LIM be a pure state on $B(\mathscr{F}_\phi)$ which extends lim.

LIM is a multiplicative, positive linear functional on $B(\mathscr{F}_\phi)$. Further, if $f(E) \geqq 0$ for all $E \leqq F$, for some fixed $F \in B(\mathscr{F}_\phi)$, then $\mathrm{LIM}(f) \geqq 0$. Indeed, if f_1 is defined by $f_1(E) = 0$ when $E \leqq F$ and $f_1(E) = f(E)$ otherwise, then $\mathrm{LIM} f_1 = \lim f_1 = 0$ and $\mathrm{LIM} f = \mathrm{LIM} f - \mathrm{LIM} f_1 = \mathrm{LIM}(f - f_1) \geqq 0$, since $f - f_1 \geqq 0$. From this it follows that if $f, g \in B(\mathscr{F}_\phi)$ and $f(E) \leqq g(E)$, for all $E \leqq F$, for some fixed F, then $\mathrm{LIM} f \leqq \mathrm{LIM} g$. For convenience, we frequently write $\mathrm{LIM}_E f(E)$ in place of $\mathrm{LIM} f$.

Let $B(\mathscr{F}_\phi, \mathscr{H})$ be the linear space of all bounded functions on $\mathscr{F}_\phi$ with values in $\mathscr{H}$. For all $x, y \in B(\mathscr{F}_\phi, \mathscr{H})$, the function $E \to \langle Ex(E), y(E)\rangle$ is in $B(\mathscr{F}_\phi)$, hence we can define a sesquilinear form on $B(\mathscr{F}_\phi, \mathscr{H})$ by

$$\langle x, y\rangle = \mathop{\mathrm{LIM}}_E \langle Ex(E), y(E)\rangle .$$

Let $\mathscr{N} = \{x \in B(\mathscr{F}_\phi, \mathscr{H}) \mid \mathrm{LIM}_E \|Ex(E)\| = 0\}$. $\mathscr{N}$ is a linear subspace of $B(\mathscr{F}_\phi, \mathscr{H})$; let $\mathscr{K}^1$ be the quotient space. Observe that if $x \in \mathscr{N}$ and $y \in B(\mathscr{F}_\phi, \mathscr{H})$ then $\langle x, y\rangle = 0$. Indeed, $|\langle x, y\rangle| \leqq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}$ and $\langle x, x\rangle = \mathrm{LIM}_E \langle Ex(E), x(E)\rangle = \mathrm{LIM}_E \|Ex(E)\|^2 = 0$. This implies that the sesquilinear form on $B(\mathscr{F}_\phi, \mathscr{H})$ induces an inner product on the quotient space $\mathscr{K}^1$. For any $x \in B(\mathscr{F}_\phi, \mathscr{H})$, let $\bar{x}$ denote the image of x in the quotient $\mathscr{K}^1$. The inner product on $\mathscr{K}^1$ is given by $\langle \bar{x}, \bar{y}\rangle = \mathrm{LIM}_E \langle Ex(E), y(E)\rangle$. Let $\mathscr{K}$ be the completion of $\mathscr{K}^1$; $\mathscr{K}$ is the Hilbert space on which we construct a representation of $\mathscr{D}_\phi$.

For each operator A in $\mathscr{A}$ define a linear mapping $\Pi_0(A)$ on $B(\mathscr{F}_\phi, \mathscr{H})$ by $(\Pi_0(A)x)(E) = EAEx(E)$, for all $E \in \mathscr{F}_\phi$. Observe first that $\Pi_0(A)$ leaves $\mathscr{N}$ invariant. Let $x \in \mathscr{N}$. Then

$$0 \leqq \mathop{\mathrm{LIM}}_E \|E\Pi_0(A)x(E)\| = \mathop{\mathrm{LIM}}_E \|EAEx(E)\| \leqq \|A\| \mathop{\mathrm{LIM}}_E \|Ex(E)\| = 0 ,$$

hence $\Pi_0(A)x \in \mathscr{N}$. From this it follows that $\Pi_0(A)$ induces an operator $\Pi(A)$ acting on the quotient space $\mathscr{K}^1$. Note that $\Pi(A)$ is determined by the condition

$$\langle \Pi(A)\bar{x}, \bar{y}\rangle = \mathrm{LIM}_E \langle EAEx(E), y(E)\rangle .$$

Since Π_0 is an algebra homomorphism on $\mathscr{A}$, so is Π. (This uses the fact that for any interval E, the mapping $A \to EAE$ is multiplicative on $\mathscr{A}$.)

We now compute the norm of $\Pi(A)$ as an operator on the pre-Hilbert space $\mathscr{K}^1$. Let $\bar{x}$ and $\bar{y}$ be unit vectors in $\mathscr{K}^1$. Let F be an arbitrary interval in $\mathscr{F}_\phi$. Then, for any $E \in \mathscr{F}_\phi$ with $E \leqq F$, we have $\|EAE\| \leqq \|FAF\|$, and hence

$$|\langle EAEx(E),\, y(E)\rangle| \leqq \|FAF\|\,\|Ex(E)\|\,\|Ey(E)\| .$$

Consequently,

$$\begin{aligned}|\langle \Pi(A)\bar{x},\, \bar{y}\rangle| &= \left|\operatorname*{LIM}_E \langle EAEx(E),\, y(E)\rangle\right| \\ &\leqq \operatorname*{LIM}_E |\langle EAEx(E),\, y(E)\rangle| \\ &\leqq \|FAF\| \operatorname*{LIM}_E \|Ex(E)\| \operatorname*{LIM}_E \|Ey(E)\| \\ &\leqq \|FAF\| .\end{aligned}$$

Since F is arbitrary, $\|\Pi(A)\| \leqq \inf\{\|FAF\| \mid F \in \mathscr{F}_\phi\} = N_\phi(A)$.

In fact, we actually have equality. Let $\varepsilon > 0$ be arbitrary. For each $E \in \mathscr{F}_\phi$, there is a vector $x(E) \in E\mathscr{H}$ such that $\|x(E)\| = 1$ and $\|EAEx(E)\| \geqq N_\phi(A) - \varepsilon$. Then,

$$\begin{aligned}\|\Pi(A)\bar{x}\|^2 &= \langle \Pi(A)\bar{x},\, \Pi(A)\bar{x}\rangle \\ &= \operatorname*{LIM}_E \langle EAEx(E),\, EAEx(E)\rangle \\ &= \operatorname*{LIM}_E \|EAEx(E)\|^2 \\ &\geqq (N_\phi(A) - \varepsilon)^2 .\end{aligned}$$

Since $\|\bar{x}\| = 1$, we have $\|\Pi(A)\| \geqq N_\phi(A) - \varepsilon$, and since ε is arbitrary, $\|\Pi(A)\| = N_\phi(A)$.

For each $A \in \mathscr{A}$, $\Pi(A)$ has a unique extension to a bounded linear operator, which we also denote by $\Pi(A)$, acting on the Hilbert space $\mathscr{K}$. Thus Π is a representation of $\mathscr{A}$ acting on $\mathscr{K}$ for which $\|\Pi(A)\| = N_\phi(A)$, for all $A \in \mathscr{A}$. From Proposition 1 it is clear that Π induces an isometric representation of $\mathscr{D}_\phi$ acting on $\mathscr{K}$.

COROLLARY 24. *There is an isometric representation of* $\mathscr{D} = \mathscr{A}/\mathscr{I}$ *as an algebra of operators acting on Hilbert space.*

Proof. For each $\phi \in X$, let Π_ϕ be an isometric representation of $\mathscr{D}_\phi$ acting on a Hilbert space $\mathscr{H}_\phi$. Let $q_\phi\colon \mathscr{A} \to \mathscr{D}_\phi$ and $q\colon \mathscr{A} \to \mathscr{D}$ be the canonical quotient maps. Let $\mathscr{H} = \sum^{\oplus}_{\phi \in X} \mathscr{H}_\phi$. Define $\Pi_0\colon \mathscr{A} \to \mathscr{L}(\mathscr{H})$ by $\Pi_0(A) = \sum^{\oplus}_{\phi \in X} \Pi_\phi(q_\phi(A))$. Then Π_0 is a representation of $\mathscr{A}$. For each $A \in \mathscr{A}$, $\|\Pi_0(A)\| = \sup_\phi \|\Pi_\phi q_\phi(A)\| = \sup_\phi N_\phi(A) = \|q(A)\|$, by Proposition 9. Hence Π_0 induces an isometric representation, Π, of $\mathscr{D}$ acting on $\mathscr{H}$.

5. Let $\mathscr{A}$ be a reflexive operator algebra with commutative subspace lattice $\mathscr{L}$. Recall that the *diagonal* of $\mathscr{A}$ is defined to be the von Neumann algebra $\mathscr{A} \cap \mathscr{A}^*$ and the *core* of $\mathscr{A}$ is defined to be the von Neumann algebra generated by $\mathscr{L}$. We shall denote the diagonal and core of $\mathscr{A}$ by $\mathscr{A}_d$ and $\mathscr{A}_c$ respectively. Observe that $\mathscr{A}_d = \mathscr{A}_c'$,

i.e., the diagonal is the commutant of the core. As before, $\mathscr{I} = \bigcap_{\phi \in X} \mathscr{A}_\phi$.

DEFINITION. We define the $\mathscr{I}$-commutant of the core to be the algebra

$$\mathscr{A}_{\mathscr{I}} = \{A \in \mathscr{A} \mid AB - BA \in \mathscr{I}, \quad \text{for all} \quad B \in \mathscr{A}_c\}.$$

REMARK. It is clear that $\mathscr{I} \subseteqq \mathscr{A}_{\mathscr{I}}$ and $\mathscr{A}_d = \mathscr{A}_c' \subseteqq \mathscr{A}_{\mathscr{I}}$, hence $\mathscr{A}_d + \mathscr{I} \subseteqq \mathscr{A}_{\mathscr{I}}$. Since $\mathscr{A}_d$ is a C^*-algebra, it is semi-simple; since $\mathscr{I} \subseteqq \mathscr{R}$, it follows that $\mathscr{I} \cap \mathscr{A}_d = (0)$, so $\mathscr{A}_d + \mathscr{I} = \mathscr{A}_d \oplus \mathscr{I}$. It is proven in [6] (Theorem 2.4) that if $\mathscr{L}$ is a nest then $\mathscr{A}_d \oplus \mathscr{I} = \mathscr{A}_{\mathscr{I}}$. We prove below that the same result holds for commutative subspace lattices which satisfy certain additional hypotheses. It seems quite possible that the result holds in the general case, but we have not been able to prove it.

We sketch briefly some of the tools needed for the theorem. Full details, proofs, and/or references can be found in [6]. Let M be an invariant mean on the (abelian) group, $\mathscr{U}$, of unitary operators in $\mathscr{A}_c$. If g is a bounded, complex valued function on $\mathscr{U}$ we frequently write $M_U g(U)$ in place of Mg. Let $\mathscr{C}_1$ denote the ideal of trace class operators in $B(\mathscr{H})$ and identify $B(\mathscr{H})$ as the dual of $\mathscr{C}_1$ via the pairing $(T, f) = Tr(Tf)$, for $T \in B(\mathscr{H})$, $f \in \mathscr{C}_1$. Define a mapping ψ on $B(\mathscr{H})$ by the formula $(\psi(T), f) = M_U(U^*TU, f)$, for $T \in \mathscr{B}(\mathscr{H})$, $f \in \mathscr{C}_1$. The translation invariance of M implies that ψ maps $B(\mathscr{H})$ into $\mathscr{A}_c' = \mathscr{A}_d$; one can verify that ψ is a norm 1 projection of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{A}_d$ which satisfies $\psi(AB) = A\psi(B)$, $\psi(BA) = \psi(B)A$, for all $A \in \mathscr{A}_d$, $B \in \mathscr{B}(\mathscr{H})$. If $T \in \mathscr{B}(\mathscr{H})$, define $\delta T: \mathscr{A}_c \to B(\mathscr{H})$ by $\delta T(A) = AT - TA$. If $D: \mathscr{A}_c \to B(\mathscr{H})$ is any derivation (i.e., D is linear and $D(AB) = AD(B) + D(A)B$, for all $A, B \in \mathscr{A}_c$) then $D = \delta T$, where T is defined by $(T, f) = M_U(U^*D(U), f)$, $f \in \mathscr{C}_1$. If $D = \delta T_1$, for some other $T_1 \in \mathscr{B}(\mathscr{H})$, then $T = T_1 - \psi(T_1)$.

Each of the following two hypotheses is satisfied by every nest on a separable Hilbert space and by some, but not all, nontotally ordered commutative subspace lattices. (Hypothesis A is actually satisfied by every nest regardless of the dimension of the Hilbert space.)

Hypothesis A. If (F_n) is an infinite sequence of mutually orthogonal intervals from $\mathscr{L}$ then there exists an element ψ in X such that if $E \in \mathscr{F}_\psi$ then $F_n \leqq E$, for some n.

Hypothesis B. For each $\phi \in X$, there is a countable family $\{F_n\}$

of intervals in $\mathscr{F}_\phi$, totally ordered by inclusion, such that, for each $E \in \mathscr{F}_\phi$, $F_n \leqq E$, for some n.

THEOREM 25. *If $\mathscr{A}$ satisfies either Hypothesis A or Hypothesis B then the $\mathscr{I}$-commutant of the core of $\mathscr{A}$ is equal to the direct sum of the diagonal of $\mathscr{A}$ and $\mathscr{I}$*; i.e., $\mathscr{A}_{\mathscr{I}} = \mathscr{A}_d \oplus \mathscr{I}$.

Proof. From the remark at the beginning of this section, it suffices to prove that if $B \in \mathscr{A}$ and if $AB - BA \in \mathscr{I}$, for all $A \in \mathscr{A}_c$ then $B \in \mathscr{A}_d + \mathscr{I}$. For this, it suffices to show that $B - \psi(B) \in \mathscr{I}$. Let $T = B - \psi(B)$ and let $D = \delta B = \delta T$. If $T \notin \mathscr{I}$, then there exists an element $\phi \in X$ such that $N_\phi(T) > 0$. Note that ϕ cannot be atomic, for otherwise $\mathscr{F}_\phi$ has a minimal projection E_0, for which $E_0 T E_0 \in \mathscr{A}_d$. But then $E_0 T E_0 = \psi(E_0 T E_0) = E_0 \psi(T) E_0 = 0$ and $N_\phi(T) = 0$, a contradiction. Thus $\mathscr{F}_\phi$ is a directed family of projections whose strong limit is 0.

Let $E \in \mathscr{F}_\phi$. The lower semi-continuity of norm and the fact that $\|ETE\| \geqq N_\phi(T) > (1/2)N_\phi(T)$ allows us to find a projection $E' \leqq E$, $E' \in \mathscr{F}_\phi$ such that $\|(E - E')T(E - E')\| > (1/2)N_\phi(T)$. Since $E - E'$ is a simple projection, it contains an interval F for which $\|FTF\| > (1/2)N_\phi(T)$. By repeating this argument, we may obtain inductively a mutually orthogonal sequence of intervals (F_n) such that $\|F_n T F_n\| > (1/2)N_\phi(T)$, all n. If Hypothesis B is satisfied we can also arrange to choose the (F_n) so that any projection E in $\mathscr{F}_\phi$ contains some F_n. If Hypothesis A is satisfied then there is an element ψ in X such that every projection in $\mathscr{F}_\psi$ contains some F_n.

We next claim that for each n, there is an operator $A_n \in \mathscr{A}_c$ such that $\|A_n\| = 1$ and $\|F_n D(A_n) F_n\| > (1/2)N_\phi(T)$. Suppose the contrary, namely that there is an integer n such that $\|F_n D(A) F_n\| \leqq (1/2)N_\phi(T)$, for all $A \in \mathscr{A}_c$ with $\|A\| \leqq 1$. Then for any unitary operator U in $\mathscr{A}_c$ we have

$$\begin{aligned} |(U^* F_n D(U) F_n, f)| &\leqq \|f\|_1 \, \|U^* F_n D(U) F_n\| \\ &\leqq \|f\|_1 \, \|F_n D(U) F_n\| \\ &\leqq \frac{1}{2} N_\phi(T) \|f\|_1 \quad \text{for all} \quad f \in \mathscr{C}_1 \,. \end{aligned}$$

Since $(F_n T F_n, f) = M_U(F_n U^* D(U) F_n, f) = M_U(U^* F_n D(U) F_n, f)$, we obtain $|(F_n T F_n, f)| \leqq (1/2)N_\phi(T)\|f\|_1$, for all $f \in \mathscr{C}_1$. This implies that $\|F_n T F_n\| \leqq (1/2)N_\phi(T)$, a contradiction. Thus the claim is established.

Since $A_n \in \mathscr{A}_c$, $A_n F_n = F_n A_n$ for all n. The fact that the (F_n) are mutually orthogonal implies that $\sum_n A_n F_n$ converges strongly in $\mathscr{A}_c$; let $A = \sum_n A_n F_n$. Observe that $F_n D(A) F_n = F_n(AT - TA)F_n =$

$F_n(A_nT - TA_n)F_n = F_nD(A_n)F_n$, for all n. Hence $\|F_nD(A)F_n\| \geqq (1/2)N_\phi(T)$. For any projection E in $\mathscr{F}_\psi$ (if Hypothesis A is satisfied) or $\mathscr{F}_\phi$ (if Hypothesis B is satisfied), we must therefore have $\|ED(A)E\| \geqq (1/2)N_\phi(T)$. Since $N_\phi(T) > 0$, this implies that $D(A) \notin \mathscr{I}$. But $D(A) = AB - BA \in \mathscr{I}$ by hypothesis, a contradiction. Thus we must have $T \in \mathscr{I}$ and the theorem is proven.

REFERENCES

1. W. Arveson, *Operator algebras and invariant subspaces*, Ann. of Math., **100** (1974), 433-532.

2. G. Birkhoff, *Lattice Theory*, 3rd ed. Amer. Math. Soc. Colloquium Publications, 25, Providence, R. I.

3. A. Hopenwasser, *The radical of a reflexive operator algebra*, Pacific J. Math., **65** (1976), 375-392.

4. J. Kelley, *General Topology*, D. Van Nostrand Company, New York.

5. E. C. Lance, *Some properties of nest algebras*, Proc. London Math. Soc., (3), **19** (1969), 45-68.

6. D. Larson, *On the structure of certain reflexive operator algebras*, To appear, J. Functional Anal.

7. J. Ringrose, *On some algebras of operators*, Proc. London Math. Soc., (3), **15** (1965), 61-83.

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THE UNIVERSITY OF ALABAMA
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LINCOLN, NE 68588

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