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**ASYMPTOTICALLY STABLE DYNAMICAL SYSTEMS ARE  
LINEAR**

ROGER MCCANN

## ASYMPTOTICALLY STABLE DYNAMICAL SYSTEMS ARE LINEAR

ROGER C. MCCANN

**If  $\pi$  is a dynamical system on a locally compact metric space  $X$  which has a globally asymptotically stable critical point, then  $\pi$  can be embedded into a dynamical system on  $l_2$  which is derived from a linear differential equation. If  $X$  is  $n$ -dimensional, then  $l_2$  may be replaced by  $R^{2n}$ .**

Throughout this paper  $R$  and  $R^+$  will denote the reals and non-negative reals respectively. A dynamical system on a topological space  $X$  is a continuous mapping:  $\pi: X \times R \rightarrow X$  such that (where  $\pi(x, t) = x\pi t$ )

- (i)  $x\pi 0 = x$  for all  $x \in X$ ,
- (ii)  $(x\pi t)\pi s = x\pi(t + s)$  for all  $x \in X$  and  $s, t \in R$ .

A point  $p \in X$  is called a critical point of  $\pi$  if  $p\pi t = p$  for every  $t \in R$ . A subset  $S$  of  $X$  is called a section with respect to  $\pi$  if  $(S\pi t) \cap S = \emptyset$  for every  $t \neq 0$ . A subset  $S$  of  $X$  is said to be a section for  $Y \subset X$  if  $S$  is a section and  $\{x\pi t: x \in S, t \in R\} = Y$ . A compact subset  $M$  of  $X$  is said to be stable with respect to  $\pi$  if for any neighborhood  $U$  of  $M$  there is a neighborhood  $V$  of  $M$  such that  $\{x\pi t: x \in V, t \in R^+\} \subset U$ . The compact subset  $M$  of  $X$  is said to be a global attractor if for any neighborhood  $U$  of  $M$  and  $x \in X$ , there is a  $c \in R$  such that  $x\pi t \in U$  whenever  $c \leq t$ . If  $M$  is a stable global attractor, then  $M$  is said to be globally asymptotically stable.

Let  $X$  and  $Y$  be topological spaces on which are defined dynamical systems  $\pi$  and  $\rho$  respectively. We say that  $\pi$  can be embedded into  $\rho$  if there is a homeomorphism  $h$  of  $X$  onto a subset of  $Y$  such that  $h(x\pi t) = h(x)\rho t$  for every  $x \in X$  and  $t \in R$ . In the special case  $h(X) = Y$  we will say that  $\pi$  is isomorphic to  $\rho$ .

The set of all sequences  $x = \{x_1, x_2, \dots, x_n, \dots\}$  of real numbers such that  $\sum_{n=1}^{\infty} x_n^2$  converges is denoted by  $l_2$ . If addition and scalar multiplication are defined coordinatewise and if a norm is defined by  $\|x\| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$ , then  $l_2$  is a real Banach space.

Throughout the remainder of this paper  $X$  will denote a locally compact metric space.

Let  $p \in X$  be a globally asymptotically stable critical point with respect to the dynamical system  $\pi$  and let  $U$  be a compact neighborhood of  $p$ . It is known ([1, Theorem 2.7.14]) that there is a continuous (Liapunov) function  $v: X \rightarrow R^+$  such that

- (i)  $v(x) = 0$  if and only if  $x = p$ ,
- (ii)  $v(x\pi t) = e^{-t}v(x)$  for  $x \in X - \{p\}$  and  $t > 0$ .

Let  $a > 0$  be so small that  $v^{-1}(a) \subset U$  and set  $S = v^{-1}(a)$ . The following lemma is also well known and is easily verified.

LEMMA 1.  $S$  is a compact section for  $X - \{p\}$ . Moreover, the mapping  $\Gamma: X - \{p\} \rightarrow R$  defined by  $x\pi\Gamma(x) \in S$  is continuous.

Since  $S$  is compact it is separable. Let  $d$  denote a metric on  $X$  and let  $\{x_n\}$  be a countable dense subset of  $S$ . We define a countable number of continuous functions  $f_n: S \rightarrow R^+$  by

$$f_n(x) = d(x, x_n).$$

LEMMA 2. If  $f_n(x) \leq f_n(y)$  for every  $n$ , then  $x = y$ .

*Proof.* Suppose that  $x \neq y$ . Let  $r = d(x, y)$  and  $B = \{z: d(z, y) \leq r/4\}$ . Since  $\{x_n\}$  is dense in  $S$  there is a  $k$  such that  $x_k \in B$ . Then

$$f_k(y) = d(y, x_k) \leq \frac{1}{4}r < \frac{3}{4}d(x, x_k) = f_k(x).$$

A similar argument shows that there is a  $j$  such that  $f_j(x) < f_j(y)$ . The desired result follows directly.

LEMMA 3. The mapping  $h: S \rightarrow l_2$  defined by

$$h(x) = \left( f_1(x), \frac{1}{2}f_2(x), \dots, \frac{1}{n}f_n(x), \dots \right)$$

is a homeomorphism of  $S$  onto  $h(S)$ .

*Proof.* Since  $S$  is compact the mapping  $d$  restricted to  $S \times S$  is uniformly continuous and bounded. Hence, the set of mappings  $\{f_n\}$  is equicontinuous and equibounded. For each  $x \in S$ ,  $h(x) \in l_2$  since  $\{f_n\}$  is equibounded. Since  $\{f_n\}$  is equicontinuous,  $h$  is continuous. It follows immediately from Lemma 2 that  $h$  is one-to-one. A continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism.

Let  $c \in (0, 1)$  and define a dynamical system  $\rho$  on  $l_2$  by  $x\rho t = c^t x$ . This dynamical system can be interpreted as being derived from the linear differential equation  $dy/dt = ky$ ,  $y(0) = x$ , where  $k = \ln c$ .

LEMMA 4. If  $x, y \in S$  are such that  $h(x) = h(y)\rho t$  for some  $t \in R$ , then  $x = y$  and  $t = 0$ .

*Proof.* Suppose that  $h(x) = h(y)\rho t = c^t h(y)$  for some  $t \in R$ . Without loss of generality we may assume that  $t \geq 0$ . Then  $f_n(x) = c^t f_n(y) \leq f_n(y)$  for every  $n$ . By Lemma 2,  $x = y$ . If  $x = y$ , clearly  $t = 0$ .

LEMMA 5. *The mapping  $H: X \rightarrow I_2$  defined by*

$$H(x) = \begin{cases} 0 & \text{if } x = p, \\ e^{-r(x)}h(x\pi Y(x)) & \text{if } x \in X - \{p\} \end{cases}$$

where  $Y$  is the mapping defined in Lemma 1, is a homeomorphism of  $X$  onto  $h(X)$ .

*Proof.* If  $H(x) = H(y)$ ,  $x \neq 0 \neq y$ , then

$$e^{-r(y)}h(y\pi Y(y)) = e^{-r(x)}h(x\pi Y(x))$$

so that

$$h(y\pi Y(y)) = h(x\pi Y(x))\rho(Y(x) - Y(y)).$$

By Lemma 4,  $y\pi Y(y) = x\pi Y(x)$  and  $Y(x) - Y(y) = 0$ . Hence,  $x = y$  and  $H$  is one-to-one. Since  $\pi$ ,  $Y$ , and  $h$  are continuous on  $X - \{p\}$ ,  $H$  is continuous on  $X - \{p\}$ . We will now show that  $H$  is continuous at  $p$ . Let  $\{z_i\}$  be a sequence in  $X - \{p\}$  which converges to  $p$ . We will first show that  $Y(z_i) \rightarrow -\infty$ . Since  $z_i\pi Y(z_i) \in S$  and  $V(z) = a$  for each  $z \in S$ , we have

$$0 < a = V(z_i\pi Y(z_i)) = e^{-r(z_i)}v(z_i).$$

We must have  $Y(z_i) \rightarrow -\infty$  since  $v(z_i) \rightarrow 0$ . Now

$$H(z_i) = e^{-r(z_i)}h(z_i\pi Y(z_i)) \longrightarrow 0$$

because  $c \in (0, 1)$ ,  $Y(z_i) \rightarrow -\infty$ , and  $h(S)$  is compact with  $0 \notin h(S)$ . This proves that  $H$  is continuous at  $p$  so that  $H$  is continuous. Note that  $H(x) = h(x\pi Y(x))\rho(-Y(x))$ . A short calculation shows that  $H^{-1}(H(x)) = h^{-1}[H(x)\rho(Y(x))]\pi(-Y(x))$  whenever  $x \neq p$ . Since  $h^{-1}$ ,  $H$ ,  $\rho$ ,  $Y$ , and  $\pi$  are continuous on their respective domains,  $H^{-1}$  is continuous on  $H(X) - \{0\}$ . Let  $\{x_i\}$  be any sequence such that  $H(x_i) \rightarrow 0$ . Since  $H(x_i) = e^{-r(x_i)}h(x_i\pi Y(x_i))$  and  $h(S)$  is compact with  $0 \notin h(S)$  we must have  $Y(x_i) \rightarrow -\infty$ . Then

$$0 < a = v(z_i\pi Y(z_i)) = e^{-r(x_i)}v(x_i)$$

so that we must have  $v(x_i) \rightarrow 0$ . Thus,  $x_i \rightarrow p$ . This proves that  $H^{-1}$  is continuous at 0.  $H$  is a homeomorphism.

THEOREM 6. *Let  $\pi$  be a dynamical system on a locally compact metric space  $X$  and let  $\rho_c$ ,  $0 < c < 1$ , be the dynamical system on  $I_2$  defined by  $x\rho_c t = c^t x$ . If  $\pi$  has a globally asymptotically stable critical point, then  $\pi$  can be embedded into  $\rho_c$ .*

*Proof.* In light of Lemma 5 it remains to show that  $H(x\pi t) =$

$h(x)\rho t$ . It is easy to show that  $\mathcal{Y}(x\pi t) = \mathcal{Y}(x) - t$ . Hence,

$$\begin{aligned} H(x\pi t) &= c^{-\mathcal{Y}(x)+t}h((x\pi t)\pi(\mathcal{Y}(x) - t)) \\ &= c^t c^{-\mathcal{Y}(x)}h(x\pi\mathcal{Y}(x)) \\ &= c^t h(x) \\ &= h(x)\rho t. \end{aligned}$$

If  $X$  is of finite dimension  $n$ , then  $l_2$  can be replaced by  $R^{2n}$  in Theorem 6. This may be proved as follows. Let  $S$  be a compact section for  $\pi$ . It is known that if  $A$  is compact and  $B$  is one dimensional, then  $\dim(A \times B) = \dim A + \dim B$ . This is cited in [2, page 34] and [5, page 302], and referenced as [3] in [5]. Since  $S\pi R$  is homeomorphic with  $S \times R$ , we have  $\dim S + 1 = \dim S + \dim R = \dim(S \times R) = \dim(S\pi R) \leq n$ . Hence  $\dim S \leq n - 1$ . It is known that a  $k$ -dimensional space can be embedded in  $R^{2k+1}$ , [2, page 60]. Hence,  $S$  can be embedded into  $R^{2n-1}$ . The one point compactification of  $R^{2n-1}$  is  $S^{2n-1}$ , the unit sphere in  $R^{2n}$ . Thus, there is an imbedding  $g: S \rightarrow S^{2n-1} \subset R^{2n}$ . Consider the dynamical system  $\alpha_c$  defined by the linear differential equation

$$\frac{dy}{dt} = ky, \quad y(0) = x$$

where  $y: R \rightarrow R^{2n}$  and  $k < 0$ . Then  $x\alpha_c t = c^t x$  for  $t \in R$ ,  $x \in R^{2n}$ , and  $c = e^k$ . Define  $G: X \rightarrow R^{2n}$  by

$$G(x) = \begin{cases} 0 & \text{if } x = p, \\ c^{-\mathcal{Y}(x)}g(x\pi\mathcal{Y}(x)) & \text{if } x \in X - \{p\}. \end{cases}$$

The proof that  $G$  is a homeomorphism is essentially the same as the proof of Lemma 5. With this result the proof of the following theorem is identical with that of Theorem 6.

**THEOREM 7.** *Let  $\pi$  be a dynamical system on an  $n$ -dimensional locally compact space  $X$  and  $\alpha_c$ ,  $0 < c < 1$ , be the dynamical system on  $R^{2n}$  defined by  $x\alpha_c t = c^t x$ . If  $\pi$  has a globally asymptotically stable critical point, then  $\pi$  can be embedded into  $\alpha_c$ .*

If  $S$  can be embedded into  $S^{k-1}$ , then obvious modifications of the proof of Theorem 7 show that  $\pi$  can be embedded into the dynamical system on  $R^k$  defined by  $x\alpha_c t = c^t x$ ,  $0 < c < 1$ . If  $X$  has dimension  $n$ , what is the smallest integer  $k$  such that  $S$  can be embedded into  $S^{k-1}$ ? The author does not know, but conjectures that if  $X = R^n$  then  $S$  can be embedded into  $S^{n-1}$ . If this conjecture were true then  $S$  would be homeomorphic to  $S^{n-1}$ . The proof of this, or the construc-

tion of a counterexample, appears to be difficult. However, in the case  $n = 2$ , the conjecture is true.

**THEOREM 8.** *Let  $\pi$  be a dynamical system on  $R^2$  which has a globally asymptotically stable point  $p$ . If  $S$  is any section for  $X - \{p\}$ , then  $S$  is homeomorphic to  $S^1$ .*

*Proof.* Evidently  $S$  is compact and connected. Let  $x$  and  $y$  be any two points of  $S$ . Since  $p$  is asymptotically stable  $L^-(x) = L^-(y) = \phi$ . It is easy to show that  $D = \{p\} \cup \{x\pi R\} \cup \{y\pi R\}$  is a curve which separates the plane into exactly two components. Moreover,  $S \cap D = \{x, y\}$ . Hence,  $S - \{x, y\}$  has exactly two components. A continuum whose connection is destroyed by the removal of two arbitrary points is a simple closed curve, [5, page 99].

**COROLLARY 9.** *Let  $\pi$  be a dynamical system on  $R^2$  and let  $\alpha_c$ ,  $0 < c < 1$ , be the dynamical system on  $R^2$  defined by  $x\alpha_c t = c^t x$ . If  $\pi$  has a globally asymptotically stable critical point, then  $\pi$  is isomorphic to  $\alpha_c$ .*

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