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Let $\mathscr{S}$ be a class of spaces in the category of Tychonoff spaces and let $co(\mathscr{S})$ be its coreflective hull in that category, with coreflector c.

Let τ be the topology of uniform convergence on the set of continuous maps $C(X, Y)$.

For $\kappa \geqq \aleph_0$ let $\mathscr{S}(\kappa)$ be the collection of all Tychonoff spaces which are pseudo-κ- compact and m-discrete for every $m < \kappa$.

THEOREM. **The following are equivalent:**

(a) $co(\mathscr{S})$ is cartesian-closed and the exponential objects for $S \in \mathscr{S}$ and $Y \in co(\mathscr{S})$ are the spaces $c\tau C(S, Y)$.

(b) The projection $\pi: c(S \times T) \to S$ is z-closed for each $S, T \in \mathscr{S}$.

(c) Either $co(\mathscr{S})$ is the category of discrete spaces, or there exists $\kappa \geqq \aleph_0$ and a finitely productive subfamily $\mathscr{S}'$ of $\mathscr{S}(\kappa)$ such that $\mathscr{S} \subseteqq \mathscr{S}' \subseteqq co(\mathscr{S})$.

Furthermore, if $\mathscr{S}$ is map-invariant, then (a) implies that all spaces in $\mathscr{S}$ are pseudocompact.

Several examples are given.

0. **Introduction.** Coreflective hulls of families of Tychonoff spaces are examined to characterize those which are cartesian-closed, that is, have an exponential law $Z^{X \times Y} = (Z^X)^Y$, especially where the exponential spaces are defined in a natural way using topologies of uniform convergence on the hom-sets $C(X, Y)$. The main result implies that if the coreflective hull of a class is cartesian-closed in this way, then that class must be a subclass of either the pseudocompact spaces or the $\aleph_0$-discrete spaces. Hence, the most important examples of such subcategories are contained in the pseudocompactly-generated class of spaces. Other equivalent conditions, involving finite productivity and fine uniform structures, are given for these subcategories.

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1. **Background.** The reader is referred to [8], [9], [12], and [13] for the material in this section. All subcategories are assumed to be full, isomorphism-closed, and to contain a nonempty space.

DEFINITION. A category $\mathscr{C}$ having finite products is *cartesian-closed* if, for each object $X \in \mathscr{C}$, the product functor $P_X: \mathscr{C} \to \mathscr{C}$ defined by $P_X(Y) = X \times Y$ has a right adjoint $E_X: \mathscr{C} \to \mathscr{C}$, written $E_X(Y) = Y^X$.

The $\mathscr{C}$-objects X^Y are called exponentials, and they satisfy the condition on hom-sets:

$$\mathscr{C}(Z \times X, Y) = \mathscr{C}(Z, Y^X)$$

for all $X, Y, Z \in \mathscr{C}$.

Another characterization is more useful. The existence of a right adjoint to P_X is equivalent to the existence of $\mathscr{C}$-objects Y^X, for each Y, and $\mathscr{C}$-morphisms $e_Y: Y^X \times X \to Y$ such that:

(1) $\{e_Y: Y \in \mathscr{C}\}$ is natural in Y, and

(2) given $Z \in \mathscr{C}$ and morphism $f: Z \times X \to Y$, there exists a unique morphism $g: Z \to Y^X$ which makes the following diagram commute:

$$\begin{tikzcd} Y^X \times X \arrow[rr, "e_Y"] & & Y \\ & Z \times X \arrow[ul, "g \times i"] \arrow[ur, "f"'] & \end{tikzcd}$$

(i is the identity map on X.)

Any topological category, such as the category of Tychonoff spaces (*Tych*) and its coreflective subcategories, is cartesian-closed if and only if its product preserves sums and quotients, by a theorem of Herrlich in [8]. The product in *Tych* preserves sums, but not quotients, so *Tych* is not cartesian-closed. A more explicit reason for this is given by Arens in [1]: if X is a Tychonoff space for which there is a weakest topology on $C(X, [0, 1])$ making the evaluation map $e: X \times C(X, [0, 1]) \to [0, 1]$ continuous, then X must be locally compact.

Now if $\mathscr{C}$ is a coreflective subcategory of *Tych*, then it has finite products, denoted by $X \otimes Y$. These are the coreflections of the usual topological products $X \times Y$. Again, the products $X \otimes Y$ preserve sums.

Let $C(X, Y)$ be the set of continuous maps from X into Y. It is easy to verify that, in a cartesian-closed coreflective subcategory of *Tych*, each exponential space Y^X must have the same cardinality as $C(X, Y)$. Therefore, one may assume without loss of generality that the underlying set of Y^X is $C(X, Y)$ and that the map $e_X: Y^X \otimes X \to Y$ is the ordinary evaluation map $e_X(f, x) = f(x)$.

If $\mathscr{S}$ is a subfamily of *Tych*, let $co(\mathscr{S})$ denote its coreflective hull in *Tych*, consisting of all quotients of sums of members of $\mathscr{S}$.

DEFINITION 1. (i) For $X, Y \in Tych$ let $\tau C(X, Y)$ be the function space equipped with the topology of uniform convergence on X with respect to the fine uniformity on Y.

(ii) If $\mathscr{S}$ is a family of Tychonoff spaces, let $\tau_{\mathscr{S}} C(X, Y)$ be the function space equipped with the topology projectively generated by all functions $\bar{f}$ defined as follows: given $S \in \mathscr{S}$ and a continuous map $f: S \to X$, define $\bar{f}: C(X, Y) \to \tau C(S, Y)$ by $\bar{f}(g) = g \circ f$.

The topologies τ defined in (i) are Tychonoff since they are associated with Hausdorff uniformities. The topologies in (ii) are also well-defined Tychonoff structures if $\mathscr{S}$ contains a nonempty space. In general, $\tau_{\mathscr{S}}$ is weaker than τ. If $\mathscr{S}$ is map-invariant, then $\tau_{\mathscr{S}} C(X, Y)$ has the topology of uniform convergence on $\mathscr{S}$-subspaces of X.

For example, let $\mathscr{K}$ be the class of all compact spaces in *Tych*. The result of Steenrod in [17], translated to *Tych*, is that $co(\mathscr{K})$ is cartesian-closed and, for $X, Y \in co(\mathscr{K})$, $Y^X = k\tau_{\mathscr{K}} C(X, Y)$, where $k: Tych \to co(\mathscr{K})$ is the coreflector. Other examples of cartesian-closed coreflective subcategories of *Tych* having similar exponential spaces are given in [3] and [18]. These examples are all contained in $co(\mathscr{K})$.

2. Cartesian-closed subcategories of *Tych*. The main problem of this paper is to characterize the coreflective, cartesian-closed subcategories of *Tych* in which topologies of uniform convergence are used to form the exponentials. Specifically, we want to characterize the coreflections $c: Tych \to \mathscr{C}$ such that:

(a) $\mathscr{C}$ is cartesian-closed, and

(b) the class $\{X \in \mathscr{C}: c\tau C(X, Y) = Y^X \text{ for all } Y \in \mathscr{C}\}$ inductively generates $\mathscr{C}$.

We first show that if $co(\mathscr{S})$ is cartesian-closed, then the exponentials Y^S, for $S \in \mathscr{S}$ and $Y \in co(\mathscr{S})$, determine all other exponentials Y^X in $co(\mathscr{S})$.

LEMMA 1. *Let $c: Tych \to \mathscr{C}$ be a coreflection and let $\mathscr{C} = co(\mathscr{S})$. Suppose that for each $S \in \mathscr{S}$ the functor $S \otimes \_\!\_: \mathscr{C} \to \mathscr{C}$ has a right adjoint, denoted by $Y \mapsto Y^S$. Then $\mathscr{C}$ is cartesian-closed and, for $X, Y \in \mathscr{C}$, the exponential Y^X is given as follows: let σ be the topology on $C(X, Y)$ projectively generated by all functions $\bar{f}: C(X, Y) \to Y^S$, for $S \in \mathscr{S}$, arising from a continuous map $f: S \to X$ by $\bar{f}(g) = g \circ f$. Then $Y^X = c\sigma C(X, Y)$.*

Proof. Since $\mathscr{S}$ contains a nonempty space, the functions $\bar{f}$ separate the points of $C(X, Y)$, so σ is a well-defined Tychonoff topology and $Y^X \in \mathscr{C}$.

It suffices to show that the sets $C(Z \otimes X, Y)$ and $C(Z, Y^X)$ are in bijective correspondence in a natural way, for $X, Y, Z \in \mathscr{C}$.

(a) First, suppose $h: Z \otimes X \to Y$ is continuous. For each $z \in Z$ the restriction of h to $\{z\} \times X$ is continuous, so we may define a function $h': Z \to Y^X$ by $h'(z)(x) = h(z, x)$ for $z \in Z$, $x \in X$. (We have used the fact that the underlying set of Y^X is $C(X, Y)$.) To show that h' is continuous, we must show that, given $S \in \mathscr{S}$ and a continuous map $f: S \to X$, the composition $\bar{f} \circ h': Z \to Y^X \to Y^S$ is continuous. Let $i: Z \to Z$ be the identity map. Then $i \times f: Z \otimes S \to Z \otimes X$ is continuous, so the map $j = h \circ (i \times f): Z \otimes S \to Y$ is continuous. Since the functor $S \otimes \_\_: \mathscr{C} \to \mathscr{C}$ has a right adjoint, the associated map $j': Z \to Y^S$ is continuous, where $j'(z)(s) = j(z, s)$. But $j' = \bar{f} \circ h'$, so h' is continuous. This defines a one-to-one function $h \mapsto h'$ from $C(Z \otimes X, Y)$ into $C(Z, Y^X)$.

(b) Now suppose that $g': Z \to Y^X$ is continuous, and let $g: Z \otimes X \to Y$ be the function $g(z, x) = g'(z)(x)$. We must show that g is continuous.

Since $\mathscr{S}$ generates $\mathscr{C}$ there exists a quotient map $q: \Sigma S_a \to Z \otimes X$, where ΣS_a is a sum of spaces S_a in $\mathscr{S}$. It suffices to show that $g \circ q$ is continuous. Let $S = S_a$ for some a. Let π_X and π_Z be the projections of $X \otimes Z$ onto X and Z, respectively. Let $q_X = \pi_X \circ q: S \to X$ and $q_Z = \pi_Z \circ q: S \to Z$.

Now the map $\bar{q}_X: Y^X \to Y^S$ is continuous by definition of Y^X. So, we have the continuous composition:

$$S \xrightarrow[q_Z]{} Z \xrightarrow[g']{} Y^X \xrightarrow[\bar{q}_X]{} Y^S .$$

Let $r': S \to Y^S$ be this composition. Then by the assumption on $S \in \mathscr{S}$, the map $r: S \otimes S \to Y$ defined by $r(s, t) = r'(s)(t)$ is continuous. Let $d: S \to S \otimes S$ be the injection onto the diagonal. Then d is continuous, and $r \circ d = g \circ q|_S$. So, the restriction of $g \circ q$ to each summand $S = S_a$ is continuous, so g is continuous since q is a quotient map.

Therefore, the natural correspondence $h \mapsto h'$ is a bijection from $C(Z \otimes X, Y)$ onto $C(Z, Y^X)$. So, for each $X \in \mathscr{C}$, the functor $X \otimes \_\_: \mathscr{C} \to \mathscr{C}$ has a right adjoint $Y \mapsto Y^X$ with $Y^X = c\sigma C(X, Y)$.

Lemma 1 may be applied to the topologies τ and $\tau_{\mathscr{S}}$ given in Definition 1:

COROLLARY 1. *Let c: Tych $\to \mathscr{C}$ be a coreflection. Suppose that there exists $\mathscr{S} \subseteq \mathscr{C}$ such that:*

(i) *$\mathscr{C} = co(\mathscr{S})$, and*

(ii) *for each $S \in \mathscr{S}$, the functor $S \otimes \_: \mathscr{C} \to \mathscr{C}$ has a right adjoint $Y \mapsto Y^S$ where $Y^S = c\tau C(S, Y)$.*

Then $\mathscr{C}$ is cartesian-closed and $Y^X = c\tau_{\mathscr{S}} C(X, Y)$ for all $X, Y \in \mathscr{C}$.

If $\mathscr{C}$ is cartesian-closed, the exponential Y^X must be the coarsest space in $\mathscr{C}$ for which the evaluation map $e: Y^X \otimes X \to Y$ is continuous. It is known that $e: \tau C((X, Y)) \times X \to Y$ is continuous for $X, Y \in$ *Tych*. (Theorem 10 (e), Chapter 7 of [11].)

LEMMA 2. *Let c: Tych $\to \mathscr{C}$ be a cartesian closed coreflection. Let $\mathscr{S}$ be a subclass of $\mathscr{C}$ which inductively generates $\mathscr{C}$. Then $c\tau_{\mathscr{S}} C(X, Y)$ is finer than Y^X, for all $X, Y \in \mathscr{C}$.*

Proof. For $S \in \mathscr{S}$, $\tau C(S, Y) = \tau_{\mathscr{S}} C(S, Y)$, and by the remark above, $e: c\tau C(S, Y) \otimes S \to Y$ is continuous, so $c\tau C(S, Y) \to Y^S$ is continuous.

For $X \in \mathscr{C}$, an argument similar to the one given in Lemma 1 to show that the evaluation map is continuous may be used here to show that $e: c\tau_{\mathscr{S}} C(X, Y) \otimes X \to Y$ is continuous. Hence $c\tau_{\mathscr{S}} C(X, Y)$ is finer than Y^X for all $X, Y \in \mathscr{C}$.

It should be stressed that the topology $\tau_{\mathscr{S}}$ depends on the generating family $\mathscr{S}$, so that the upper bound for Y^X in Lemma 2 is sharper for smaller families $\mathscr{S}$.

DEFINITION 2. Let c: *Tych* $\to \mathscr{C}$ be a cartesian-closed coreflection. Define the subclass $\mathscr{S}_\tau$ of $\mathscr{C}$ by

$$\mathscr{S}_\tau = \{X \in \mathscr{C}: Y^X = c\tau C(X, Y) \ \forall \ Y \in \mathscr{C}\} .$$

LEMMA 3. *Let c: Tych $\to \mathscr{C}$ be a cartesian-closed coreflection. If $S \in \mathscr{S}_\tau$ and T is a continuous image of S, then $cT \in \mathscr{S}_\tau$. In particular, $\mathscr{S}_\tau$ is quotient-invariant.*

Proof. Let $f: S \to T$ be a continuous map onto T. Then $f: S \to cT$ is continuous. Let $Y \in \mathscr{C}$. Since $\mathscr{C}$ is cartesian-closed, the natural map $\bar{f}: Y^{cT} \to Y^S$ is continuous, and $Y^S = c\tau C(S, Y)$ since $S \in \mathscr{S}_\tau$.

Let V be a fine uniform neighborhood of the diagonal in Y. For $h \in Y^{cT}$, it suffices to show that the set

$$U_h \equiv \{g \in Y^{cT}: (g(t), h(t)) \in V \ \forall t \in cT\}$$

is open in Y^{cT}. Now the set

$$U_{hf} \equiv \{j \in Y^S: (j(s), h \circ f(s)) \in V \ \forall \ s \in S\}$$

belongs to τ, so it is open in Y^S. Therefore $\bar{f}^{-1}(U_{hf})$ is open in Y^{cT}. But

$$\bar{f}^{-1}(U_{hf}) = \{g \in Y^{cT}: (g \circ f(s), h \circ f(s)) \in V \;\; \forall s \in S\} ,$$

and this set is just U_h, since f is onto. Hence U_h is open, so $Y^{cT} \to c\tau C(cT, Y)$ is continuous. By Lemma 2 the inverse is also continuous, so $Y^{cT} = c\tau C(cT, Y)$, and therefore $cT \in \mathscr{S}_\tau$.

We will show that the class $\mathscr{S}_\tau$ is also finitely productive in *Tych*.

DEFINITION 3. A function $f: X \to Y$ is *z-closed* if $f(Z)$ is closed in Y for any zero set Z in X.

If $X \in$ *Tych*, let αX be the (topologically) fine uniform space associated with X.

THEOREM 1. *Let c: Tych $\to \mathscr{C}$ be a cartesian-closed coreflection. Let $\mathscr{S}_\tau$ be the subclass of $\mathscr{C}$ given in Definition 2. Then:*

(i) *All uniform products $\alpha S \times \alpha T$ are topologically fine for $S, T \in \mathscr{S}_\tau$.*

(ii) *The projections $\pi: S \otimes S \to T$ are z-closed for all $S, T \in \mathscr{S}_\tau$.*

(iii) *$\mathscr{S}_\tau$ is finitely productive in Tych.*

Proof. We first show that if $S, T \in \mathscr{S}$ and $Z \in \mathscr{C}$ and if f: $S \otimes T \to Z$ is continuous, then $f: \alpha S \times \alpha T \to \alpha Z$ is uniformly continuous.

To do this, it suffices to show that the families $\{f_y: y \in T\}$ and $\{f_x: x \in S\}$ are equi-uniform on αS and αT, respectively, where $f_y(x) = f_x(y) = f(x, y)$. Clearly, the functions f_y and f_x are uniformly continuous on αS and αT since they are the restrictions of f to the subspaces $S \times \{y\}$ and $\{x\} \times T$ of $S \otimes T$.

The family $\{f_y: y \in T\}$ will be equi-uniform on the fine space αS if it is equi-continuous at each point of S, by Theorem 38, Chapter 3 of [10]. So, let $x_0 \in S$ and let V be a uniform neighborhood of the diagonal for αZ. The function $\bar{f}: S \to Z^T$ is continuous. Let U be the basic neighborhood of $\bar{f}(x_0)$ in Z^T associated with V:

$$U = \{g \in Z^T: (g(y), \bar{f}(x_0)(y)) \in V \;\; \forall\, y \in T\} .$$

Then $\bar{f}^{-1}(U) = \{x \in S: (f_y(x), f_y(x_0)) \in V \; \forall y \in T\}$, and this set is a neighborhood of x_0 by continuity of $\bar{f}$. It follows that $\{f_y: y \in T\}$ is equi-continuous at x_0. Hence, the family is equi-uniform on αS.

By symmetry of the product $S \otimes T$, it follows that $\{f_x: x \in S\}$ is equi-uniform on αT. Therefore, $f: \alpha S \times \alpha T \rightarrow \alpha Z$ is uniformly continuous.

For (i), we simply note that if $S, T \in \mathscr{S}$, $Z \in Tych$, and if f: $S \times T \rightarrow Z$ is continuous, then $f: S \otimes T \rightarrow cZ$ is continuous, so it follows from the argument above that $f: \alpha S \times \alpha T \rightarrow \alpha cZ$ is uniformly continuous. Therefore, $\alpha S \times \alpha T$ is topologically fine.

It also follows from the argument above that $S \otimes T = S \times T$ for $S, T \in \mathscr{S}_\tau$.

For (ii), we use a result of Hager ([7]) and Noble ([15]): if $\alpha X \times \alpha Y$ is topologically fine, then the projections from the topological product $X \times Y$ onto X and Y are both z-closed. For $S, T \in \mathscr{S}_\tau$, we know that $S \times T = S \otimes T$, and $\alpha S \times \alpha T$ is fine by (i), so the projections from $S \otimes T$ onto S and T are z-closed.

For (iii), let $S, T \in \mathscr{S}_\tau$ and $Y \in \mathscr{C}$. We must show that $Y^{S \times T} = c\tau C(S \times T, Y)$. Let F: $Y^{S \times T} \rightarrow (Y^S)^T$ be the natural correspondence; by the exponential law, F is a homeomorphism.

Let $f \in Y^{S \times T}$ and let U_f be a basic τ-neighborhood of f associated with some uniform cover $\mathscr{U}$ of αY. We will show that U_f is a neighborhood of f in $Y^{S \times T}$. Let $\mathscr{V}$ star-refine $\mathscr{U}$. Now $\mathscr{V}$ and $\mathscr{U}$ determine covers $\mathscr{V}_S$ and $\mathscr{U}_S$ of Y^S belonging to the uniformity of uniform convergence, and $\mathscr{V}_S$ star-refines $\mathscr{U}_S$. Since the topology on Y^S contains τ, for $S \in \mathscr{S}_\tau$, the covers $\mathscr{V}_S$ and $\mathscr{U}_S$ belong to the fine uniformity associated with Y^S, so $\mathscr{V}_S$ in turn determines a cover $\mathscr{V}_{ST}$ of $(Y^S)^T$ belonging to the uniformity of uniform convergence. Since $T \in \mathscr{S}_\tau$, the members of $\mathscr{V}_{ST}$ are open sets in $(Y^S)^T$.

Now, returning to the set U_f, we have

$$U_f = \{j \in Y^{S \times T}: j(s, t) \in st(f(s, t), \mathscr{U}) \ \ \forall (s, t) \in S \times T\} .$$

Let $g = F(f)$ and let V_g be the basic neighborhood of g in $\mathscr{V}_{ST}$, so

$$V_g = \{j \in (Y^S)^T: j(t) \in st(g(t), \mathscr{V}_S) \ \ \forall t \in T\} .$$

Let $U_{g(t)}$ be the basic neighborhood of $g(t)$ from the cover $\mathscr{U}_S$, so

$$U_{g(t)} = \{h \in Y^S: h(s) \in st(g(t)(s), \mathscr{U}) \ \ \forall s \in S\} .$$

Now $\mathscr{V}_S$ star-refines $\mathscr{U}_S$, so that

$$V_g \subseteqq \{j \in (Y^S)^T: j(t) \in U_{g(t)} \ \ \forall t \in T\} .$$

Hence $V_g \subseteqq \{j \in (Y^S)^T: j(t)(s) \in st(g(t)(s), \mathscr{U}) \ \ \forall s \in S, t \in T\}$.

Therefore, $F^{-1}(V_g) \subseteqq U_f$, so the continuity of F implies that U_f is a neighborhood of f in $Y^{S \times T}$. Hence, the topology of $Y^{S \times T}$ contains τ, so using Lemma 2 it follows that $S \times T \in \mathscr{S}_\tau$. This shows that the subclass $\mathscr{S}_\tau$ of $\mathscr{C}$ is finitely productive in $Tych$.

DEFINITION 4. Let κ be an infinite cardinal.

(a) A space X is *pseudo-κ-compact* if each locally finite family of open subsets of X has power less than κ.

(b) A space X is *κ-discrete* if every intersection of κ or fewer open subsets of X is open.

For $\kappa \geqq \aleph_0$ let $\mathscr{S}(\kappa)$ be the class of all Tychonoff spaces which are pseudo-κ-compact and μ-discrete for every $\mu < \kappa$. For example, $\mathscr{S}(\aleph_0)$ is the class of all pseudocompact spaces. If κ is singular then $\mathscr{S}(\kappa)$ consists of the discrete spaces of power less than κ, but if κ is regular then $\mathscr{S}(\kappa)$ contains nondiscrete spaces.

Let $\mathscr{D}$ be the collection of all discrete spaces. Any coreflective subcategory of *Tych* contains $\mathscr{D}$.

THEOREM 2. *Let c: Tych $\rightarrow \mathscr{C}$ be a cartesian-closed coreflection. Suppose that there exists a subfamily $\mathscr{S}$ of $\mathscr{C}$ such that:*

(1) $Y^S = c\tau C(S, Y)$ for all $S \in \mathscr{S}$, $Y \in \mathscr{C}$, and

(2) $\mathscr{S}$ inductively generates $\mathscr{C}$.

Then either $\mathscr{C} = \mathscr{D}$ or there exists $\kappa \geqq \aleph_0$ such that $\mathscr{S} \subseteqq \mathscr{S}(\kappa)$. If $\mathscr{S}$ is map-invariant in Tych, then $\mathscr{S} \subseteqq \mathscr{S}(\aleph_0)$, the pseudocompact spaces.

Proof. Suppose $\mathscr{S}$ contains a nondiscrete space X. By Theorem 1, $\alpha X \times \alpha X$ is fine. By a result of Isbell (Theorem 32, Chapter 7 of [10]), this implies that there exists $\kappa \geqq \aleph_0$ such that $X \in \mathscr{S}(\kappa)$. Now if $Y \in \mathscr{S}$ and $Y \neq X$, then $\alpha X \times \alpha Y$ is fine, so by the same theorem in [10], $Y \in \mathscr{S}(\kappa)$ also. Therefore $\mathscr{S} \subseteqq \mathscr{S}(\kappa)$.

Now suppose that $\mathscr{S}$ is map-invariant, and suppose that there exists a nonpseudocompact space $X \in \mathscr{S}$. By the first part, either X is discrete or there exists $\kappa > \aleph_0$ such that $X \in \mathscr{S}(\kappa)$. In either case, X admits $\aleph_0$. Also, X is infinite, so the countable discrete space N is a continuous image of X. Since $\mathscr{S}$ is map-invariant, N and all other countable spaces belong to $\mathscr{S}$. However, if N^* is the one-point compactification of N, then the projection $\pi: N \times N^* \rightarrow N^*$ is not z-closed, and this contradicts Theorem 1 (ii). Therefore $\mathscr{S} \subseteqq \mathscr{S}(\aleph_0)$.

The only possibilities for a subcategory $\mathscr{C}$ satisfying the hypotheses of Theorem 2 are the subcategories of either the pseudocompactly-generated spaces or the $\aleph_0$-discrete spaces.

We now consider sufficient conditions for cartesian-closedness.

THEOREM 3. *Let c: Tych $\rightarrow \mathscr{C}$ be a coreflection and let $\mathscr{S}$ be a generating family for $\mathscr{C}$. If the projections $\pi: S \otimes T \rightarrow S$ are z-*

closed for all S, $T \in \mathscr{S}$, then $\mathscr{C}$ is cartesian-closed and its exponentials are defined by $Y^X = c\tau_{\mathscr{S}}C(X, Y)$ for X, $Y \in \mathscr{C}$.

Proof. By Corollary 1 it suffices to show that for each $S \in \mathscr{S}$ the functor $S \otimes \_$ has a right adjoint $Y \mapsto Y^S$ defined by $Y^S = c\tau C(S, Y)$. By Lemma 2, it is enough to show that for X, $Y \in \mathscr{C}$ and $S \in \mathscr{S}$, if $f: X \otimes S \to Y$ is continuous, then the associated map $\bar{f}: X \to Y^S$ is also continuous.

First, suppose $X \in \mathscr{S}$. Let $x_0 \in X$ and let U be a basic neighborhood of $\bar{f}(x_0)$ in $\tau C(S, Y)$. Then U is associated with a continuous pseudometric d on Y, so that

$$U = \{g \in Y^S: d(g(s), \bar{f}(x_0)(s)) < 1 \ \forall s \in S\} .$$

Now $\bar{f}(x_0)(s) = f(x_0, s)$, so

$$\bar{f}^{-1}(U) = \{x \in X: d(f(x, s), f(x_0, s)) < 1 \ \forall s \in S\} .$$

Then $X - \bar{f}^{-1}(U) = \{x \in X: \exists s \in S \ni d(f(x, s), f(x_0, s)) \geqq 1\}$.

We will show that $X - \bar{f}^{-1}(U)$ is closed, using the following composition of maps:

$$X \otimes S \xrightarrow[G_f]{} S \times Y \xrightarrow[j]{} Y \otimes Y \xrightarrow[d]{} R .$$

Here G_f, the graph of f, is the map $(x, s) \mapsto (s, f(x, s))$; the map j is the product of $\bar{f}(x_0)$ with the identity map on Y. Let F be the above composition. Since all maps involved are continuous, so is F. Also, $X - \bar{f}^{-1}(U) = \pi_X(F^{-1}([1, \infty)))$. Now $F^{-1}([1, \infty))$ is a zero set in X, so its projection onto X is closed if $X \in \mathscr{S}$, by assumption. So, $\bar{f}^{-1}(U)$ is open, so $\bar{f}: X \to \tau C(S, Y)$ is continuous. Then if $Y^S = c\tau C(S, Y)$, $\bar{f}: X \to Y^S$ is continuous.

In general, for $X \in \mathscr{C}$, there exists a subfamily $\{T_a\}$ of $\mathscr{S}$ and a quotient map $q: \Sigma T_a \to X$. Let H be the composition:

$$\Sigma(T_a \otimes S) \xrightarrow[j]{} (\Sigma T_a) \otimes S \xrightarrow[q \times i]{} X \otimes S \xrightarrow[f]{} Y ,$$

where j is the natural bijection and i is the identity map on S. Then H is continuous. Let H_a be the restriction of H to $T_a \otimes S$. Then $H_a: T_a \otimes S \to Y$ is continuous, so by the first part, the associated map $\bar{H}_a: T_a \to Y^S$ is continuous. Taking sums, the map $\Sigma\bar{H}_a: \Sigma T_a \to Y^S$ is continuous, and it is not hard to verify that $\Sigma\bar{H}_a = \bar{f} \circ q$. Therefore $\bar{f}$ is continuous since q is a quotient map.

We now summarize these results.

THEOREM 4. *Let $\mathscr{S}$ be a family of Tychonoff spaces. The following are equivalent:*

(a) *$co(\mathscr{S})$ is cartesian-closed and $Y^S = c\tau C(S, Y)$ for $S \in \mathscr{S}$, $Y \in co(\mathscr{S})$.*

(b) *$co(\mathscr{S})$ is cartesian-closed and $Y^X = c\tau_{\mathscr{S}} C(X, Y)$ for $X, Y \in co(\mathscr{S})$.*

(c) *The projections $\pi: S \otimes T \to S$ are z-closed for $S, T \in \mathscr{S}$.*

(d) *Either $co(\mathscr{S}) = \mathscr{D}$ or there exists $\kappa \geqq \aleph_0$ such that all finite products of spaces in $\mathscr{S}$ belong to $\mathscr{S}(\kappa) \cap co(\mathscr{S})$.*

Furthermore, if $\mathscr{S}$ is map-invariant, then conditions (a)-(d) *are equivalent to*:

(e) *All finite products of members of $\mathscr{S}$ belong to $\mathscr{S}(\aleph_0) \cap co(\mathscr{S})$.*

Proof. We have already seen that conditions (a), (b) and (c) are equivalent.

Suppose $\mathscr{S}$ satisfies (d) or (e). If $co(\mathscr{S}) = \mathscr{D}$, then clearly $\mathscr{D}$ is cartesian-closed, so (a) holds. Otherwise, $\mathscr{S} \subseteqq \mathscr{S}(\kappa)$ for some $\kappa \geqq \aleph_0$, and if $X, Y \in \mathscr{S}$, then $X \times Y \in \mathscr{S}(\kappa)$. This implies that the uniform product $\alpha X \times \alpha Y$ is topologically fine, by a result in Chapter 7 of [10]. Then, by a result in [7] and [15], the projections from $X \times Y$ onto X and Y are z-closed. Since $X \times Y = X \otimes Y$ for $X, Y \in \mathscr{S}$, condition (c) follows.

Now suppose that $\mathscr{S}$ satisfies (a). Let $\mathscr{S}_\tau$ be the subfamily of $co(\mathscr{S})$ given in Definition 2. Then $\mathscr{S} \subseteqq \mathscr{S}_\tau$, so $co(\mathscr{S}) = co(\mathscr{S}_\tau)$. By Theorem 2, either $co(\mathscr{S}) = \mathscr{D}$ or there exists $\kappa \geqq \aleph_0$ such that $\mathscr{S}_\tau \subseteqq \mathscr{S}(\kappa)$. Suppose $co(\mathscr{S}) \neq \mathscr{D}$. By Theorem 1 $\mathscr{S}_\tau$ is finitely productive, so if $X, Y \in \mathscr{S}$ then $X \times Y \in \mathscr{S}_\tau \subseteqq co(\mathscr{S}) \cap \mathscr{S}(\kappa)$. Hence (d) holds.

If $\mathscr{S}$ is map-invariant, then $\mathscr{S} \subseteqq \mathscr{S}(\aleph_0)$ by Theorem 2, and also $\mathscr{S}_\tau \subseteqq \mathscr{S}(\lambda)$ for some $\lambda \geqq \aleph_0$. If $co(\mathscr{S}) \neq \mathscr{D}$, then $\mathscr{S}$ contains an infinite pseudocompact space, and any such space is not κ-discrete for any infinite cardinal κ, so it cannot belong to $\mathscr{S}(\kappa)$ for any $\kappa > \aleph_0$. Hence $\lambda = \aleph_0$, so $\mathscr{S}_\tau \subseteqq \mathscr{S}(\aleph_0)$. The rest of condition (e) follows from the finite productivity of $\mathscr{S}_\tau$.

This result may be stated as follows: $co(\mathscr{S}) \neq \mathscr{D}$ is cartesian-closed with exponentials obtained from $\tau_{\mathscr{S}}$ if and only if there exists $\kappa \geqq \aleph_0$ and a finitely productive family $\hat{\mathscr{S}} \subseteqq \mathscr{S}(\kappa)$ such that $\hat{\mathscr{S}} \subseteqq \mathscr{S} \subseteqq co(\mathscr{S})$.

EXAMPLES. (1) Let $\mathscr{T}$ be the collection of all pseudocompact spaces which have pseudocompact product with any other pseudocompact space. (This class is characterized by Frolík in [4].) It is easy to see that $\mathscr{T}$ is finitely productive; in fact, $\mathscr{T}$ is productive by a result of Noble in [14]. In any event, $co(\mathscr{T})$ is cartesian-closed.

(2) If $co(\mathscr{S})$ is cartesian-closed and $\mathscr{S} \subseteq \mathscr{S}(\aleph_0)$, it does not follow that $\mathscr{S} \subseteq \mathscr{T}$. In [5] a space X is constructed so that its finite, but not infinite, powers are pseudocompact. If $\mathscr{S}$ is the set of all finite powers of X, then $co(\mathscr{S})$ is cartesian-closed. Because $\mathscr{T}$ is productive, $\mathscr{S} \cap \mathscr{T} = \varnothing$. (This space X is similar to spaces constructed in [2]; all are subspaces of βN.)

(3) Let κ be an uncountable regular cardinal. Let $\mathscr{K}(\kappa)$ be the collection of all spaces which are κ-compact (every open cover has a subcover of power less than κ) and μ-discrete for all $\mu < \kappa$. Then $\mathscr{K}(\kappa) \subseteq \mathscr{S}(\kappa)$, and $\mathscr{K}(\kappa)$ is finitely productive, so its coreflective hull is cartesian-closed.

COROLLARY 2. *Let $c: Tych \rightarrow \mathscr{C}$ be a cartesian-closed coreflection. If $Y^X = c\tau C(X, Y)$ for all $X, Y \in \mathscr{C}$, then $\mathscr{C} = \mathscr{D}$.*

Proof. If $\mathscr{C} \neq \mathscr{D}$, then by Theorem 4 there exists $\kappa \geqq \aleph_0$ such that $\mathscr{C} \subseteq \mathscr{S}(\kappa)$. This is impossible since $\mathscr{C}$ contains all discrete spaces, but no discrete space of power κ or greater belongs to $\mathscr{S}(\kappa)$.

We conclude with some problems which appear to be unsettled:

(1) Is $co(\mathscr{S}(\kappa))$ cartesian-closed for any $\kappa \geqq \aleph_0$? For $\kappa = \aleph_0$, the class $\mathscr{S}(\aleph_0)$ is not finitely productive: there exists a pseudocompact space X such that $X \times X$ is not pseudocompact. (This example is due to Novák in [16], and it also appears in Chapter 9 of [6].) Therefore, if $co(\mathscr{S}(\aleph_0))$ is cartesian-closed, then there exists a space Y in $co(\mathscr{S}(\aleph_0))$ such that $Y^X \neq c\tau C(X, Y)$.

(2) If $\mathscr{C} \neq \mathscr{D}$ and $\mathscr{C}$ is coreflective and cartesian-closed, is $\mathscr{C} \subseteq co(\mathscr{S}(\kappa))$ for some $\kappa \geqq \aleph_0$? This question can probably be answered in the negative by a counterexample.

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