A CONSTRUCTIVE PROOF OF THE INFINITE VERSION OF THE BELLUCE-KIRK THEOREM

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TECK-CHEONG LIM

In [5], we proved the following infinite version of the Belluce-Kirk theorem [1]:

**Theorem 1** [5]. Let $K$ be a nonempty weakly compact convex subset of a Banach space and assume that $K$ possesses normal structure. Let $F$ be a commutative family of nonexpansive self-mappings of $K$. Then $\mathcal{F}$ has a common fixed point.

Fuchssteiner [3] recently proved an iteration theorem on partially ordered sets and derived several known fixed point theorems as consequences. This note is to respond to a final remark in [3]. We show that Theorem 1, indeed a more general one, can be proved without making use of the axiom of choice. We shall make use of the following theorem which can be proved constructively [2, Theorem I.2.5].

**Theorem 2** (Zermelo [7]). Let $f: E \to E$ have the property that $f(x) \preceq x$ where $(E, \preceq)$ is a nonempty partially ordered set with the additional properties:

(i) If $a \preceq b$ and $b \preceq a$ then $a = b$;
(ii) Every chain in $E$ has a least upper bound. Then $f$ has a fixed point in $E$.

Let $(X, d)$ be a metric space and let $\{B_{\alpha}: \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of $X$, i.e., $\Lambda$ is a directed set and $B_{\alpha} \subseteq B_{\beta}$ if $\alpha \geq \beta$. For each $x \in X$, let

$$r(x) = \lim_{\alpha} \sup_{a} d(x, y): y \in B_{\alpha} = \inf_{a} \sup_{a} d(x, y): y \in B_{\alpha}$$

and

$$r = \inf\{r(x): x \in X\}.$$  

The set $\{x \in X: r(x) = r\}$ (the number $r$) will be called the asymptotic center (asymptotic radius) of $\{B_{\alpha}: \alpha \in \Lambda\}$ w.r.t. $X$. For a set $C$ in a topological space, $\text{cl}(C)$ will denote its closure. A topological semigroup $S$ is said to be left reversible if any two nonempty closed right ideals of $S$ have a nonvoid intersection (cf. [4]). An action of a topological semigroup $S$ on $X$ is a mapping $\psi$ from $S \times X$ into $X$ denoted by $\psi(s, x) = s(x)$ such that $(s_{1}s_{2})(x) = s_{1}(s_{2}(x))$ for all $s_{1}, s_{2} \in S$. 

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The action is separately continuous if $\psi$ is continuous in each of the variables when the other is held fixed. An action of $S$ on $X$ is nonexpansive if for each $s \in S$, the mapping from $X$ into $X$ defined by $x \to s(x)$ is nonexpansive. If $S$ is a left reversible topological semigroup, and we put $s \geq t$ if $sS \subseteq \text{cl}(tS)$, then $(S, \geq)$ becomes a directed set (see [4]).

The proof of the next lemma makes use of Theorem 1 in [6]. Note that this theorem was proved constructively.

**Lemma 1.** Let $K$ be defined as in Theorem 1 and let $S$ be a left reversible topological semigroup of nonexpansive, separately continuous actions on $K$. For each $s \in S$, let $W_s = \text{cl}(sS(K)) = \text{cl}\{st(x) : t \in S, x \in K\}$. If $K$ contains more than one point, then the family $W = \{W_s : s \in S\}$ is a decreasing net of subsets in $K$ whose asymptotic center in $K$ is a closed convex $S$-invariant proper subset of $K$.

**Proof.** If $s \geq t$, then by making use of the continuity of $s \to s(x)$ for a fixed $x$, one can easily show that $sS(x) \subseteq \text{cl}(tS(x))$ and hence $W_s \subseteq W_t$. Thus $\{W_s : s \in S\}$ forms a decreasing net of sets in $K$. By Theorem 1 in [6], the asymptotic center $C$ of $W$ w.r.t. $K$ is a closed convex proper subset of $K$. Assume that $r$ is the asymptotic radius and that $x$ is in the asymptotic center. If $||x - y|| \leq r + \varepsilon$ for every $y \in W_s$, then for each $s \in S$, $||s(x) - z|| \leq r + \varepsilon$ for all $z \in W_s$ by the nonexpansiveness of $s$. It follows that $C$ is an $S$-invariant set.

**Theorem 3 [6].** Let $K$ and $S$ be defined as in Lemma 1. Then $S$ has a common fixed point.

**Proof.** Let $X = \{Y \subseteq K : \phi \neq Y = \overline{\text{Co}(Y)}, S(Y) \subseteq Y\}$. Order $X$ by putting $Y_1 \leq Y_2$ if and only if $Y_1 \supseteq Y_2$. $(X, \leq)$ satisfies the conditions in Theorem 2. For each $Y \in X$, let $f(Y)$ be the asymptotic center of $\{W_s : s \in S\}$ w.r.t. $Y$, where $W_s = \text{cl}(sS(Y))$. Since $f(Y) \geq Y$ for $Y \in X$, it follows from Theorem 2 that $f$ has fixed point, i.e., there exists $Y_0 \in X$ such that $f(Y_0) = Y_0$. By Lemma 1, $Y_0$ is a singleton. Therefore, $Y_0$ consists of one common fixed point of $S$.

**Remark.** Obviously, Theorem 3 can also be proved by the iteration theorem in [3]. Theorem 1 is a special case of Theorem 3 when $S$ is a discrete commutative semigroup generated by $\mathcal{F}$. 
References


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University of Chicago

Chicago, IL 60637
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