

Pacific Journal of Mathematics

**A CONSTRUCTIVE PROOF OF THE INFINITE VERSION OF
THE BELLUCE-KIRK THEOREM**

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In [5], we proved the following infinite version of the Belluce-kirk theorem [1]:

THEOREM 1 [5]. *Let K be a nonempty weakly compact convex subset of a Banach space and assume that K possesses normal structure. Let F be a commutative family of nonexpansive self-mappings of K . Then \mathcal{F} has a common fixed point.*

Fuchssteiner [3] recently proved an iteration theorem on partially ordered sets and derived several known fixed point theorems as consequences. This note is to respond to a final remark in [3]. We show that Theorem 1, indeed a more general one, can be proved without making use of the axiom of choice. We shall make use of the following theorem which can be proved constructively [2, Theorem I.2.5].

THEOREM 2 (Zermelo [7]). *Let $f: E \rightarrow E$ have the property that $f(x) \geq x$ where (E, \leq) is a nonempty partially ordered set with the additional properties:*

- (i) *If $a \leq b$ and $b \leq a$ then $a = b$;*
- (ii) *Every chain in E has a least upper bound. Then f has a fixed point in E .*

Let (X, d) be a metric space and let $\{B_\alpha: \alpha \in A\}$ be a decreasing net of bounded subsets of X , i.e., A is a directed set and $B_\alpha \subseteq B_\beta$ if $\alpha \geq \beta$. For each $x \in X$, let

$$r(x) = \limsup_a \{d(x, y): y \in B_\alpha\} = \inf_a \sup \{d(x, y) | y \in B_\alpha\}$$

and

$$r = \inf \{r(x): x \in X\} .$$

The set $\{x \in X: r(x) = r\}$ (the number r) will be called the asymptotic center (asymptotic radius) of $\{B_\alpha: \alpha \in A\}$ w.r.t. X . For a set C in a topological space, $\text{cl}(C)$ will denote its closure. A topological semigroup S is said to be left reversible if any two nonempty closed right ideals of S have a nonvoid intersection (cf. [4]). An action of a topological semigroup S on X is a mapping ψ from $S \times X$ into X denoted by $\psi(s, x) = s(x)$ such that $(s_1 s_2)(x) = s_1(s_2(x))$ for all $s_1, s_2 \in S$,

$x \in X$. The action is separately continuous if ψ is continuous in each of the variables when the other is held fixed. An action of S on X is nonexpansive if for each $s \in S$, the mapping from X into X defined by $x \rightarrow s(x)$ is nonexpansive. If S is a left reversible topological semigroup, and we put $s \geq t$ if $sS \subseteq \text{cl}(tS)$, then (S, \geq) becomes a directed set (see [4]).

The proof of the next lemma makes use of Theorem 1 in [6]. Note that this theorem was proved constructively.

LEMMA 1. *Let K be defined as in Theorem 1 and let S be a left reversible topological semigroup of nonexpansive, separately continuous actions on K . For each $s \in S$, let $W_s = \text{cl}(sS(K)) = \text{cl}\{st(x) : t \in S, x \in K\}$. If K contains more than one point, then the family $W = \{W_s : s \in S\}$ is a decreasing net of subsets in K whose asymptotic center in K is a closed convex S -invariant proper subset of K .*

Proof. If $s \geq t$, then by making use of the continuity of $s \rightarrow s(x)$ for a fixed x , one can easily show that $sS(x) \subseteq \text{cl}(tS(x))$ and hence $W_s \subseteq W_t$. Thus $\{W_s : s \in S\}$ forms a decreasing net of sets in K . By Theorem 1 in [6], the asymptotic center C of W w.r.t. K is a closed convex proper subset of K . Assume that r is the asymptotic radius and that x is in the asymptotic center. If $\|x - y\| \leq r + \varepsilon$ for every $y \in W_t$, then for each $s \in S$, $\|s(x) - z\| \leq r + \varepsilon$ for all $z \in W_{st}$ by the nonexpansiveness of s . It follows that C is an S -invariant set.

THEOREM 3 [6]. *Let K and S be defined as in Lemma 1. Then S has a common fixed point.*

Proof. Let $X = \{Y \subseteq K : \phi \neq Y = \overline{\text{Co}}(Y), S(Y) \subseteq Y\}$. Order X by putting $Y_1 \leq Y_2$ if and only if $Y_1 \supseteq Y_2$. (X, \leq) satisfies the conditions in Theorem 2. For each $Y \in X$, let $f(Y)$ be the asymptotic center of $\{W_s : s \in S\}$ w.r.t. Y , where $W_s = \text{cl}(sS(Y))$. Since $f(Y) \geq Y$ for $Y \in X$, it follows from Theorem 2 that f has fixed point, i.e., there exists $Y_0 \in X$ such that $f(Y_0) = Y_0$. By Lemma 1, Y_0 is a singleton. Therefore, Y_0 consists of one common fixed point of S .

REMARK. Obviously, Theorem 3 can also be proved by the iteration theorem in [3]. Theorem 1 is a special case of Theorem 3 when S is a discrete commutative semigroup generated by \mathcal{S} .

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