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THE DIMENSION OF THE KERNEL OF A PLANAR SET

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Let S be a compact subset of R^2 . We establish the following: For $1 \leq k \leq 2$, the dimension of $\ker S$ is at least k if and only if for some $\varepsilon > 0$, every $f(k)$ points of S see via S a common k -dimensional neighborhood having radius ε , where $f(1) = 4$ and $f(2) = 3$. The number $f(k)$ in the theorem is best possible.

We begin with some definitions: Let S be a subset of R^d . For points x and y in S , we say x sees y via S if the segment $[x, y]$ lies in S . The set S is *starshaped* if there is some point p in S such that, for every x in S , p sees x via S . The set of all such points p is called the (convex) *kernel* of S , denoted by $\ker S$.

A well-known theorem of Krasnosel'skii [5] states that if S is a compact set in R^d , then S is starshaped if and only if every $d + 1$ points of S see a common point via S .

Although various results have been obtained concerning the dimension of the set $\ker S$ (Hare and Kenelly [3], Toranzos [6], Foland and Marr [2], Breen [1]), it still remains to set forth an appropriate analogue of the Krasnosel'skii theorem for sets whose kernel is at least k -dimensional, $1 \leq k \leq d$. Hence the purpose of this work is to investigate such an analogue for subsets of the plane.

The following terminology will be used. Throughout the paper, $\text{conv } S$, $\text{cl } S$, $\text{int } S$, $\text{bdry } S$, and $\ker S$ denote the convex hull, closure, interior, boundary, and kernel, respectively, of the set S . If S is convex, $\dim S$ represents the dimension of S . Finally for $x \neq y$, $R(x, y)$ denotes the ray emanating from x through y and $L(x, y)$ is the line determined by x and y .

2. The results. We begin with the following theorem for sets whose kernel is 1-dimensional.

THEOREM 1. *Let S be a compact set in R^2 . The dimension of $\ker S$ is at least 1 if and only if for some $\varepsilon > 0$, every 4 points of S see via S a common segment of radius ε . The number 4 is best possible.*

Proof. The necessity of the condition is obvious. Hence we need only establish its sufficiency.

By Krasnosel'skii's theorem in R^2 , S is starshaped, so we may select a point z in $\ker S$. Moreover, we assert that every 4 points of S see a common segment of length ε having z as endpoint (we refer to such a segment as an ε -interval at z): For x_1, x_2, x_3, x_4 in S , these points see a common 2ε -interval $[a, b]$ in S , and since $z \in \ker S$, $\text{conv}\{z, x_i, a, b\} \subseteq S$ for each $1 \leq i \leq 4$. Hence x_i sees $\text{conv}\{z, a, b\}$ for every i . Certainly one of the edges $[z, a], [z, b]$ of the triangle (possibly degenerate) $\text{conv}\{z, a, b\}$ has length at least ε , and this edge satisfies our assertion.

To complete the proof, we consider two cases.

Case 1. Assume that $z \in \text{int } S$. Let N be a disk about z of radius $r \leq \varepsilon$ contained in S . If $N = S$ the result is immediate, so assume that $S \sim N \neq \phi$. For $y \in S \sim N$, we define C_y to be the subset of N seen by y . Since S is starshaped, S is simply connected, so C_y is convex. Let $[a_y, b_y]$ be the intersection of C_y with the line perpendicular to $L(y, z)$ at z , and let δ_y be the smaller of the lengths of the segments $[a_y, z]$ and $[b_y, z]$, say the length of $[a_y, z]$.

If $\text{glb } \delta_y > 0$, then $\cap C_y$ contains a neighborhood of z , contained in $\ker S$. Hence we may assume $\text{glb } \delta_y = 0$.

Let $\{y_n\}$ be a sequence of points in S such that $\delta_{y_n} \rightarrow 0$ as $n \rightarrow \infty$. Let y_0 be a limit point of $\{y_n\}$ and assume y_n converges to y_0 . Set $L = L(y_0, z)$ and call the open halfplanes into which L divides the plane L_1 and L_2 . Without loss of generality, we assume that for each n , the corresponding a_n lies in the closed halfplane $\text{cl } L_2$ determined by L .

We now show that every two points of S see a common ε -interval at z in $\text{cl } L_1$: Otherwise, some members x_1 and x_2 of S would see no such interval, and there would exist points q_1 and q_2 in $\text{bdry } N \cap L_2$ such that every ε -interval at z seen by both x_1 and x_2 would lie in the convex region bounded by rays $R(z, q_1)$ and $R(z, q_2)$. However, for δ_n sufficiently small, y_n sees no ε -interval at z in this region, impossible since x_1, x_2, y_n see a common ε -interval at z . Thus the result is established.

Assume that the points of $\text{bdry } N \cap \text{cl } L_1$ are ordered in a clockwise direction from s_0 to t_0 , where s_0 and t_0 denote the endpoints of the interval $N \cap L$. For each y in S , there exist s_y and t_y on $\text{bdry } N \cap \text{cl } L_1$ such that y sees $[s_y, z] \cup [t_y, z]$ via S and such that s_y and t_y are, respectively, the first and last points on $\text{bdry } N \cap \text{cl } L$ having this property. Finally, let E_y denote the convex hull of all segments

$[z, a_y]$ seen by y , where $a_y \in \text{bdry } N \cap \text{cl } L_1$. Certainly y sees E_y via S .

We say $a < b$ on $\text{bdry } N \cap \text{cl } L_1$ if a precedes b in our clockwise order. Since every pair of points of S sees a common ε -interval at z in $\text{cl } L_1$, it follows that $\text{lub } s_y \leq \text{glb } t_y$. Let $s_1 = \text{lub } s_y$ and $t_1 = \text{glb } t_y$. Then for each y we have $s_0 \leq s_y \leq s_1 \leq t_1 \leq t_y \leq t_0$. If $s_0 = s_1$ or $t_1 = t_0$, the proof is complete. Hence we assume that $s_0 \neq s_1$ and $t_1 \neq t_0$, so that $\text{conv } \{s_1, z, t_1\} \cap L = \{z\}$. If for some positive number r' , the set $\cap E_y$ contains an interval of length r' in $\text{conv } \{s_1, z, t_1\}$, the proof is finished. Otherwise, for every $1/n$ there is some w_n in S for which $E_{w_n} = E_n$ does not contain $M(z, 1/n) \cap \text{conv } \{s_1, z, t_1\}$, where $M(z, 1/n)$ denotes the $1/n$ -disk centered at z . Hence the sequence of sets E_n converges to $[s_0, t_0]$.

In this case, every point of S sees some ε -interval at z on L : Suppose on the contrary that for some x in S , x sees neither $[s_0, z]$ nor $[z, t_0]$ via S . Then there exist points p_1 and p_2 in $\text{bdry } N \cap L_1$ and points p'_1 and p'_2 in $\text{bdry } N \cap L_2$ such that every ε -interval at z seen by x lies either in the convex region bounded by $R(z, p_1) \cup R(z, p_2)$ or in the convex region bounded by $R(z, p'_1) \cup R(z, p'_2)$. However, for n sufficiently large, the points y_n and w_n defined previously see no common ε -interval at z in either of these regions, impossible since every 4 points of S see a common ε -interval at z . Thus the assertion is proved.

Finally, we have to show that for at least one of the segments $[s_0, z]$ and $[z, t_0]$, every point of S sees this segment via S : Otherwise, there would exist points $u, v \in S$, $p_1, p_2 \in \text{bdry } N \cap L_1$ and $p'_1, p'_2 \in \text{bdry } N \cap L_2$ such that the ε -segments at z seen by both u and v would be either in the convex region bounded by $R(z, p_1) \cup R(z, p_2)$ or in the convex region bounded by $R(z, p'_1) \cup R(z, p'_2)$. This contradicts the fact that u, v, w_n, y_n see a common ε -segment at z for each value of n . We conclude that $\ker S$ is a full 1-dimensional, and the proof for Case 1 is complete.

Case 2. Assume that $z \in \text{bdry } S$. There are two possibilities to consider.

Case 2a. Suppose that there exist points s, t, u in S such that $z \in \text{int conv } \{s, t, u\}$. Then for two of these points, say s and t , no point of $[s, z]$ sees any point of $[t, z]$ via S . Then s and t see a common ε -interval at z in the closed region R' bounded by rays $R(t, z) \sim [t, z]$ and $R(s, z) \sim [s, z]$. We define R to be that minimal sector of a circle containing all ε -intervals at z seen by both s and

t . Then R is bounded by segments $[z, s_0]$ and $[z, t_0]$ in S , and since s, t, s_0, t_0 see a common ε -interval at z in R , certainly $\text{conv}\{s_0, z, t_0\} \subseteq S$. As before, order the points of $\text{bdry } R \sim ([z, s_0] \cup [z, t_0])$ in a clockwise direction, and say $a < b$ on $\text{bdry } R \sim ([z, s_0] \cup [z, t_0])$ if a precedes b in our clockwise ordering.

Assume that s_0 and t_0 are first and last points in our ordering. For each y in S , define D_y to be the convex hull of all ε -intervals at z in R seen by y , and let s_y and t_y be the first and last points of D_y in $\text{bdry } R \sim ([z, s_0] \cup [z, t_0])$. Clearly $s_1 \equiv \text{lub } s_y \leq \text{glb } t_y \equiv t_1$. Furthermore, a simple geometric argument reveals that every y in S sees the region $\text{conv}\{s_0, z, t_0\} \cap D_y$ via S . But $s_0 \leq s_y \leq s_1 \leq t_1 \leq t_y \leq t_0$ on $\text{bdry } R$, so $\text{conv}\{s_0, z, t_0\} \cap \text{conv}\{s_1, z, t_1\} \subseteq \text{conv}\{s_0, z, t_0\} \cap D_y$, and y sees $\text{conv}\{s_0, z, t_0\} \cap \text{conv}\{s_1, z, t_1\}$ via S . This set is at least 1-dimensional and so $\dim \ker S \geq 1$, the required result.

Case 2b. Suppose that $z \in \text{bdry conv } S$. Then there must exist a line H supporting S at z , with S in the closed halfplane $\text{cl } H_1$ determined by H . Order the points $\{x: x \in \text{cl } H_1 \text{ and } \text{dist}(z, x) = \varepsilon\}$ in a clockwise direction, and assume that s_0 and t_0 are the first and last points of S in our ordering. Then $\text{conv}\{s_0, z, t_0\} \subseteq S$, since s_0 and t_0 see a common ε -interval at z .

If points s_0, z, t_0 are not collinear, then the argument in Case 2a above may be used to complete the proof. Hence consider the case in which s_0, z, t_0 lie in H . If $s_0 = t_0$, the proof is trivial, so assume $s_0 < z < t_0$. If s_0 and t_0 see a common interval at z in $H_1 \cup \{z\}$, then for some neighborhood N of z , $N \cap S$ is convex, and the argument of Case 1 may be adapted to finish the proof. In case s_0 and t_0 see no such interval, then using the fact that every 4 points see a common ε -interval at z , it is easy to show that for at least one of the segments $[s_0, z]$ and $[t_0, z]$, every point of S sees this segment via S . Hence we conclude that $\dim \ker S \geq 1$ in Case 2, and the proof of Theorem 1 is complete.

The following example illustrates that the number 4 in Theorem 1 is best possible.

EXAMPLE 1. Let S be the set in Figure 1. Then every 3 points of S see via S at least one of the segments $[z, x_i]$, $1 \leq i \leq 4$, yet $\ker S = \{z\}$.

Example 2 shows that the uniform lower bound ε on the segments seen by 4 points is necessary.

EXAMPLE 2. Let S be the set in Figure 2. Then every 4 points see a common segment on the x -axis, but $\ker S$ is the origin.

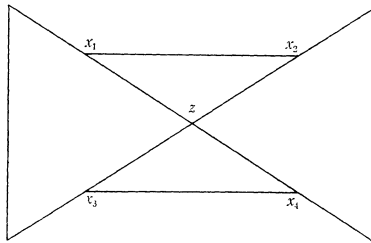


FIGURE 1

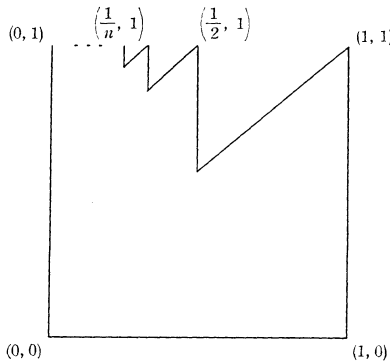


FIGURE 2

Our second theorem is not limited to the plane and is essentially a quantitative version of Krasnosel'skii's theorem.

THEOREM 2. *Let S be a compact set in R^2 . The dimension of $\ker S$ is 2 if and only if for some $\varepsilon > 0$, every 3 points of S see via S a common neighborhood of radius ε . The number 3 is best possible.*

Proof. Again we need only establish the sufficiency of the condition. Clearly S is starshaped, so select z in $\ker S$. We observe that for every 3 points x_1, x_2, x_3 in S , there corresponds a connected subset T of S such that $\text{dist}(z, t) = \varepsilon$ for each t in T and $\text{conv}(T \cup \{z\})$ is a 2-dimensional subset of S . To verify this, let N be a neighborhood of radius ε seen by x_1, x_2, x_3 . Then since $z \in \ker S$, $\text{conv}(\{x_i, z\} \cup N) \subseteq S$ for each i , so x_i sees $\text{conv}(\{z\} \cup N)$ via S . Letting $T = \{y: y \in \text{conv}(\{z\} \cup N), \text{dist}(z, y) = \varepsilon\}$, T satisfies the requirements given above.

Furthermore, letting D denote the closed ε -disk about z , notice that $\text{conv}(T \cup \{z\})$ is either D or a nondegenerate sector of D . If we associate with each set T the corresponding arc length $\delta(T)$ along bdy D , since S is compact, the numbers $\delta(T)$ are bounded below by some positive number δ . Therefore, for each $y \in S$, we may consider the collection G_y of all sectors of D seen by y for which the corresponding arc length on D is at least δ . Then using the sets G_y , the

argument in Theorem 1 may be appropriately modified and in fact simplified to complete the proof. The details are straightforward and hence are omitted.

To see that the number 3 of Theorem 2 is best possible, consider the following easy example.

EXAMPLE 3. Let S be the set in Figure 3. Then every two points of S see one of the regions A_i via S , $1 \leq i \leq 3$, yet $\ker S = \phi$.

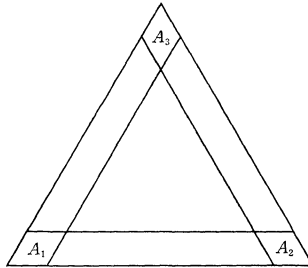


FIGURE 3

In conclusion, it is interesting to notice that both Theorems 1 and 2 fail completely and in fact no $f(k)$ is possible without the requirement that S be compact.

EXAMPLE 4. To see that our set must be closed, let S denote the unit disk with its center removed. Then every j -member subset of S sees via S an open sector having arc length $2\pi/2^j$, and every denumerable set of points sees a radius of S . Yet the set is not starshaped.

EXAMPLE 5. To show that S must be bounded, consider the following example by Hare and Kenelly [4]: Define $T_n = \{(x, y) : n - 1 \leq y \leq n, n \leq x + y\}$, and let $S = \bigcup T_n$. Then every finite subset of S sees via S a common disk of radius $1/2$ in T_1 , yet S is not starshaped.

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