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In [5], we proved the following infinite version of the Belluce-kirk theorem [1]:

THEOREM 1 [5]. *Let K be a nonempty weakly compact convex subset of a Banach space and assume that K possesses normal structure. Let F be a commutative family of nonexpansive self-mappings of K. Then $\mathscr{F}$ has a common fixed point.*

Fuchssteiner [3] recently proved an iteration theorem on partially ordered sets and derived several known fixed point theorems as consequences. This note is to respond to a final remark in [3]. We show that Theorem 1, indeed a more general one, can be proved without making use of the axiom of choice. We shall make use of the following theorem which can be proved constructively [2, Theorem I.2.5].

THEOREM 2 (*Zermelo* [7]). *Let $f: E \to E$ have the property that $f(x) \geqq x$ where $(E, \leqq)$ is a nonempty partially ordered set with the additional properties*:

(i) *If $a \leqq b$ and $b \leqq a$ then $a = b$;*

(ii) *Every chain in E has a least upper bound. Then f has a fixed point in E.*

Let (X, d) be a metric space and let $\{B_\alpha: \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of X, i.e., Λ is a directed set and $B_\alpha \subseteqq B_\beta$ if $\alpha \geqq \beta$. For each $x \in X$, let

$$r(x) = \lim_\alpha \sup\{d(x, y): y \in B_\alpha\} = \inf_\alpha \sup\{d(x, y) y \in B_\alpha\}$$

and

$$r = \inf\{r(x): x \in X\} .$$

The set $\{x \in X: r(x) = r\}$ (the number r) will be called the asymptotic center (asymptotic radius) of $\{B_\alpha: \alpha \in \Lambda\}$ w.r.t. X. For a set C in a topological space, cl(C) will denote its closure. A topological semigroup S is said to be left reversible if any two nonempty closed right ideals of S have a nonvoid intersection (cf. [4]). An action of a topological semigroup S on X is a mapping ψ from $S \times X$ into X denoted by $\psi(s, x) = s(x)$ such that $(s_1 s_2)(x) = s_1(s_2(x))$ for all $s_1, s_2 \in S$,

$x \in X$. The action is separately continuous if ψ is continuous in each of the variables when the other is held fixed. An action of S on X is nonexpansive if for each $s \in S$, the mapping from X into X defined by $x \to s(x)$ is nonexpansive. If S is a left reversible topological semigroup, and we put $s \geqq t$ if $sS \subseteqq \mathrm{cl}(tS)$, then $(S, \geqq)$ becomes a directed set (see [4]).

The proof of the next lemma makes use of Theorem 1 in [6]. Note that this theorem was proved constructively.

LEMMA 1. *Let K be defined as in Theorem 1 and let S be a left reversible topological semigroup of nonexpansive, separately continuous actions on K. For each $s \in S$, let $W_s = \mathrm{cl}(sS(K)) = \mathrm{cl}\{st(x): t \in S, x \in K\}$. If K contains more than one point, then the family $W = \{W_s: s \in S\}$ is a decreasing net of subsets in K whose asymptotic center in K is a closed convex S-invariant proper subset of K.*

Proof. If $s \geqq t$, then by making use of the continuity of $s \to s(x)$ for a fixed x, one can easily show that $sS(x) \subseteqq \mathrm{cl}(tS(x))$ and hence $W_s \subseteqq W_t$. Thus $\{W_s: s \in S\}$ forms a decreasing net of sets in K. By Theorem 1 in [6], the asymptotic center C of W w.r.t. K is a closed convex proper subset of K. Assume that r is the asymptotic radius and that x is in the asymptotic center. If $\|x - y\| \leqq r + \varepsilon$ for every $y \in W_t$, then for each $s \in S$, $\|s(x) - z\| \leqq r + \varepsilon$ for all $z \in W_{st}$ by the nonexpansiveness of s. It follows that C is an S-invariant set.

THEOREM 3 [6]. *Let K and S be defined as in Lemma 1. Then S has a common fixed point.*

Proof. Let $X = \{Y \subseteqq K: \phi \neq Y = \overline{\mathrm{Co}}(Y), S(Y) \subseteqq Y\}$. Order X by putting $Y_1 \leqq Y_2$ if and only if $Y_1 \supseteqq Y_2$. $(X, \leqq)$ satisfies the conditions in Theorem 2. For each $Y \in X$, let $f(Y)$ be the asymptotic center of $\{W_s: s \in S\}$ w.r.t. Y, where $W_s = \mathrm{cl}(sS(Y))$. Since $f(Y) \geqq Y$ for $Y \in X$, it follows from Theorem 2 that f has fixed point, i.e., there exists $Y_0 \in X$ such that $f(Y_0) = Y_0$. By Lemma 1, Y_0 is a singleton. Therefore, Y_0 consists of one common fixed point of S.

REMARK. Obviously, Theorem 3 can also be proved by the iteration theorem in [3]. Theorem 1 is a special case of Theorem 3 when S is a discrete commutative semigroup generated by $\mathscr{T}$.

REFERENCES

1. L. P. Belluce and W. A. Kirk, *Fixed-point theorems for families of contraction mappings*, Pacific J. Math., **18** (1966), 213-217.

2. N. Dunford and T. Schwartz, *Linear Operators*, New York-London-Sydney, Interscience Publishers, (1958).

3. B. Fuchssteiner, *Iterations and fixpoints*, Pacific J. Math., **68** (1977), 73-80.

4. R. D. Holmes and A. T. Lau, *Nonexpansive actions of topological semigroups and fixed points*, J. London Math. Soc., (2) **5** (1972), 330-336.

5. T. C. Lim, *A fixed-point theorem for families of nonexpansive mappings*, Pacific J. Math., **53** (1974), 487-493.

6. ———, *Characterizations of normal structure*, Proc. Amer. Math. Soc., **43** (1974), 313-319.

7. E. Zermelo, *Neuer Beweis fur die Moglichkeit einer Wohlordnung*, Math. Ann., **65**, (1908), 107-128.

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