AN IMPLICIT FUNCTION THEOREM IN BANACH SPACES

IAIN RAEBURN
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We prove the following theorem:

**Theorem:** Suppose \( X, Y, \) and \( Z \) are complex Banach spaces, \( U \) and \( V \) are open sets in \( X \) and \( Y \) respectively, and \( x \in U, y \in V. \) Suppose \( f: U \to V \) and \( k: V \to Z \) are holomorphic maps with \( f(x) = y, \) \( k \circ f \) constant and range \( f'(x) = \ker k'(y) \neq \{0\}. \) Let \( D \) be a domain in \( C^n, \) \( z \in D \) and \( g: D \to Y \) be a holomorphic map with \( g(z) = y \) and \( k \circ g \) constant. Then there is an open neighborhood \( W \) of \( z \) and a holomorphic map \( h: W \to X \) such that \( h(z) = x \) and \( g|_w = f \circ h. \)

We use this result to prove an Oka principle for sections of a class of holomorphic fibre bundles on Stein manifolds whose fibres are orbits of actions of a Banach Lie group on a Banach space.

**Introduction.** Suppose \( U \) is an open set in \( C^n, \) \( x \in U, \) and \( f: U \to \text{C}^n \) is a holomorphic map such that \( f'(x) \) is surjective. Then a form of the implicit function theorem tells us that there is a neighborhood \( V \) of \( f(x) \) and a holomorphic map \( \rho: V \to U \) such that \( \rho(f(x)) = x \) and \( f \circ \rho \) is the identity on \( V. \) This theorem remains true if \( f \) is a holomorphic map of an open set \( U \) in a Banach space \( X \) into a Banach space \( Y, \) provided that \( \ker f'(x) \) is a complemented subspace of \( X. \) That this is also a necessary condition follows from the fact that \( f'(x) \circ \rho'(f(x)) \) is the identity operator on \( Y, \) so that \( \rho'(f(x)) \circ f'(x) \) is a projection of \( X \) onto \( \ker f'(x). \)

In general, implicit function theorems work well in a Banach space setting, provided that we impose suitable complementation conditions (see, for example [4]). In practice it can be very hard to find out whether a given subspace of a Banach space is complemented; our main theorem is an implicit function theorem which has no complementation hypothesis. Before we state our theorem, we shall reword the result mentioned above. Let \( X \) and \( Y \) be complex Banach spaces, \( U \) be open in \( X, \) \( x \in U, \) and \( f: U \to Y \) be a holomorphic map such that \( f'(x) \) is surjective and \( \ker f'(x) \) is a complemented subspace of \( X. \) Then if \( V \) is an open set in a Banach space \( W, \) \( w \in V, \) and \( g \) is a holomorphic map of \( V \) into \( Y \) such that \( g(w) = f(x) \), there is a neighborhood \( N \) of \( w \) and a holomorphic map \( h(= \rho \circ g) \) of \( N \) into \( X \) such that \( f \circ h = g \) on \( N. \) Our main theorem asserts that provided \( W \) is finite-dimensional, this theorem is still true without the hypothesis that \( \ker f'(x) \) be complemented. More generally, suppose there is a third Banach space \( Z \) and a holomorphic map \( k: Y \to Z \) such that \( k \circ f \) is constant and range \( f'(x) = \ker k'(f(x)). \) Let \( D \) be
an open set in $C^n$, and let $z \in D$. Then our main theorem says that if $g$ is a holomorphic map of $D$ into $\ker^{-1}(k(f(x)))$ with $g(z) = f(x)$, then there is a holomorphic map $h$ of a neighborhood $N$ of $z$ into $X$ such that $f \circ h = g|_N$. We shall prove this theorem in § 2.

Grauert [2] has proved an Oka principle for sections of a holomorphic fibre bundle over a Stein manifold with fibre a complex Lie group. Ramspott [10] has generalized this result to allow homogeneous spaces as fibres, and Bungart [1] has extended it to the case where the fibres are infinite-dimensional Lie groups. In § 3, as an application of our implicit function theorem, we shall extend the theorems of Ramspott and Bungart to allow for infinite-dimensional fibres which are the orbits of suitable actions $(g, x) \rightarrow g \cdot x$ of an infinite-dimensional Lie group $G$ on a Banach space $X$; more specifically, we demand that such an orbit $M$ also be the level set of a holomorphic map $k$ in such a way that the derivatives of the orbit map $g \rightarrow g \cdot x_0$ and $k$ form an exact sequence at $x_0 \in M$.

1. Preliminaries. Let $X$ and $Y$ be complex Banach spaces, let $U$ be an open set in $X$ and let $f$ be a continuous map of $U$ into $Y$. We say $f$ is holomorphic in $U$ if at each point of $U$ $f$ has a Fréchet derivative which is a complex linear map of $X$ into $Y$. Equivalently, $f$ is holomorphic in $U$ if for each $x \in U$ and $h \in X$ the function $z \rightarrow f(x + zh)$ is holomorphic in a neighborhood of 0 in $C$. If $f: U \rightarrow Y$ is holomorphic in $U$, then $f$ has complex Fréchet derivatives of all orders; that is, for $x \in U$ and all $n$ the $n$th derivative $f^{(n)}(x)$ exists as a complex multilinear map of $X^n$ to $Y$. We give $X^n$ the norm $\| (x_1, \ldots, x_n) \| = \sup \{ \| x_i \| \}$ and put the corresponding operator norm on $L^n(X^n, Y)$, the space of complex $n$-linear maps of $X^n$ into $Y$. If $f: U \subseteq X \rightarrow Y$ is holomorphic, it is well-known that $\limsup (\| f^{(n)}(x) \|/n!)^{1/n}$ is finite for each $x \in U$. For further details of this material, we refer to [7].

We shall use many times two differentiation techniques which are well-known in one variable; namely, the chain rule and Liebnitz' formula. Let $U$ be open in $X$, $V$ be open in $Y$, and let $f: U \rightarrow V$ and $g: V \rightarrow Z$ be differentiable. Then the chain rule [5, p. 99] says that $g \circ f$ is differentiable, and, for $x_0 \in U$, the derivative $(g \circ f)'(x_0) \in L(X, Z)$ is given by

$$(g \circ f)'(x_0)x = g'(f(x_0))[f'(x_0)x] \quad \text{for} \quad x \in X.$$ 

Let $U$ be an open set in $C$, and let $f: U \rightarrow L(Y, Z)$ and $g: U \rightarrow L(X, Y)$ be $n$ times continuously differentiable maps. Then we can define $fg: U \rightarrow L(X, Z)$ by $fg(u) = f(u) \circ g(u)$ for $u \in U$, and a special case of the product formula [5, p. 97] gives that $fg$ is differentiable and
\[(fg)'(u) = f(u) \circ g'(u) + f'(u) \circ g(u).\]

Proceeding exactly as in the scalar case, an induction argument gives us our version of Liebnitz’ formula: the function \(fg\) is \(n\) times continuously differentiable and

\[
(fg)^{(n)}(u) = \sum_{r=0}^{n} \binom{n}{r} f^{(r)}(u) \circ g^{(n-r)}(u) \quad \text{for} \quad u \in U.
\]

2. The implicit function theorem.

**Theorem 2.1.** Suppose \(X, Y,\) and \(Z\) are complex Banach spaces, \(U\) and \(V\) are open sets in \(X\) and \(Y\) respectively, and \(x \in U, y \in V.\)

Suppose \(f: U \to V\) and \(k: V \to Z\) are holomorphic maps with \(f(x) = y,\) \(k \circ f\) constant and range \(f'(x) = \ker k'(y) \neq \{0\}.\)

Let \(D\) be a domain in \(C^n, z \in D,\) and \(g: D \to Y\) be a holomorphic map with \(g(z) = y\) and \(k \circ g\) constant. Then there is an open neighborhood \(W\) of \(z\) and a holomorphic map \(h: W \to X\) such that \(h(z) = x\) and \(g \mid _W = f \circ h.\)

**Proof.** We shall assume for simplicity that \(x, y,\) and \(z\) are all 0.

By shrinking \(D\) if necessary, we may assume that \(g\) has a power series representation

\[g(z) = \sum_{|I| = 0}^{\infty} \frac{g^{(I)}(0)}{I!} z^I \quad \text{for} \quad z \in D,
\]

where \(I\) denotes the multiindex \((i_1, \ldots, i_n), z^I = z_1^{i_1} \cdots z_n^{i_n}, I! = i_1! i_2! \cdots i_n!,\) and

\[g^{(I)}(0) = \frac{\partial^{i_1}}{\partial z_1^{i_1}} \frac{\partial^{i_2}}{\partial z_2^{i_2}} \cdots \frac{\partial^{i_n}}{\partial z_n^{i_n}} g(0).
\]

We shall suppose first that such an \(h\) exists; then \(f \circ h\) is a holomorphic map of \(D\) into \(Y.\) Let \(I\) be a nonzero multiindex, and assume without loss of generality that \(i_1 > 0.\) If \(I' = (i_1 - 1, i_2, \ldots, i_n),\) then by the chain rule applied to the function \(z_i \to f \circ h(z_i, 0, \ldots, 0)\) we have

\[g^{(I)}(0) = (f \circ h)^{(I)}(0) = \left( (f' \circ h) \frac{\partial h}{\partial z_i} \right)^{(I')} (0).
\]

Now \(f' \circ h\) is a holomorphic map of \(D\) into \(L(X, Y)\) and we can regard \(\partial h/\partial z_i\) as a holomorphic map of \(D\) into \(L(C, X) \cong X,\) so our Liebnitz formula applies; we obtain

\[g^{(I)}(0) = \sum_{r=0}^{i_1-1} \binom{i_1-1}{r} \left[ \frac{\partial^r}{\partial z_i^r} (f' \circ h) \frac{\partial^{i_1-r}}{\partial z_i^{i_1-r}} h \right]^{(I - (i_1, 0, \ldots, 0))} (0).
\]
By successively applying the Liebnitz formula to the different variables, we obtain

\[ g^{(I)}(0) = \sum_{J \subseteq I} \left[ \begin{array}{c} I' \\ J \end{array} \right] (f' \circ h)^{(J)}(0) \circ h^{(I-J)}(0) \]

where

\[ \left[ \begin{array}{c} I' \\ J \end{array} \right] = \left[ \begin{array}{cccc} i_1 - 1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{array} \right] . \]

Hence if such an \( h \) exists, for all multiindices \( I \) its derivatives satisfy

\[ (f' \circ h)(0)h^{(I)}(0) = g^{(I)}(0) - \sum_{0 \leq J \subseteq I} \left[ \begin{array}{c} I' \\ J \end{array} \right] (f' \circ h)^{(J)}(0)[h^{(I-J)}(0)] . \]  

We observe that by repeating this process on the term \((f' \circ h)^{(J)}(0)\), we find that each \((f' \circ h)^{(J)}(0)h^{(I-J)}(0)\) can be written as a linear combination of points of \( Y \) of the form

\[ (f^{(j)} \circ h)(0)[h^{(L_1)}(0), \cdots, h^{(L_j)}(0)] \]

for some \( j \geq 2 \) and multiindices \( L_1, \cdots, L_j \) with \( L_i > 0 \) for all \( i \) and \( \sum_{i=1}^j L_i = I \).

We first define \( h(0) = 0 \). Then \((f' \circ h)(0) = f'(0): X \rightarrow Y\), and range \( f'(0) = \ker k'(0) \), a closed linear subspace of \( Y \). Then by the open mapping theorem there is a constant \( C \) such that for each \( y \in \text{range } f'(0) \) there exists \( x \in X \) with \( f'(0)x = y \) and \( ||x|| \leq C \cdot ||y|| \). We shall assume that \( C \cdot ||f'(0)|| \geq 1 \). We shall define \( h^{(I)}(0) \) inductively so that (1) holds and

\[ ||h^{(I)}(0)|| \leq C \left( g^{(I)}(0) - \sum_{0 \leq J \subseteq I} \left[ \begin{array}{c} I' \\ J \end{array} \right] (f' \circ h)^{(J)}(0)[h^{(I-J)}(0)] \right) \]

where by \((f' \circ h)^{(I)}(0)h^{(I-J)}(0)\) we mean the linear combination described above. We observe that for \( |I| = 1 \), \((k \circ g)^{(I)}(0) = k'(0)g^{(I)}(0)\), and since \( k \circ g \) is constant we have \( g^{(I)}(0) \in \ker k'(0) = \text{range } f'(0) \), so that we can choose \( h^{(I)}(0) \) as required. Suppose now that for all \( J \) with \( |J| < |I| \) the right hand side of (1) is in the range of \( f'(0) \) and we have chosen \( h^{(J)}(0) \) satisfying (1) and (2). For notational convenience we shall regard \( h \) as the polynomial

\[ h(z) = \sum_{0 \leq J < I} \frac{h^{(J)}(0)}{J!} z^J \text{ for } z \in D, \]

so that for \( J < I \) the terms \((f' \circ h)^{(J)}(0)\), \((k' \circ f \circ h)^{(J)}(0)\) and so on all make sense, and all such terms agree with those given by expanding and using (1). To show that we can define \( h^{(I)}(0) \) as required it is enough to show that
\[(3) \quad k'(0) \left[ g^{(I)}(0) - \sum_{0 < J \leq I'} \left[ \frac{I'}{J} \right] (f' \circ h)^{(J)}(0)[h^{(I-J)}(0)] \right] = 0.\]

Since \(k \circ g = 0\), \((k \circ g)^{(I)}(0) = 0\), and so as before

\[k'(0)g^{(I)}(0) = - \sum_{0 < K \leq I'} \left[ \frac{I'}{K} \right] (k' \circ g)^{(K)}(0)[g^{(I-K)}(0)].\]

By the inductive hypothesis

\[g^{(I-K)}(0) = \sum_{0 < L \leq (I-K)} \left[ \frac{I'}{L} \right] (f' \circ h)^{(L)}(0)[h^{(I-K-L)}(0)]\]

for all \(K \leq I\). Hence

\[(4) \quad k'(0)g^{(I)}(0) = - \sum_{0 < K \leq I'} \sum_{0 \leq L \leq (I-K)} \left[ \frac{I'}{K} \right] \left[ \frac{I'}{L} \right] (k' \circ f \circ h)^{(K)}(0)(f' \circ h)^{(L)}(0)[h^{(I-K-L)}(0)]\]

since all derivatives of \(f \circ h\) of less than \(I\)th order are those of \(g\). Now \((k \circ f)' = 0\), and so

\[0 = (k \circ f)' \circ h = (k' \circ f \circ h)(f' \circ h).\]

Thus for every \(J < I\)

\[0 = ((k' \circ f \circ h)(f' \circ h))^{(J)}(0) = \sum_{0 < M \leq J} \left[ \frac{I'}{M} \right] (k' \circ f \circ h)^{(M)}(0)(f' \circ h)^{(J-M)}(0),\]

and so as elements of \(L(X, Z)\),

\[k'(0)(f' \circ h)^{(J)}(0) = - \sum_{0 < M \leq J} \left[ \frac{I'}{M} \right] (k' \circ f \circ h)^{(M)}(0)(f' \circ h)^{(J-M)}(0).\]

Thus

\[- \sum_{0 < J \leq I'} \left[ \frac{I'}{J} \right] k'(0) \circ (f' \circ h)^{(J)}(0)[h^{(I-J)}(0)] = \sum_{0 < J \leq I'} \sum_{0 < M \leq J} \left[ \frac{I'}{J} \right] \left[ \frac{J}{M} \right] (k' \circ f \circ h)^{(M)}(0)(f' \circ h)^{(J-M)}(0)[h^{(I-J)}(0)]\]

\[= \sum_{0 < M \leq I'} \sum_{M \leq J < I'} \left[ \frac{I'}{M} \right] \left[ \frac{I'}{J} \right] \left[ \frac{J}{M} \right] (k' \circ f \circ h)^{(M)}(0)(f' \circ h)^{(J-M)}(0)[h^{(I-J)}(0)]\]

\[= - k'(0)g^{(I)}(0) \quad \text{by (4)}.\]

Thus we have proved (3) and we can define \(h^{(I)}(0)\) to satisfy (1) and (2).
Now define
\[ h(z) = \sum_{|I|=0}^{\infty} \frac{h^{(I)}(0)}{I!} z^I \quad \text{for} \quad z \in \mathbb{C}^n. \]

If we can show that this series converges absolutely in some neighborhood of 0, then we shall be done. Now, by (2), if \( I \) is a multi-index
\[ \| h^{(I)}(0) \| \leq k \left\{ \| g^{(I)}(0) \| + \sum_{0 < j \leq I} I_j \right\}, \]
where \( I_j \) is a linear combination of terms of the form
\[ x = \| f^{(I)}(0) \| \| h^{(L_1)}(0) \| \cdots \| h^{(L_j)}(0) \|, \]
for some \( j \geq 2, L_i, \cdots, L_j > 0 \) with \( \sum_{i=1}^j L_i = I \). Define \( F(0) = \| f(0) \| = 0, F^{(1)}(0) = \| f'(0) \|, \) and \( F^{(n)}(0) = -\| f^{(n)}(0) \| \) for \( n > 1 \). Since \( f: U \to Y \) is holomorphic, we have that \( \limsup \left( \frac{\| f^{(n)}(0) \|}{n!} \right)^{1/n} \) is finite, and so there is an open neighborhood \( V \) of 0 such that
\[ F(t) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} t^n \quad \text{for} \quad t \in V, \]
defines a holomorphic function of \( V \) into \( C \). Similarly we define a holomorphic map \( G \) of \( D \) into \( C \) by \( G^{(I)}(0) = \| g^{(I)}(0) \| \) for all multi-indices \( I \) and
\[ G(z) = \sum_{|I|=0}^{\infty} \frac{G^{(I)}(0)}{I!} z^I \quad \text{for} \quad z \in D. \]

Since \( F''(0) = \| f''(0) \| \neq 0 \), by the inverse function theorem for one variable there is a neighborhood \( W \) of 0 in \( D \), and a holomorphic map \( H \) of \( W \) into \( C \) with \( F \circ H = G \big|_W \) and \( H(0) = 0 \). By differentiating \( F \circ H \) using the chain rule and Liebnitz' formula, we obtain
\[ F''(0)H^{(I)}(0) = G^{(I)}(0) - \sum_{0 < J \subseteq I} I_j \left( F'' \circ H \right)^{(J)}(0) H^{(I-J)}(0). \]

Again, we expand each \( (F' \circ H)^{(J)}(0) \) in the same way and obtain
\[ F''(0)H^{(I)}(0) = G^{(I)}(0) + \sum_{0 < J \subseteq I} I_j \left( F'' \right)^{(J)} \xi^{J'}, \]
where \( \xi^{J'} \) is identical to \( \chi^{J'} \) with each \( \| h^{(L_i)}(0) \| \) replaced by \( H^{(L_i)}(0) \).

We shall now prove that there is a constant \( M \) such that for all multiindices \( I \)
\[ \| h^{(I)}(0) \| \leq M^{2|I| - 1} H^{(I)}(0). \]
In fact, take $M = C \|f'(0)\|$ which was chosen to be $\geq 1$. The inequality is trivially true for $I = 0$. Suppose (6) holds for all $J$ with $|J| < |I|$. Then for each term $\chi$ of $\chi_j'$

\[(7)\quad \chi \leq \|f^{(j)}(0)\| \prod_{i=1}^{\lfloor |J|/2 \rfloor} H^{(L_i^j)}(0) \cdots H^{(L_j^j)}(0) \leq M^{2|I|-2} \|f^{(j)}(0)\| H^{(|I|)}(0) \cdots H^{(L_j^j)}(0),\]

since $j \geq 2$ and $\sum L_i = I$. The right hand side of (7) is $M^{2|I|-2}$ times the term of $\xi_j''$ corresponding to $\chi$, and so we have

\[
\| h^{(I)}(0) \| \leq k \left\{ \| g^{(J)}(0) \| + \sum_{0 < |J| < |I|} \left[ \frac{I'}{J} \right] M^{2|I|-2} \xi_j^{I'} \right\} \\
\leq k M^{2|I|-2} F'(0) H^{(|I|)}(0) = k M^{2|I|-2} \| f'(0) \| H^{(|I|)}(0)
\]

as required. Since $H$ is holomorphic in a polydisc, from (6) it follows that the power series (5) converges in a polydisc about 0, and the proof is complete.

3. Sections of holomorphic fibre bundles. We shall start this section with a couple of technical results which we shall need later. The first is an application of the mean value theorem [5, p. 103].

**Lemma 3.1.** Let $X$ and $Y$ be Banach spaces, let $U$ be open in $X$, and let $f: U \to Y$ be continuously differentiable. Let $K$ be a compact Hausdorff space and define $\tilde{f}: C(K, U) \to C(K, Y)$ by $(\tilde{f}\phi)(k) = f(\phi(k))$ for $\phi \in C(K, U)$ and $k \in K$. Then $\tilde{f}$ is continuously differentiable and for $\phi \in C(K, U)$

\[(\tilde{f}'(\phi))\psi)(k) = [\tilde{f}'(\phi(k))]\psi(k) \quad \text{for all } \psi \in C(K, X), \quad k \in K.
\]

Let $X$ and $Y$ be Banach spaces, $T \in L(X, Y)$ and suppose $T$ has closed range. Then by the open mapping theorem $T: X \to \text{range } T$ has a bounded inverse $T^{-1}$. Call $\| T^{-1} \|$ the inversion constant of $T$. Let $K$ be a compact Hausdorff space, and let $T: K \to L(X, Y)$ be a continuous map. Then $T$ induces a bounded linear map $\tilde{T}: C(K, X) \to C(K, Y)$, where

\[(\tilde{T}f)(k) = T(k)f(k) \quad \text{for } f \in C(K, X), \quad k \in K.
\]

**Lemma 3.2.** Suppose that $T(k)$ has closed range for each $k \in K$ and suppose that the inversion constant of $T(k)$ is less than $M$ for each $k \in K$. Then

1. If $g \in C(K, Y)$ satisfies $g(k) \in \text{range } T(k)$ for all $k \in K$, then for each $\varepsilon > 0$ there is $f \in C(K, X)$ with $\| f \| \leq M \| g \|$ and $\| \tilde{T}f - g \| < \varepsilon$. 

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(2) $T$ has closed range and the inversion constant of $T$ is less than $2M$.

Proof. Part (1) follows by a standard partition of unity argument. To prove part (2) it is enough to show that for each $g \in C(K, Y)$ with $g(k) \in \text{range } T(k)$ for all $k \in K$, there is some $f \in C(K, X)$ with $Tf = g$ and $\|f\| \leq 2M \|g\|$. Let such a $g$ be given. Then by (1) we can choose $f_1$ such that $\|f_1\| \leq M \|g\|$ and $\|Tf_1 - g\| \leq 1/2 \|g\|$. Then $(g - Tf_1)(k) \in \text{range } T(k)$ for each $k \in K$, and so by (1) we can find $f_2 \in C(K, X)$ such that $\|f_2\| \leq M \|g - Tf_1\| \leq M(1/2) \|g\|$ and $\|Tf_2 - Tf_1 - g\| \leq 1/4 \|g\|$. In this way we can find a sequence $\{f_n\} \subset C(K, X)$ such that $\|f_n\| \leq M \|g\|/2^n - 1$ and $\|\bar{T}(\sum_{i=1}^n f_i) - g\| \leq \|g\|/2^n$. Then $f = \sum_{i=1}^\infty f_i$ is the required function.

Let $G$ be a Banach Lie group, and suppose that $G$ is acting holomorphically on a Banach space $X$. Let $x_0 \in X$, write $\pi(g) = g \cdot x_0$ for $g \in G$, and set $F = \pi(G)$. We shall say $F$ is a homogeneous space under the action of $G$ if there is a Banach space $Y$ and a holomorphic map $\pi : X \to F$ satisfying

1. $\pi(kx) = y_0$ for all $x \in F$ and some $y_0 \in Y$;
2. range $\pi'(1) = \ker k'(x_0)$;
3. there is a neighborhood $N$ of 1 in $G$ such that $k'(g \cdot x_0)$ has closed range for $g \in N$ and inversion constant uniformly bounded over $N$;
4. $H = \{g \in G : g \cdot x_0 = x_0\}$ is a Banach Lie group.

Examples. (1) Let $A$ and $B$ be Banach algebras with identity, and let $\text{Hom} (A, B)$ be set of continuous homomorphisms of $A$ into $B$. If $\phi \in \text{Hom} (A, B)$ we set

$$F_\phi = \{\psi \in \text{Hom} (A, B) : \exists b \in B^{-1} \text{ with } \psi(a) = b\phi(a)b^{-1} \text{ for } a \in A\}.$$ Denote by $B$ the two sided Banach $A$-module consisting of $B$ with the products

$$a \cdot b = \phi(a)b, \quad b \cdot a = b\phi(a) \quad \text{for } a \in A, \ b \in B.$$ Then if the Hochschild cohomology groups $H^1(A, B_\phi)$ and $H^2(A, B_\phi)$ vanish (for the definitions, see [3]), $F_\phi$ is a homogeneous space under the action of $B^{-1}$. That conditions (1), (2), and (3) hold is checked in [9, §3]; (4) follows from the observation that $\{b \in B^{-1} : b\phi(a)b^{-1} = \phi(a) \text{ for } a \in A\}$ is the set of invertible elements in $\phi(A)'$, the commutant of $\phi(A)$, which is a closed subalgebra of $B$.

(2) Let $F_1$ be the set of continuous algebra multiplications on $A$ which give algebras isomorphic to $A$. Then if the Hochschild groups


$H^r(A, A)$ and $H^r(A, A)$ vanish, $F_1$ is a homogeneous space under the action of $L(A)^{-1}$ given by

$$\phi \cdot m(a, b) = \phi^{-1}(m(\phi(a)\phi(b))) \quad \text{for} \quad a \in A, \ b \in B,$$

where $\phi \in L(A)^{-1}$ and $m$ is a multiplication on $A$. Again, (1), (2), and (3) are checked in [9, §4]; (4) follows since the isotropy group of the usual multiplication is the set of algebra automorphisms of $A$, which is a Banach Lie group with Lie algebra the set of bounded derivations of $A$.

**Theorem 3.3.** Let $F$ be a homogeneous space under the action of a Banach Lie group $G$. Let $M$ be a Stein manifold, $N$ be a closed submanifold of $M$ and suppose $E$ is a holomorphic fibre bundle over $M$ with fibre $F$ and structure group $G$. Then

(I) If $s: M \to E$ is a continuous section such that $s|_N$ is holomorphic, then $s$ is homotopic in the space of sections which extend $s|_N$ to a holomorphic section $\tilde{s}: M \to E$.

(II) If two holomorphic sections $f_1$ and $f_0$ of $E$ over $M$ are homotopic in the space of continuous sections, then they are homotopic in the space of holomorphic sections.

**Proof.** Let $s: M \to E$ be a continuous section whose restriction to $N$ is holomorphic, and let $p: E \to M$ denote the bundle projection. We shall show that there is an open cover $\{U_j\}_{j \in J}$ of $M$ by holomorphically convex sets such that $E|_{U_j}$ is trivial for each $j$, and satisfying:

(*) Let $\Phi_j: U_j \times F \to p^{-1}(U_j)$ be a trivialization of $E|_{U_j}$, and for $m \in U_j$ define $\Phi_{j,m}: F \to p^{-1}(m)$ by $\Phi_{j,m}(e) = \Phi_j(m, e)$ for $e \in F$. Then $p_j(e) = \Phi^{-1}_{j,p(e)}(e)$ for $e \in p^{-1}(U_j)$ defines a holomorphic map $p_j$ of $p^{-1}(U_j)$ into $F$. There exist continuous maps $s_j: U_j \to G$ such that $\pi \circ s_j = p_j \circ s|_{U_j}$ for all $j$ and such that $s_j|_{U_j \cap N}$ is holomorphic.

Let $m \in M$; it is enough to show that $m$ has a neighborhood $U$ satisfying (*). Choose a neighborhood $V$ of $m$ such that

(a) $V$ is relatively compact;
(b) $E|_V$ is trivial via $\Phi: V \times F \to p^{-1}(V)$;
(c) $V \cap N$ is a co-ordinate neighborhood in $N$.

Since $G$ acts transitively on the fibre $F$, there is some $g \in G$ with $\pi(g) = \Phi^{-1}_{\pi}(s(m))$. Define a continuous map $t: V \to F$ by

$$t(m') = g^{-1} \cdot (\Phi^{-1}_{m}(s(m'))) \quad \text{for} \quad m' \in V.$$

Then $t|_{V \cap N}$ is a holomorphic map of $V \cap N$ into $F \subset X$. By Theorem 2.1, there is a neighborhood $W \subset V$ of $m$ in $M$ and a holomorphic map $f$ of $W \cap N$ into $G$ such that $\pi \circ f = t|_{W \cap N}$. 

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Let $K = \tilde{W}$. Then if $G$ has Lie algebra $\mathfrak{g}$, $C(K, G)$ is a Banach Lie group with Lie algebra $C(K, \mathfrak{g})$. As in Lemma 3.1, the sequence $G \xrightarrow{\pi} X \xrightarrow{\hat{k}} Y$ induces a sequence

$$(1) \quad C(K, G) \xrightarrow{\pi} C(K, X) \xrightarrow{\hat{k}} C(K, Y)$$

of holomorphic maps. Since $(\hat{k} \circ \pi)(g) = y_0$ for $g \in C(K, G)$, where $y_0$ denotes the constant function value $y_0$, the derivatives form a complex

$$(2) \quad C(K, \mathfrak{g}) \xrightarrow{\pi'(1)} C(K, X) \xrightarrow{\hat{k}'(x_0)} C(K, Y).$$

Now, since, near 1, $C(K, G)$ can be identified with $C(K, \mathfrak{g})$, we can apply Lemma 3.1 to deduce that

$$(\pi'(1)\psi)(k) = \pi'(1)\psi(k) \quad \text{for } \psi \in C(K, \mathfrak{g}), \ k \in K$$

and

$$(\hat{k}'(x_0)\alpha)(k) = k'(x_0)\alpha(k) \quad \text{for } \alpha \in C(K, X), \ k \in K.$$}

Now range $\pi'(1) = \ker k'(x_0)$, and so in particular range $\pi'(1)$ is closed. Thus (see, for example, [6]) there is a continuous map $\eta$: range $\pi'(1) \to X$ such that $\pi'(1) \circ \eta$ is the identity on range $\pi'(1)$. Now let $\alpha \in \ker \hat{k}'(x_0)$. Then $\alpha(k) \in \ker k'(x_0)$ for every $k$ in $K$, and so $\eta \circ \alpha$ is a continuous map of $K$ into $X$ such that $\hat{k}'(1)(\eta \circ \alpha) = \alpha$, proving that the complex (2) is exact. For $\alpha \in C(K, X)$ close to $x_0$, Lemma 3.1 gives

$$(\hat{k}'(\alpha)\beta)k = k'(\alpha(k))\beta(k) \quad \text{for } \beta \in C(K, X), \ k \in K.$$}

Thus, by Lemma 3.2, for $\alpha$ sufficiently close to $x_0$, $k'(\alpha)$ has closed range and bounded inversion constant. Hence we can apply [9, Theorem 1] to the complex (1) and deduce that there is $\varepsilon > 0$ such that if $\psi \in C(K, X)$ satisfies $\hat{k}(\psi) = y_0$ and $\|\psi - x_0\| < \varepsilon$, $\psi$ has a preimage in $C(K, G)$.

Now choose a neighborhood $W' \subset W$ of $m$ such that $\|t(m') - t(m)\| < \varepsilon$ for $m' \in W'$, and choose a neighborhood $U$ of $m$ such that $U \subset \text{int } W'$ and $U$ is holomorphically convex. Since $K$ is a compact Hausdorff space, by Urysohn's lemma there is a continuous function $\phi$: $K \to [0, 1]$ with $\phi = 0$ off $\tilde{W}'$ and $\phi = 1$ on $\tilde{U}$. Then $\phi t + (1 - \phi)x_0$ is within $\varepsilon$ of $x_0$ on $K$, and so there is a continuous map $\tilde{\eta}$: $K \to G$ such that $\pi \circ \tilde{\eta} |_{\tilde{U}} = t |_{\tilde{U}}$. Now $\tilde{\eta}^{-1}f$ is a continuous map of $\tilde{U} \cap N$ into $H$, and so by shrinking $U$ if necessary, we can assume $\tilde{\eta}^{-1}f$ is a continuous map of $\tilde{U} \cap N$ into a Banach space. Thus by Dugundji's extension theorem $\tilde{\eta}^{-1}f$ extends to a continuous map $u$ of $U \cap N$ into $H$. Then $v = \tilde{\eta}u$ is a continuous map of $U$ into $G$ with $\pi \circ v = t$ and $v |_{U \cap N} = f$ holomorphic. The map $\tilde{s}$ defined by $\tilde{s}(m') = gv(m')$ for $m' \in U$ is the required lift of $s$. 
We are now in the situation that Ramspott is in after the first paragraph of §4 of [10]. We can use the rest of his proof, using Theorem 8.4 of [1] and Theorems A and B of [8, §3] in place of the corresponding finite-dimensional theorems of Grauert. We note that the hypothesis—which has not been used so far—that the isotropy group of $x_0$ is a Banach Lie group is required to apply the lemma in [10, §5].

Note. The results of Grauert, Ramspott, and Bungart apply to bundles over Stein spaces; since our basic technique involves lifting of power series it does not immediately apply in this more general setting.

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University of New South Wales, Kensington
New South Wales, Australia 2033
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