RIGHT SELF-INJECTIVE RINGS WHOSE ESSENTIAL RIGHT IDEALS ARE TWO-SIDED

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A ring $R$ of the kind described by the title is called a right $q$-ring and is characterized by the property that each of its right ideals is quasi-injective as a right $R$-module. The principal results of this paper are Theorem 6, which describes how an arbitrary right $q$-ring is constructed from division rings, local rings, and right $q$-rings with no primitive idempotent, and Theorem 5 which shows that a right $q$-ring cannot have an infinite set of orthogonal noncentral idempotents.

Ivanov described the structure of indecomposable, nonlocal right $q$-rings and conjectured that every right $q$-ring must be a direct sum of such rings together with a ring all of whose idempotents are central. Our results imply that though the structure of right $q$-rings is slightly more complicated than this (there are chain $q$-rings), one can still reduce the study of $q$-rings to ones which have only central idempotents. More precisely, the study of right $q$-rings is reduced to the study of right self-injective duo rings which are either local or have no primitive idempotent.

The work done here is an extension and generalization of Ivanov's investigations. We develop the finiteness conditions inherent in that work without the assumption of indecomposability and the structure of an arbitrary right $q$-ring is developed at the same time. Throughout the paper all rings have identity $1 \neq 0$ and all modules are unital.

Preliminaries. If one has a decomposition $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ of a right $R$-module $A$ as a finite direct sum of submodules then one has a representation of $\text{End}_R A$, the ring of $R$-endomorphisms of $A$, as a ring of $n \times n$ "matrices" of the form $(\alpha_{ij})$ where $\alpha_{ij}$ belongs to $\text{Hom}_R (A_j, A_i)$. In particular, when one has a finite decomposition of the module $R_R$ one also has a representation of the ring $R \cong \text{End}_R R$ as a ring of matrices. A decomposition of $R_R = A \oplus B$ as a direct sum of two modules $A$ and $B$ which are unrelated in the sense that $\text{Hom}_R (A, B)$ and $\text{Hom}_R (B, A)$ are both the trivial group yields a representation of $R$ as the product of the rings $\text{End}_R A$ and $\text{End}_R B$. For a direct sum decomposition of $R_R$, such unrelated summands may be achieved by summing over classes of related summands. When a module $M$ is a direct sum of simple
modules then a sum over a class of related summands is called an isotypic component of $M$, since two simple modules are related if and only if they are isomorphic.

If $A$ is a right $R$-module then $E(A)$ denotes the injective hull of $A$. When $R$ is right self-injective we will assume that $E(A)$ is a right ideal of $R$ whenever $A$ is a right ideal of $R$. The fact that the rings described by the title are the rings whose right ideals are quasi-injective is a consequence of the fact [6, 1.1 Theorem] that $A$ is quasi-injective if and only if $A$ is fully invariant in $E(A)$, that is End$_RE(A) \cdot A \subseteq A$.

Reduction to basic rings. A $q$-ring $R$ will be called basic if each of the nonzero isotypic components of the socle of $R$ is simple, i.e., $R$ has no two distinct isomorphic minimal right ideals. We shall show that a right $q$-ring is the ring direct sum of a semisimple ring and a basic ring.

The following lemma of [3] is fundamental to our study.

**LEMMA 1.** Let $R$ be a right $q$-ring and $A$ and $B$ be independent right ideals of $R$. If $f$ belongs to Hom$_R(A, B)$ then $f(A)$ is semisimple.

**Proof.** Recall that the socle of $B$ is the intersection of the essential submodules of $B$. Let $B_1$ be an arbitrary essential submodule of $B$. It follows that $A \oplus B_1$ is essential in $A \oplus B$. Since $R$ is a right $q$-ring, it follows that $A \oplus B_1$ is fully invariant in $A \oplus B$. Letting $g$ be the endomorphism of $A \oplus B$ defined by $g(a + b) = f(a)$ for $a$ in $A$ and $b$ in $B$, we see that $f(A) \subseteq B_1$.

**Corollary.** If $A$ and $B$ are independent isomorphic right ideals of $R$ then each is injective and semisimple. An isotypic component of the socle of $R$ which is not simple is injective.

**Proof.** Assume that $A$ and $B$ are independent isomorphic right ideals. Since $E(A)$ and $E(B)$ are also independent and isomorphic then the above lemma implies that each is semisimple. It follows that $A = E(A)$ and $B = E(B)$.

Let $H$ be an isotypic component which is not simple. If $H$ is the direct sum of an infinite set of simple modules then $H = H_1 \oplus H_2$ where $H_1 \simeq H$ and $H_2 \simeq H$. Since $H_1$ and $H_2$ are injective then so is $H$. It follows that $H$ must be a finite direct sum of at least two copies of a minimal right ideal $S$ of $R$. Then $S$ is injective and so is $H$. 

Kenneth A. Byrd
PROPOSITION 1. Let $\Gamma$ be an independent set of right ideals of a right $q$-ring $R$. Suppose that for each member $A$ of $\Gamma$ there is a minimal right ideal $S(A)$ of $R$ so that

1. if $A \neq B$ then $S(A) \not\cong S(B)$,
2. for each $A$, $\text{Hom}_R(A, S(A)) \neq 0$,
3. $\Sigma\{A \mid A \in \Gamma\} \cap \Sigma\{S(A) \mid A \in \Gamma\} = 0$.

Then $\Gamma$ is finite.

Proof. According to (2) there is for each $A$ in $\Gamma$ an epimorphism $\alpha_A: A \rightarrow S(A)$ and this induces on the direct sum, the epimorphism $\alpha: \Sigma A \rightarrow \Sigma S(A)$. Choose hulls in $R$ and extend $\alpha$ to the mapping $\beta: E(\Sigma A) \rightarrow E(\Sigma S(A))$. From (3) and Lemma 1 we know that the image of $\beta$ is $\Sigma S(A)$. On the other hand the image of $\beta$ must be cyclic since $E(\Sigma A)$ is a direct summand of $R$. It follows that there are only finitely many nonisomorphic $S(A)$ for $A$ in $\Gamma$, so (1) implies that $\Gamma$ is finite.

THEOREM 1. A right $q$-ring is isomorphic to the direct product of a semisimple ring and a basic right $q$-ring.

Proof. The above proposition implies that $\text{Soc } R$ has only a finite set $\{A_1, \ldots, A_k\}$ of isotypic components which are not simple. Since each of the $A_i$ is injective we have a decomposition $R_R = (\Sigma A_i) \oplus B$. It follows easily that $R$ is isomorphic to the product of the semisimple ring $\text{End}_R(\Sigma A_i)$ and the ring $\text{End}_R B$. If $B = eR$ where $e^2 = e$ then $\text{End}_R B \simeq eRe$. Since $eR(1 - e) \simeq \text{Hom}_R(\Sigma A_i, B) = 0$ then $eRe = eR$ so that right ideals of $eRe$ are the same as $R$-submodules of $B$. Then since $B$ has no distinct pair of isomorphic simple submodules, it follows that $eRe$ is basic.

DEFINITION. If $A$ and $B$ are right ideals of $R$ then the notation $A \rightarrow B$ will indicate that $A \cap B = 0$ and $\text{Hom}_R(A, B) \neq 0$. We shall write $A \rightarrow$ if $A \rightarrow B$ for some $B$, and we shall write $\rightarrow B$ if, for some $A$, $A \rightarrow B$.

The following finiteness condition is due to Ivanov [3, Lemma 3].

THEOREM 2. Let $R$ be a right $q$-ring. If $\Gamma$ is an independent set of right ideals of $R$ so that $A \rightarrow$ for each $A$ in $\Gamma$ then $\Gamma$ is finite.

Proof. By Theorem 1 we may assume that $R$ is basic. By Lemma 1 we can find for each $A$ in $\Gamma$ an epimorphism $\alpha_A$ from $A$ onto a minimal right ideal $S(A)$ such that $A \cap S(A) = 0$. Also, by taking injective hulls, we may assume that each $A$ in $\Gamma$ is a direct
Suppose that $S(A) = S(B)$ for some $B, B$ in $\Gamma$ where $A \neq B$. From the projectivity of $A$ there is a mapping $\beta: A \to B$ so that $\alpha_B \beta = \alpha_A$. It follows from Lemma 1 that $\text{Im} \beta$ contains a copy of $S(B)$, so that $S(B) \subseteq B$ since $R$ is basic. This contradiction implies that if $A, B \in \Gamma$ and $A \neq B$ then $S(A) \neq S(B)$.

Let $\Gamma_1$ be the set of all $A_i$ in $\Gamma$ so that $S(A_i) \subseteq \Sigma\{A_i | A_i \in \Gamma\}$. Since $R$ is basic and the sum is direct then for each member $A_1$ of $\Gamma_1$ there is a unique member $\gamma(A_1)$ of $\Gamma$ so that $S(A_1) \subseteq \gamma(A_1)$. We use the mapping $\gamma: \Gamma_1 \to \Gamma$ to form the partition $\{\gamma^{-1}(A) | A \in \text{Im} \gamma\}$ of $\Gamma_1$. Since $A \notin \gamma^{-1}(A)$ for each $A$ in $\text{Im} \gamma$, it follows from Proposition 1 that each member of this partition is a finite set.

Assume that $\Gamma_1$ is infinite and let $\phi$ be a function which chooses a member from each nonempty subset of $\Gamma_1$. If $X$ is a finite subset of $\Gamma_1$ then $X \cup \gamma(X) \cup \gamma^{-1}(X)$ is also finite where $\gamma^{-1}(X) = \bigcup \{\gamma^{-1}(B) | B \in X\}$ and $\gamma^{-1}(B) = \emptyset$ if $B \notin \text{Im} \gamma$. Denote by $X'$ the set complement of $X \cup \gamma(X) \cup \gamma^{-1}(X)$ in $\Gamma_1$. We note $X' \neq \emptyset$ for all finite subsets $X$ of $\Gamma_1$. Define the sequence $(A_i)_{i=1}^\infty$ in $\Gamma_1$ by setting $A_i = \phi(\Gamma_i)$ and if $A_1, \ldots, A_n$ are already chosen then $A_{n+1} = \phi([A_1, \ldots, A_n])$. Suppose that $(\Sigma A_i) \cap (\Sigma S(A_i)) \neq 0$. Since $R$ is basic this means that for some $j, k$ one has $\gamma(A_j) = A_k$ and this cannot happen by the construction of the sequence. The existence of such a sequence contradicts Proposition 1 so we conclude that $\Gamma_1$ is finite.

Since $\Gamma - \Gamma_1$ is clearly finite by Proposition 1 then $\Gamma$ is finite.

Injective hulls of minimal right ideals. Let $\mathcal{S}$ be the set of minimal right ideals of a basic right $q$-ring $R$ and let $E(\mathcal{S}) = \{E(S) | S \in \mathcal{S}\}$ be a chosen set of injective hulls in $R$ for the members of $\mathcal{S}$. For each $S$ in $\mathcal{S}$ there is a primitive idempotent $e_S$ of $R$ such that $e_SR = E(S)$. According to Lemma 1 if $e$ is a primitive idempotent of $R$ and $-eR$ then $eR$ is isomorphic to a member of $E(\mathcal{S})$. In fact if $eR$ is not isomorphic to a member of $E(\mathcal{S})$ then $e$ is central as the next proposition shows.

**Proposition 2.** Let $e$ be a primitive idempotent of a basic right $q$-ring $R$. If $eR \to$, then $-eR$.

**Proof.** Suppose the proposition is false so that $(1 - e)R \cong \text{Hom}_R(eR, (1 - e)R) \neq 0$ but $eR(1 - e) = 0$. Since $e$ is primitive $eR = eRe$ is a local ring and since $eR(1 - e) = 0$ then the right ideals of $eRe$ are precisely the $R$-submodules of $eR$. If $J$ is the Jacobson radical of $R$ then $eJ$ is the unique maximal right ideal of $eRe$. If $eJ = 0$ then $eR$ is simple and since $R$ is basic it follows that
(1 − e)Re = 0 contrary to the assumptions. So eJ = 0 and it follows that eJ contains a nonzero cyclic submodule L for which there is a eRe-epimorphism β: L → eRe/eJ which is also an R-epimorphism. The assumption that (1 − e)Re = 0 together with Lemma 1 implies that the simple image eRe/eJ of eR embeds in (1 − e)R. Since (1 − e)R is injective there is an R-homomorphism α: eR → (1 − e)R so that α|L = β. Since Im α is semisimple it follows that α(eJ) = 0 so that β = 0 which is a contradiction.

PROPOSITION 3. If e is a primitive idempotent of a basic right q-ring R and →eR, then (1) S = eR(1 − e) is a minimal right ideal of R, (2) eRe ∼ EndR_S, and (3) S is the only proper nonzero submodule of eR.

Proof. (1) Since →eR then eR(1 − e) is nonzero and it is contained in the socle of eR. Since e is primitive it follows that eR = E(S) for some minimal right ideal S containing eR(1 − e). If HomR(eR, S) ≠ 0 then there is a copy of S in (1 − e)R contradicting the fact that R is basic. It follows that se = 0 for every s ∈ S, that is S ⊆ eR(1 − e). Thus S = eR(1 − e).

(2) If J is the Jacobson radical of R then eRe has radical eJe = \{x ∈ eRe | xS = 0\}. Since eR = S ⊕ eRe as abelian groups one has

\[(eJe)R = (eJe)(eR) \subseteq (eJe)S + eJe = eJe\]

so that eJe is a right R-submodule of eR. Since S ∩ eJe = 0 then eJe = 0 so that eRe is a division ring. Restriction to S is an isomorphism from EndReR onto EndR_S.

(3) If K is a nonzero submodule of eR then S ⊆ K and K = Ke ⊕ K(1 − e). It follows that S = K(1 − e). Since Ke is a right ideal of eRe then either Ke = 0 or Ke = eRe. Thus K = S or K = eR.

Let \(\mathcal{A}(R) = \{E(S) ∈ E(\mathcal{A}) | E(S)\} \). We consider the restriction of the →→ relation to \(\mathcal{A}(R)\). Note that E(S₁) → E(S₂) for E(S₁), E(S₂) members of \(\mathcal{A}\) means that the top, E(S₁)/S₁, of E(S₁) is isomorphic to the bottom, S₂, of E(S₂).

Let D be the domain and T be the range of the restriction of → to the set \(\mathcal{A}\). It is easy to show that → is a one-to-one function from D onto T. Define \(α: \mathcal{A} → \mathcal{A}\) by \(α(E₁) = E₂\) if \(E₁ ∈ D\) and \(E₁ → E₂\) and \(α(E₂) = E₁\) if \(E₂ ∈ D\). Similarly \(α⁻¹: \mathcal{A} → \mathcal{A}\) is defined by \(α⁻¹(E₂) = E₁\) if \(E₂ ∈ R\) and \(E₁ → E₂\) and \(α⁻¹(E₁) = E₂\) if \(E₁ ∈ T\). Then for each \(E ∈ \mathcal{A}\) let \(\bar{E} = \{α⁻¹(E_k) | k ∈ Z\}\). It is easy to see that (1) \(E ∈ \bar{E}\) since \(α^0\) is the identity mapping, and (2) if \(F ∈ \bar{E}\) then \(\bar{F} = \)}
so that the set $\mathcal{A}$ of $\rightarrow$-classes $\mathcal{E}$ for $E \in \mathcal{A}$ is a partition of $\mathcal{A}$. In fact the associated equivalence relation on $\mathcal{A}$ is just the smallest equivalence relation on $\mathcal{A}$ which contains the restriction of $\rightarrow$ to $\mathcal{A}$.

It is immediate from Theorem 2 that the set of classes $\mathcal{E}$ with more than one member is a finite set and also that each class $\mathcal{E}$ is itself finite. It is straightforward to show that these classes $\mathcal{E}$ are of two kinds namely:

(1) Chain: $\rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_l$ where $E_i \in T$ and $E_l \in D$.

(2) Loop: $E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_l \rightarrow E_1$.

In each case the cardinality $l$ of $\mathcal{E}$ will be called the length of $\mathcal{E}$.

**Lemma 2.** Suppose that $e$ is a primitive idempotent of a basic right $q$-ring $R$ and $\rightarrow eR$ so that $S = eR(1 - e) \neq 0$.

(1) The right annihilator $S^r = \{x \in R | Sx = 0\}$ is a maximal right ideal.

(2) If $f$ is also a primitive idempotent of $R$ and $eR \rightarrow fR$ then $eRe \simeq fRf$.

**Proof.** (1) If $s$ is a nonzero element of $S$ then $s^r = M$ is a maximal right ideal. The right ideal $M$ must be essential since otherwise $S = eR$ and since $R$ is basic this contradicts the assumption $eR(1 - e) \neq 0$. It follows that $M$ is a two-sided ideal of $R$. For any nonzero element $s_i$ of $S$ one has $s_i = sr$ for some $r$ in $R$ so $s_iM = srM \subseteq sM = 0$. Thus $M = S^r$.

(2) Let $T$ be the simple submodule of $fR$. Since $T = fRe$ is a simple right $R$-module, it is a 1-dimensional $eRe$-space on the right. Jacobson's density theorem [5, p. 28] and (1) imply that $T$ is also a 1-dimensional $fRf$-space on the left. Choose a nonzero element $t$ of $T$. The correspondence $a \leftrightarrow b$ if and only if $at = tb$ is an isomorphism between $fRf$ and $eRe$.

The $\rightarrow$-classes $\mathcal{E}$ of $R$ are determined "up to isomorphism" by our choice of a representative set of injective hulls of minimal right ideals of $R$. However, the sum of an $\rightarrow$-class is independent of this choice. This is a consequence of the following proposition.

**Proposition 4.** Let $e$ be an idempotent of the basic right $q$-ring $R$. There is a one-to-one correspondence between the set $(1 - e)Re$ and the set of copies of $eR$ in $R$ such that to the element $z$ of $(1 - e)Re$ corresponds the module $(1 + z)eR$.

**Proof.** If $z$ belongs to $(1 - e)Re$ then $f = (1 + z)e$ is idempotent and since $ef = e$ and $fe = f$ it follows that $fR \simeq eR$. If $(1 + z)eR=$
(1 + z)eR for z, z in (1 − e)Re then for some r in R, (1 + z)e = (1 + z)er. It follows that e = er and z = z = z = z. Thus the correspondence is one-to-one.

Let E be a copy of eR in R. Since (1 − e)R contains no non-zero copy of a submodule of eR then the kernel, E ∩ (1 − e)R, of the projection x → ex of E into eR is zero. It follows that eR = eE ⊕ A for some submodule A of eR. But A must be zero since there is a copy of A in eE. Thus the projection of E into eR is an isomorphism onto eR. Choose a in E so that ea = e and let z = (1 − e)ae. If x ∈ E then e(ax − x) = 0 and it follows that ax = x for all x in E. Then for x in E one has

\[ x = ex + (1 − e)x = ex + (1 − e)ax = ex + zx = (1 + z)ex. \]

Thus one has E = (1 + z)eR.

In particular if eR → fR with eR and fR members of \( \mathcal{M} \) then every copy of eR in R is contained in eR ⊕ fR because (1 − e)Re = fRe. Thus the sum of an \( \to \)-class is independent of the choice of the injective hulls.

**Definition.** A basic right q-ring R is called a loop q-ring if R has only one \( \to \)-class, that class is a loop, and R is the sum of its loop.

**Notation.** Let D be a division ring. We denote by \( D_0 \) the \( D \to D \) bimodule D equipped with the zero multiplication.

**Theorem 3.** If R is a loop q-ring of length l then there is a division ring D so that R is isomorphic to the ring \( H(l, D) \) of \( l \times l \) matrices with elements on the diagonal from D and elements in the positions (2, 1), (3, 2), \( \ldots \), (l, l − 1), (1, l) from \( D_0 \) and zero entries elsewhere. Conversely every ring \( H(l, D) \) is a loop q-ring.

**Proof.** The first statement is an immediate consequence of the matrix representation of \( R = \text{End}_R R \) where \( R_R = \sum_{i=1}^l E_i \) and \( E_1 \to E_2 \to \cdots \to E_i \to E_i \). One may take \( D = \text{End}_R E_i \) and use Lemma 2 (2). The converse is proved in [3, Theorem 3].

The following theorem may be proved by a straightforward induction on the number of loops of R.

**Theorem 4.** Let R be a basic right q-ring. There is a set \{l_1, l_2, \ldots, l_k\} of integers \( \geq 2 \) and a set of division rings \{D_i,
$D_2, \ldots, D_k$ so that

$$R \cong \prod_{i=1}^{k} H(l_i, D_i) \times R_i$$

where $R_i$ is a basic right q-ring which has no loops.

Chain q-rings. Assume that $R$ is a basic right q-ring with no loops. Suppose that $\mathscr{C} = \{E_i | 1 \leq i \leq m\}$ is a finite set of chains of $R$ where $E_i$ is $\rightarrow E_{i1} \rightarrow E_{i2} \rightarrow \cdots \rightarrow E_{it}$ with $E_{ij} = e_{ij}R$ for a primitive idempotent $e_{ij}$ of $R$. Let $f = 1 - \Sigma e_{ij}$. Then for each $i$ one has $fR \rightarrow E_{ij}$ exactly when $j = 1$. Also since $fR(1 - f) = 0$ then $fR = fRf$ is a ring with identity $f$.

**PROPOSITION 5.** With the notation above, the ring $fR$ is a basic right q-ring. The set of arrow classes of $R$ is the disjoint union of the set of arrow classes of $fR$ with the set $\mathscr{C}$. For each $i$, the $fR$-module $e_{ii}Rf$ is simple, injective and is not embeddable in $fR$.

**Proof.** The first two statements are straightforward consequences of the facts that the right ideals of $fR$ coincide with the $R$-submodules of $fR$ and $\text{Hom}_{fR}(K, L) = \text{Hom}_R(K, L)$ for any right ideals $K$ and $L$ of $R$ on which $f$ acts as a right identity. Since $e_{ii}Rf$ is a simple $R$-module it is a simple $fR$-module and as $R$ is basic it cannot be isomorphic to a right ideal of $fR$. The $fR$-injectivity of $e_{ii}Rf$ follows from Baer's criterion and Lemma 1.

Suppose that $fR = gR + hR$ where $g$ and $h$ are orthogonal idempotents of $fR$. For each $E_i$ in $\mathscr{C}$ exactly one of $gR \rightarrow E_{i1}$ or $hR \rightarrow E_{i1}$ is true because if both $gR$ and $hR$ mapped onto the simple submodule of $E_{i1}$ then projectivity of $gR$ would imply that $hR$ contained a copy of that simple module thus violating the agreement that $R$ is basic. If, say, $gR \rightarrow E_{i1}$ we say the chain $E_i$ is associated with $gR$. In this way each decomposition of $f$ as a sum of orthogonal idempotents induces a corresponding partition of the set of chains $\mathscr{C}$. The proof of the next proposition describes a procedure for decomposing $f$ in such a way that each component summand of $fR$ has associated with it exactly one chain from $\mathscr{C}$.

**PROPOSITION 6.** Let $\Lambda$ be an independent set of right ideals of a right q-ring $R$. If there is a right ideal $A$ of $R$ such that (1) $A \rightarrow B$ for every $B \in \Lambda$ and (2) $A \cap (\Sigma B) = 0$, then $\Lambda$ is finite.

**Proof.** We may assume that $R$ is basic, that the members of
Suppose that $B_1$ and $B_2$ belong to $A$ and $B_1 \neq B_2$. Since $R$ is basic $B_1 \neq B_2$ and it follows from Lemma 2(1) that $B_i \neq B_j$. From $eR \rightarrow B$ for each $B \in A$ it follows that $Be \neq 0$. In particular $e \in B_i$ for $i = 1, 2$ and since $R/B_i$ is a division ring it follows that $1 - e \in B_i \cap B_j$. The modular law implies that

$$B_i = (1 - e)R + (B_i \cap eR)$$

and

$$B_i \cap eR = B_i \cap (eRe + eR(1 - e)) = eR(1 - e) + (B_i \cap eRe).$$

Thus if $B_1 \neq B_2$ then $eRe \cap B_i \neq eRe \cap B_j$.

Choose $x \in (eRe \cap B_i) - B_i$. Let $J$ denote the Jacobson radical of $R$. Since by [1, Theorem 3.1] $eRe/eJe$ is a regular ring there is an element $y$ of $eRe$ such that $x - xyz$ belongs to $eJe$. Since idempotents of $eRe$ lift modulo $eJe$ by [1, Theorem 4.1] then there is an idempotent $g$ of $eRe$ such that $xy - g$ belongs to $eJe$. We note that $g \in B_i - B_j$. Thus one has the decomposition $A = (e - g)R \oplus gR$ where $gR \rightarrow B_2$ and $(e - g)R \rightarrow B_1$.

Assume that $A$ is infinite. Choose one of $gR$ and $(e - g)R$ which has infinitely many members of $A$ as homomorphic images and call it $A'_1$ and call the other $A_i$ so that $A = A_1 \oplus A'_1$. Replace $A$ by $A'_1$ and repeat the above process so that $A'_1 = A_2 \oplus A'_2$ where $A'_2$ has infinitely many homomorphic images in $A$. In this way we construct an infinite sequence $\{A_i\}_{i=1}^\infty$ which satisfies the three conditions of Proposition 1 and this is a contradiction.

**DEFINITION.** A basic right $q$-ring $R$ is called a chain $q$-ring if $R = fR \oplus E_i \oplus \cdots \oplus E_i$ where $\rightarrow E_i \rightarrow E_i \rightarrow \cdots \rightarrow E_i$ is the only $\rightarrow$-class of $R$ and $fR \rightarrow E_i$. We call $fR$ the corner of $R$ in this case.

Note that in a chain $q$-ring $fR$ is a basic right $q$-ring all of whose idempotents are central since $fR$ has no $\rightarrow$-classes. Also $fR$ is not a right cogenerator since the simple module $E_i f$ does not embed in $fR$. For instance $fR$ might be an infinite product of division rings.

**PROPOSITION 7.** If $R$ is a basic loopless right $q$-ring then $R$ is isomorphic to the product of a finite set of chain $q$-rings each of which has as a corner an infinite product of division rings together with a basic loopless $q$-ring which has no projective minimal right
ideal.

Proof. In the basic loopless q-ring \( R \) let \( \{ e_i R | i \in I \} \) be the set of projective minimal right ideals. For each \( i \) in \( I \) one has \( e_i R = e_i R e_i \) is a division ring. Consider the usual embedding \( \alpha \) of the direct sum \( \sum e_i R \) into \( \prod e_i R \) where \( \alpha \) maps \( e_j \) to \( \delta_{ij} e_i \). Since \( e_i R = e_i R e_i \) and \( e_i R e_j = 0 \) for \( i \neq j \) then \( \alpha \) is an essential embedding. It follows that there is an \( R \)-monomorphism \( \phi : \prod e_i R \to R \) so that \( \phi \cdot \alpha \) is the inclusion of \( \sum e_i R \) in \( R \). Let \( \psi \) be the splitting map for \( \phi \) so \( \psi \phi = 1 \). One may show that \( \psi (1) = \langle e_i \rangle_{i \in I} \).

If \( g = \phi \psi (1) \) then the image of \( \phi \) is \( g R \) and \( \psi (g) = \langle e_i \rangle_{i \in I} \). Since \( R \) is basic, \( g R (1 - g) = 0 \) so \( g R = g R g \). One has for \( r, s \) in \( R \)

\[
\psi (g r \cdot g s) = \psi (g r s) = \psi (g) r s = (e_i)_{i \in I} r s = (e_i r s)_{i \in I} = (e_i r)_{i \in I} \cdot (e_i s)_{i \in I}
\]

where the last multiplication is componentwise. Thus \( \phi \) is a ring isomorphism from \( \prod e_i R \) onto \( g R \).

Proposition 6 implies that the set of chains \( \langle g R \rangle \) of \( R \) associated with \( g R \) is finite, and that there is a decomposition \( g = g_1 + g_2 + \cdots + g_k \) so that the \( g_i \) are orthogonal idempotents associated one-to-one with the chains of \( \langle g R \rangle \), i.e., each \( \langle g R \rangle \) is a singleton. Let \( \bar{g}_i \) be an idempotent such that \( \bar{g}_i R = g_i R \oplus \sum \langle g R \rangle \). One checks that for each \( i = 1, \ldots, k \), \( \bar{g}_i \) is central. For instance \( \bar{g}_i R (1 - \bar{g}_i) = 0 \) since otherwise \( (1 - \bar{g}_i) R \) has a simple image in \( \bar{g}_i R \) and by projectivity must contain a copy of that simple module, thus contradicting the fact that \( R \) is basic. Thus \( R = \bar{g}_i R \oplus (1 - \bar{g}) R \) where \( \bar{g} = \sum \bar{g}_i \) and each \( \bar{g}_i \) is central. For each \( i \), \( \bar{g}_i R \) is a chain q-ring with corner, \( g_i R \), a product of division rings. Also \( (1 - \bar{g}) R \) is a basic loopless q-ring which has no projective minimal right ideal since any such must be contained in \( \bar{g} R \) by construction.

Matrix representation of chain q-rings. A chain q-ring \( R \) is a q-ring with orthogonal idempotents \( f, e_1, e_2, \ldots, e_l \) such that the \( e_i, 1 \leq i \leq l \), are primitive, \( f R \to e_1 R \to e_2 R \to \cdots \to e_l R, f R = f R f \), and \( R = f R \oplus e_1 R \oplus \cdots \oplus e_l R \). Since the \( \to \)-relations shown are the only ones which exist between the modules \( f R, e_1 R, \ldots, e_l R \) one has the matrix representation

\[
R \cong \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & e_1 R f & e_1 R e_1 & \cdots & 0 \\
0 & 0 & e_2 R e_1 & \cdots & 0 \\
0 & 0 & \cdots & e_{l-1} R e_{l-1} & 0 \\
0 & 0 & \cdots & e_l R e_{l-1} & e_l R
\end{bmatrix}.
\]
Since $e_i R f$, is 1-dimensional as a left $e_i R e_i$-space if we select $x_i \in e_i R f$, $x_i \neq 0$ and if $M = x_i^\perp$ then $d_i x_i = x_i d_i$ implements a ring isomorphism $d_i \mapsto d_i$ from $e_i R e_i$ onto $f R / M$ and at the same time $e_i R f \simeq f R / M$ as an $f R$-module. If we use these isomorphisms to identify $e_i R e_i$ with $f R / M$ and $e_i R f$ with the $f R$-module $f R / M$ then the left action of $e_i R e_i$ on $e_i R f$ corresponds to the natural left $f R / M$-module structure of $f R / M$. Similarly for $i \geq 1$ each $e_{i+1} R e_i$ is 1-dimensional on each side so that selecting $x_{i+1} \in e_{i+1} R e_i$, $x_{i+1} \neq 0$ we have isomorphisms $d_{i+1} x_{i+1} = x_{i+1} d_{i+1}$. If we denote by $(f R / M)_0$ the abelian group of $f R / M$ with its usual left and right module structures over the rings $f R$ and $f R / M$ and with the zero multiplication then it is easy to see that

$$
R \simeq \begin{pmatrix}
    f R & 0 & 0 & \cdots & 0 \\
    (f R / M)_0 & f R / M & 0 & \cdots & 0 \\
    0 & (f R / M)_0 & f R / M & \ddots & \\
    \vdots & 0 & \ddots & \ddots & 0 \\
    0 & \vdots & \cdots & (f R / M)_0 & f R / M
\end{pmatrix}.
$$

The following proposition shows that, conversely, every ring of this form is a right $q$-ring.

**Definition.** Let $A$ be a right $q$-ring with an essential maximal right ideal $M$ such that $A / M$ is injective and does not embed in $A$. We denote by $C(A, M, l)$ the ring of $(l + 1) \times (l + 1)$ matrices with entries in the $(1, 1)$ position from $A$, entries in the other main diagonal positions from $A / M$, entries on the sub-diagonal from $(A / M)_0$, and zero entries elsewhere. (It is convenient to allow $l$ to be any integer $\geq 0$.)

**Proposition 8.** For any $l \geq 0$, the ring $C(A, M, l)$ as defined above is a right $q$-ring.

**Proof.** Let $A$ and $M$ be as described above. For each $l \geq 1$ let $A_l = C(A, M, l)$ and let $M_l$ denote the ideal of $A_l$ whose members are those matrices with zero entry in the $(l + 1, l + 1)$ position. We wish to show by induction that for every $l \geq 1$ the ring $A_l$ is a right $q$-ring with the essential maximal right ideal $M_l$ such that $A_l / M_l$ is $A_l$-injective and does not embed in $A_l$.

If $l \geq 1$ there is an obvious ring isomorphism between $A_{l+1}$ and $C(A_l, M_l, 1)$. Using this, the proof by induction is reduced to prov-
ing that the statement holds when \( l = 1 \).

Let \( e_i, \ i = 1, 2 \) be the idempotent matrix of \( A_1 \) with zero entries except at the \((i, i)\) position where the entry is 1. Since \( e_1 A_l e_2 = 0 \) then a minimal right ideal of \( A_1 \) is either a minimal right ideal of \( e_i A_1 \) (i.e., a minimal right ideal of \( A \)) or it is a simple submodule of \( e_2 A_1 \). The kernel of the ring homomorphism from \( A_1 \) onto \( A/M \) which sends a matrix \( X \) to \( X e_2 \) is \( M_1 = A e_1 \). Since \( A/M \) is a division ring, the ideal \( M_1 \) is a maximal right ideal of \( A_1 \). It is easy to check that \( S = e_2 A_1 e_1 \) is an essential submodule of \( e_2 A_1 \) so that \( M_1 \) is an essential right ideal of \( A_1 \). Since \( e_2 A_1/S \cong A_1/M_1 \) it follows that \( S \) is the only proper nonzero submodule of \( e_2 A_1 \).

Suppose that \( K \) is an essential right ideal of \( A_1 \). If \( K \supseteq e_2 A_1 \) then \( K = e_2 A_1 \oplus (K \cap e_1 A_1) \). Otherwise \( K \cap e_2 A_1 = S \) so \( Ke_2 = 0 \) and \( K \subseteq M_1 = e_1 A_1 \oplus S \). It follows that \( K = S \oplus (K \cap e_1 A_1) \). Since \( A \) is a right \( q \)-ring it is easy to see that \( K \cap e_1 A_1 \) is a two-sided ideal of \( e_1 A_1 \) and it follows easily that in either of the above cases, \( K \) is a two-sided ideal of \( A_1 \).

To see that \( A_1 \) is right self-injective it suffices to apply Baer's criterion as follows. Let \( \phi : K \to A_1 \) be an \( A_1 \)-homomorphism where \( K \) is an essential right ideal of \( A_1 \). Since \( K \) is an ideal \( K = e_1 K \oplus e_2 K \). Let \( \phi_i \) be the restriction of \( \phi \) to \( e_i K \). Since \( e_i A_1 = e_1 A_1 e_i \) then \( \text{Im} \phi_i \subseteq e_i A_1 = e_1 A_1 \oplus S \). The injectivity of \( A/M \) as an \( A \)-module implies that \( S \) is an injective \( e_i A_1 \)-module and by assumption \( e_i A_1 \cong A \) is right self-injective. Since \( \phi_i \) is an \( e_i A_1 \)-homomorphism it follows that there is an element \( a \) of \( A \) and an element \( s \) of \( A/M \) so that for every \( X \) in \( e_i K \) one has \( \phi_i(X) = \begin{pmatrix} a & 0 \\ s & 0 \end{pmatrix} \cdot X \). Let \( \phi_2 \) be the restriction of \( \phi \) to \( e_2 K \). Since no submodule of \( e_1 A_1 \) has a nonzero image in \( e_1 A_1 \) then the image of \( \phi_2 \) must be contained in \( e_2 A_1 \). It is then easy to see that there is an element \( d \) of \( A/M \) so that for each \( Y \) in \( e_2 K \) one has \( \phi_2(Y) = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \cdot Y \). It follows that for all \( Z \) in \( K \), \( \phi(Z) = \begin{pmatrix} a & 0 \\ s & d \end{pmatrix} \cdot Z \). So \( A_1 \) is right self-injective.

If \( A_1/M_1 \) embeds in \( A_1 \) then either \( A_1/M_1 \) embeds in \( e_1 A_1 \) or \( A_1/M_1 \cong S \). Since \( A_1/M_1 \cong e_1 A_1/S \) and \( e_1 A_1 e_2 = 0 \) then \( A_1/M_1 \) does not embed in \( e_1 A_1 \). If \( A_1/M_1 \cong S \) then there is an epimorphism from \( e_2 A_1 \) onto \( S \). But since \( Se_2 = 0 \) there is no such mapping. Thus \( A_1/M_1 \) does not embed in \( A_1 \). To see that \( A_1/M_1 \) is \( A_1 \)-injective suppose that \( \phi : K \to A_1/M_1 \) is an epimorphism where \( K \) is an essential right ideal of \( A_1 \). Since \( K \) is an ideal then \( K = e_1 K \oplus e_2 K \). Because \( e_1 A_1 e_2 = 0 \) and \( A_1/M_1 \) is an image of \( e_2 A_1 \) it follows that \( \phi(e_1 K) = 0 \). If \( e_1 K = e_2 K \) then \( \phi \) extends immediately to \( A_1 \). Otherwise, \( e_1 K = S \) so that \( \phi \) is an isomorphism between \( S \) and \( A_1/M_1 \) which we have just shown to be impossible.
The finiteness condition. The finiteness results Propositions 1 and 6 and Theorem 2 will be subsumed in the following theorem whose proof will be given as a sequence of lemmas.

**Theorem 5.** A right q-ring has no infinite set of orthogonal, noncentral idempotents.

It suffices to prove the result for basic rings where from Lemma 1 and Proposition 2, the theorem is equivalent to the assertion that \( \mathcal{A}(R) \) is a finite set. From Theorem 4 and Proposition 7 we may assume \( R \) has no loops and no projective minimal right ideals. We now reduce the problem to the case where \( R \) has no chain of length \( l > 1 \). Let \( \{ E_i | 1 \leq i \leq m \} \) be the set of chains of \( R \) of length \( l > 1 \), where \( E_i \) is \( \rightarrow E_{i_1} \rightarrow E_{i_2} \rightarrow \cdots \rightarrow E_{i_l} \) with \( E_{i_j} = e_{i_j}R \) for a primitive idempotent \( e_{i_j} \) of \( R \). Let \( f = 1 - \Sigma e_{i_j} \) so that \( fR \rightarrow E_{i_1} \) for each \( i \) and \( fR = fRf \). It follows that \( fR \) is a right q-ring which is basic, loopless, without projective minimal right ideals and whose \( \rightarrow \) -classes are exactly the chains of \( R \) of length 1.

**Lemma 3.** Let \( R \) be a basic right q-ring whose only \( \rightarrow \) -classes are chains of length 1 and let \( \mathcal{S} \) be the set of minimal right ideals of \( R \). To each subset \( A \) of \( \mathcal{S} \) we associate an idempotent \( e_A \) so that \( e_AR \) is an injective hull of \( \Sigma \{ S | S \in A \} \). If \( A \subseteq \mathcal{S} \) then there is a subset \( A_1 \) of \( \mathcal{S} \) so that \( A \Delta A_1 \) is finite and \( e_{A_1} \) is central. (Here \( \Delta \) denotes symmetric difference of sets.)

**Proof.** Let \( A \subseteq \mathcal{S} \) and \( e_AR \) be a hull of \( \Sigma \{ S | S \in A \} \). Let \( B \subseteq \mathcal{S} \) be the finite set of simple images of \( e_AR \) in \( (1 - e_A)R \) and let \( C \subseteq \mathcal{S} \) be the finite set of simple images of \( (1 - e_A)R \) in \( e_AR \). One has \( e_AR = eR + e_cR \) and \( (1 - e_A)R = fR + e_bR \) where \( e, f, e_c \) and \( e_b \) are pairwise orthogonal idempotents. If \( A_1 = (A - C) \cup B \) then \( A \Delta A_1 \) is finite and \( (e + e_b)R \) is a hull of \( \Sigma \{ S | S \in A_1 \} \). It is routine to check that \( e + e_b \) is central. For instance, to see that \( \text{Hom}_R((1-e-e_b)R, (e + e_b)R) = \text{Hom}_R(fR + e_cR, eR + e_bR) = 0 \) one argues as follows: Any simple submodule of \( e_bR \) is an image of \( e_AR \). It cannot also be an image of \( (1-e_A)R \) so it cannot be an image of \( fR \) and thus \( \text{Hom}_R(fR, e_bR) = 0 \). Since the \( \rightarrow \) -classes are all chains of length one then \( \text{Hom}_R(e_cR, eR) = \text{Hom}_R(e_cR, e_bR) = 0 \) and \( \text{Hom}_R(fR, eR) = 0 \) by the definition of \( C \). The argument that \( \text{Hom}_R((e + e_b)R, (1-e-e_b)R) = 0 \) is similar.

**Proof of Theorem 5.** The proof is by contradiction. Assume the theorem is false. Then there is a right q-ring \( R \) such that
\( \mathcal{S} = \{ S_i | i \in I \} \), the set of minimal right ideals of \( R \), is infinite, \( S_i \neq S_j \) for \( i \neq j \), and if for each \( i \) we let \( e_iR \) be a hull of \( S_i \) then 
\[
(1 - e_i)Re_i = 0 \quad \text{and} \quad e_iR(1 - e_i) \neq 0.
\]
If we let \( A \) be a countably infinite subset of \( \mathcal{S} \) then by the Lemma 3 there is a countably infinite subset \( A_i \) so that the hull of \( \sum S_i | S_i \in A_i \) is generated by a central idempotent \( e_{A_i} \). It follows that we can assume that \( \mathcal{S} \) is countable (so we take \( I \) to be the set of positive integers) and that \( R \) is the hull of \( \sum_{i=1}^{\infty} S_i \), i.e., the socle of \( R \) is essential in \( R \). The proof is given as a sequence of eleven assertions proved individually.

(1) If \( J \) is the Jacobson radical of \( R \) then
\[
J = \{ r \in R | re_i = 0, \forall i \} = \bigcap_{i=1}^{\infty} R(1 - e_i).
\]

**Proof of (1).** Since \( (1 - e_i)Re_i = 0 \) then \( Re_i = e_iRe_i \). Since \( e_iRe_i \) is a division ring then \( Re_i \) is a minimal left ideal of \( R \) so that \( R(1 - e_i) \) is a maximal left ideal. Thus \( J \subseteq \bigcap R(1 - e_i) \). For the other containment, if \( r \in \bigcap R(1 - e_i) \) then \( rS_i = 0 \) for all \( i \) so \( r(Soc R) = 0 \). Then since \( Soc R \) is essential in \( R \) it follows that \( r \in J \) by [1, Theorem 3.1]. Thus (1) is proved.

(2) The mapping \( \alpha : R_R \to \Pi e_iR \) defined by \( \alpha(r) = (e_ir)_{i \in I} \) is an \( R \)-monomorphism.

**Proof of (2).** If \( r \in R \) and \( r \neq 0 \) then since \( Soc R \) is essential in \( R \) there is \( r_i \in R \) so that \( 0 \neq rr_i \in Soc R \). For some \( j \in I \) one has \( e_jrr_i \neq 0 \) so \( e_jr \neq 0 \) and \( \alpha(r) \neq 0 \). Thus (2) is proved.

We will identify the module \( \Pi S_i \) with its image in \( \Pi e_iR \) under the mapping induced by the inclusions \( S_i \hookrightarrow e_iR \). Since \( \alpha(R) \) is injective then \( \Pi e_iR = \alpha(R) \oplus L \) for some submodule \( L \).

(3) \( L \subseteq \Pi S_i \).

**Proof of (3).** If \( L \not\subseteq \Pi S_i \) then there is \( (x_i)_{i=1}^{\infty} \in L \) such that \( x_j \notin S_j \) for some \( j \). Since \( S_j = e_jR(1 - e_j) \) and \( x_je_j = e_jx_je_j \) it follows that \( x_je_j \neq 0 \). Then since \( x_je_j = (x_i)_{i=1}^{\infty} e_j = \alpha(x_je_j) \in L \cap \alpha(R) \) which is a contradiction. Thus (3) is proved.

From (3) and the modular law one has
\[
\Pi S_i = \Pi S_i \cap (\alpha(R) \oplus L) = (\Pi S_i \cap \alpha(R)) \oplus L.
\]

(4) \( \Pi S_i \cap \alpha(R) = \alpha(J) \).

**Proof of (4).** Since \( J = \bigcap R(1 - e_i) \) from (1), it is clear that \( \alpha(J) \subseteq \Pi S_i \cap \alpha(R) \). For the other containment suppose that \( r \in R \) and \( \alpha(r) \in \Pi S_i \). Then for each \( i \) one has \( e_ir = e_i(1 - e_i) \) so that \( re_i = e_i re_i = 0 \). It follows from (1) that \( r \in J \). Thus (4) is proved.

We have \( \Pi S_i = \alpha(J) \oplus L \) and since \( (\Pi S_i)J = 0 \), it follows from
(2) that \( J^2 = 0 \).

(5) The rings \( R/J \) and \( \Pi e_i R_e_i \) are isomorphic.

Proof of (5). For each \( i, e_i R = S_i \oplus e_i R_e_i \) as abelian groups. The projections onto the second summands induce an abelian group epimorphism \( \pi: \Pi e_i R \to \Pi e_i R_e_i \). Let \( \beta = \pi \alpha \) map \( R \) into \( \Pi e_i R_e_i \). Using \( Re_i = e_i R_e_i \) one can see that \( \beta \) is a ring homomorphism. Since by (3) \( L \subseteq \Pi S_i \) then \( \pi(L) = 0 \) and since \( \Pi e_i R = \alpha(R) \oplus L \) it follows that \( \text{Im } \pi = \pi(\alpha(R)) = \text{Im } \beta \) so \( \beta \) is an epimorphism. We note that \( \text{Ker } \beta = \alpha^{-1}(\text{Ker } \pi) = \alpha^{-1}(\Pi S_i) = J \) by (4). Thus (5) is proved.

From Lemma 2, each \( S_i \) is a 1-dimensional left vector space over \( e_i R_e_i \). The componentwise multiplication \( (e_i r_i e_i)_{i=1}^\infty = (e_i r_i e_i s_i)_{i=1}^\infty \) makes \( \Pi S_i \) a left \( \Pi e_i R_e_i \)-module. Since each \( S_i \) is a left ideal of \( R \) and \( J S_i = 0 \) then \( \Pi S_i \) is naturally a left \( R/J \)-module where the multiplication is given by \( (r + J)(s_i)_{i=1}^\infty = (rs_i)_{i=1}^\infty \). We denote by \( D_i \) the ring \( e_i R_e_i \).

(6) As left \( R/J \)-modules, \( \Pi S_i \) is isomorphic to \( R/J \).

Proof. In each \( S_i \) select a nonzero element \( x_i \). This produces a map \( \delta: \Pi S_i \to \Pi D_i \) where \( \delta(s_i)_{i=1}^\infty = (d_i)_{i=1}^\infty \) when \( s_i = d_i x_i \). The mapping \( \delta \) is clearly a \( \Pi D_i \)-isomorphism. The mapping \( \beta \) of (5) induces a ring isomorphism \( \beta: R/J \to \Pi D_i \). One checks that if \( \bar{r} = r + J \) for \( r \in R \) and \( s \in \Pi S_i \) then \( \delta(\bar{r}s) = \beta(\bar{r}) \delta(s) \) so that if we identify \( R/J \) and \( \Pi D_i \) via \( \beta \) then \( \delta \) yields the desired isomorphism. Thus (6) is proved.

Since \( J^2 = 0 \) then \( J \) is a left \( R/J \)-module.

(7) The restriction of \( \alpha \) to \( J \) is an \( R/J \)-monomorphism from \( R/J \) into \( \Pi S_i \).

Proof of (7). If \( s \in \Pi S_i \) and \( \bar{r} = r + J \) for \( r \in R \) then one always has \( \bar{r}s = \beta(\bar{r}) \cdot s \) where \( \cdot \) denotes componentwise multiplication, since \( \beta(\bar{r}) \cdot s = \beta(r) \cdot s = (e_i r_i e_i) \cdot (s_i) = (e_i r_i s_i) = (re_i s_i) = (rs_i) = r(s_i) = \bar{r}s \). Let \( j \) belong to \( J \). Then \( \alpha(\bar{r}j) = \alpha(rj) = (e_i r_i j)_{i=1}^\infty = (e_i r_i j + e_i r(1-e_i)j)_{i=1}^\infty = (e_i r_i j)_{i=1}^\infty = (e_i r_i j)_{i=1}^\infty = (e_i r_i j)_{i=1}^\infty = (e_i r_i j)_{i=1}^\infty = \beta(\bar{r}) \cdot \alpha(j) = \bar{r} \alpha(j) \). Thus (7) is proved.

(8) The mapping \( \beta^{-1} \delta \alpha \) is an essential embedding of \( J \) into \( R/J \) as left \( R/J \)-modules.

Proof of (8). From (6) and (7), \( \beta^{-1} \delta \alpha \) is an \( R/J \)-embedding of \( J \) into \( R/J \). To show that \( \beta^{-1} \delta \alpha(J) \) is essential in \( R/J \) is equivalent to showing that \( \delta \alpha(J) \) is an essential left ideal of \( \Pi D_i \). It suffices to show that for each \( j \) the idempotent \( E_j \) where \( E_j = (\delta_i j)_{i=1}^\infty \) belongs to \( \delta \alpha(J) \). If \( x_i \in S_i \) then for each \( k \) one has \( x_i e_k = 0 \) so that \( \Sigma S_i \subseteq J \). Then with the elements \( x_i \in S_i \) chosen in (6) one has \( \delta \alpha(x_i) \in \delta \alpha(J) \) and \( \delta \alpha(x_i) = (\delta(x_i x_j))_{i=1}^\infty = \delta(\delta_i x_j)_{i=1}^\infty = E_j \). Thus (8) is proved.
Consider the bimodule \( R/J \). The right hand action of elements of \( R/J \) on \( J \) produces a ring homomorphism \( \gamma : R/J \to \text{End}(R/J) \) whose kernel is \( \{ r + J | J r = 0 \} \).

(9) If \( r \) belongs to \( R \) then \( J r = 0 \) if and only if \( \text{Supp} \ r = \{ i \in I | re_i \neq 0 \} \) is finite.

**Proof of (9).** Suppose that \( \text{Supp} \ r \) is finite. Since \( r_1 = r - \sum_{i \in \text{Supp} \ r} re_i \) left annihilates all the \( e_i \) then by (1) \( r_1 \) belongs to \( J \) and hence \( J r_1 = 0 \) since \( J^2 = 0 \). But clearly \( J \sum_{i \in \text{Supp} \ r} re_i = J \sum_{i \in \text{Supp} \ e_i} e_i re_i = 0 \). Thus \( J r = 0 \).

Suppose that \( J r = 0 \) and that \( \text{Supp} \ r \) is infinite. From Lemma 3 there is a central idempotent \( f \) of \( R \) so that if \( I_1 = \{ i \in I | e_i \in f R \} \) then \( \text{Supp} \ r \cap I_1 \) is finite. Replacing \( R \) by \( f R \) we can assume that \( \text{Supp} \ r \) is cofinite in \( I \). Let \( r_2 = r + \sum_{i \in \text{Supp} \ r} e_i \). Since \( J(\Sigma e_i) = 0 \) then \( J r_2 = 0 \). But for all \( i \) in \( I \), \( r_2 e_i \neq 0 \) so \( \beta(r_2) \) is a unit of \( \Pi D_i \).

Then there is an element \( t \) of \( R \) so that \( 1 - r_2 t \in J \) and hence \( J = J(1 - r_2 t) \subseteq J^2 = 0 \), a contradiction. Thus (9) is proved.

It follows from (9) that \( \text{Ker} \ \gamma = \{ \bar{r} | J r = 0 \} = \{ \bar{r} | re_i = e_i re_i \text{ a.e.} \} = \{ \bar{r} | \beta(r) \in \text{Soc} \Pi D \} = \text{Soc} (R/J) \). Let \( D = \Pi D_i = \Pi e_i R e_i \). Then \( \gamma \) induces a ring monomorphism from \( D/\text{Soc} \ D \) into \( \text{End}(R/J) \). Since as a left \( R/J \)-module \( J \) is isomorphic to an essential left ideal of \( R/J \) by (8), then \( \text{End}(R/J) \simeq R/J \) because \( R/J \) is a left self-injective regular ring (in fact, a product of division rings). It follows that \( \gamma \) induces a ring monomorphism from \( D/\text{Soc} \ D \) into \( D \). We then arrive at a contradiction from the following two facts.

(10) If \( G \) is a set of nonzero orthogonal idempotents of \( D = \Pi D_i \) then \( |G| \leq \aleph_0 \).

(11) The ring \( D/\text{Soc} \ D \) has a set of orthogonal idempotents of cardinality \( c \).

**Proof of (10).** For each \( i = 1, 2, \cdots \) let \( e_i \) be the sequence of \( D \) with \( i \)th slot \( e_i \) and zero elsewhere. If \( g_1 \) and \( g_2 \) belong to \( G \) and \( e_i g_1 = e_i \) and \( e_i g_2 = e_i \) then \( g_1 = g_2 \) since otherwise we have \( e_i^2 = 0 \). It follows that if we let \( E = \{ e_i | e_i g = e_i \text{ for some } g \in G \} \) then the mapping from \( E \) to \( G \) which maps \( e_i \) to \( g \) if \( e_i g = e_i \) is well-defined and it is clearly a surjection. It follows that \( |G| \leq |E| \leq \aleph_0 \).

**Proof of (11).** The set \( N \) of natural numbers has a set \( \mathcal{X} \) of \( c \) subsets of \( N \), each of cardinality \( \aleph_0 \), any two of which have finite intersection. (Match \( N \) with the set of rational numbers and choose for each real number a strictly increasing sequence of rational numbers converging to it.) For each subset \( X \) of \( N \) let \( e_X \) be the idempotent of \( D \) such that \( e_X(i) = e_i \) if \( i \) belongs to \( X \) and
The set $\mathcal{R} = \{e_x + \text{Soc} \, D \mid X \in \mathcal{R}\}$ is a set of pairwise orthogonal idempotents of $D/\text{Soc} \, D$. Since $X \neq Y$ is infinite when $X$ and $Y$ are distinct members of $\mathcal{R}$ then $e_x + \text{Soc} \, D \neq e_y + \text{Soc} \, D$. It follows that $\mathcal{R}$ has cardinality $c$.

Thus (10) and (11) hold and Theorem 5 is proved.

**Corollary.** Let $R$ be a basic right $q$-ring which has no projective minimal right ideals and has no loops. Then $R$ is a finite product of chain $q$-rings whose corners are right $q$-rings with no noncentral idempotents.

**Proof.** It follows from Theorem 5 that $\mathcal{R}(R)$ is a finite set all of whose members are chains. If $\mathcal{C}(R)$ is the union of the sets in $\mathcal{R}(R)$ then $\Sigma \mathcal{C}(R)$ is injective so there is an idempotent $g$ of $R$ such that $R_g = gR + \Sigma \mathcal{C}(R)$. If the chains are denoted $E_i$ or $E_{i_1} \rightarrow E_{i_2} \rightarrow \cdots \rightarrow E_{i_m}$ for $1 \leq i \leq m$ then $gR \rightarrow E_{i_1}$ for each $i$. As in Proposition 6 we can find orthogonal idempotents $g_i$, $1 \leq i \leq m$ so that $g = \sum_{i=1}^m g_i$ and $g_iR \rightarrow E_{i_1}$ if and only if $i = j$. If $g_i$ is an idempotent such that $g_iR = g_iR \oplus \sum_{j \neq i} E_{i,j}$ then $g_i$ is central in $R$. As a ring $g_iR$ is a chain right $q$-ring such that the corner $g_iR = g_iR \oplus E_{i,j}$ is a right $q$-ring with $\mathcal{R}(g_iR) = \emptyset$. It follows from Lemma 1 that each idempotent of $g_iR$ is central.

**Proposition 9.** If $R$ is a right $q$-ring with no projective minimal right ideals all of whose idempotents are central then $R \cong Z \times L$ where $Z$ is a right $q$-ring with no primitive idempotent and $L$ is a product of local right $q$-rings none of which is a division ring.

**Proof.** Let $\{e_i \mid i \in I\}$ be the set of primitive idempotents of $R$. As in the proof of Proposition 7 there is an idempotent $g$ of $R$ so that $gR$ is ring-isomorphic to the product of local rings $L = H e_0 R$, in such a way that $e_i R \subseteq gR$ corresponds to its usual image in $H e_i R$. Clearly, $(1 - g)R$ has no primitive idempotent. We note that local rings in the product $L$ which are division rings would correspond to projective minimal right ideals of $R$.

**Proposition 10.** Let $R$ be a chain right $q$-ring without projective minimal right ideals and with corner $gR$ a ring with all idempotents central. Then $R \cong R_1 \times L$ where $R_1$ is a chain right $q$-ring with corner $Z$ a ring with no primitive idempotents and $L$ is a product of local right $q$-rings none of which is a division ring.

**Proof.** By Proposition 9, $gR = g_1R \oplus g_2 R$ where $Z = g_1R$ has
no primitive idempotent and \( g_2 R \simeq L \) is a product of local right \( q \)-rings none of which is a division ring. The chain of \( R \) is associated with \( Z \) and not with \( L \). This follows from Proposition 5 and the fact that if \( L \) is a product of local rings which are not division rings then there is no simple, injective right \( L \)-module which is not embeddable in \( L \). For suppose that \( L = \Pi L_i \) where each \( L_i \) is a local right \( q \)-ring so that \( J_i \neq 0 \) where \( J_i \) is the Jacobson radical of \( L_i \). Suppose \( L/M \) is simple, injective \( L \)-module and is not embeddable in \( L \). Then the maximal right ideal \( M \) of \( L \) is essential and therefore \( M \) is an ideal of \( L \). Choose \( u \in L \) so that \( u = (u_i) \) where for each \( i, u_i \in J_i \) and \( u_i \neq 0 \). The right annihilator \( u^\ast \) of \( u \) in \( L \) is contained in the radical \( \Pi J_i \) of \( L \) so that in particular \( u^\ast \subseteq M \). Thus the mapping \( ua \mapsto a + M \) from \( uL \) to \( L/M \) is a well-defined epimorphism. Since \( L/M \) is injective there is an element \( x \in L/M \) so that for each \( a \in L \), \( a + M = x(ua) \). But \( u \in \Pi J_i \subseteq M \) so that \( xu = 0 \) and we have a contradiction.

We can summarize all of the structure theorems of the paper in the following way.

**Theorem 6.** A right \( q \)-ring is isomorphic to a finite product of rings of the following kinds:

1. Semisimple artinian ring.
2. Loop \( q \)-ring: \( H(\lambda, D) \).
3. \( \Pi D_i \)-chain \( q \)-ring: \( C(\Pi D_i, M, l) \) where the corner \( \Pi D_i \) is an infinite product of division rings.
4. \( Z \)-chain \( q \)-ring: \( C(Z, M, l) \) where the corner \( Z \) is a right \( q \)-ring with no primitive idempotent.
5. A product of local right \( q \)-rings none of which is a division ring.

**Final remarks.** The further study of \( q \)-rings would examine the structure of the local ones and the ones which have no primitive idempotent. The latter clearly have zero right socle and for both kinds, all idempotents are central so that one would expect the investigation of them to require methods very different from those of the present paper.

With regard to the symmetry question for the \( q \)-ring condition, it is easy to see that a chain right \( q \)-ring (of length \( \geq 1 \)) is not left self-injective so that a right \( q \)-ring need not be also a left \( q \)-ring. For consider \( R = C(A, M, 1) \) and let \( E_1 \) and \( E_2 \) be the idempotent matrices with zero entries except for entries of 1 in the \((1, 1)\) and \((2, 2)\) positions respectively. It is easy to see that the obvious correspondence between \( S = E_2 RE_1 \) and \( RE_2 \) where \( \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \)
corresponds to \( \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \) is an isomorphism of left \( R \)-modules. If \( R \) were left self-injective then by Baer's criterion the isomorphism from \( S \) to \( RE_2 \) could be realized as a right multiplication by some element of \( RE_2 \), but \( SRE_2 = 0 \). One might rephrase the question thus: Is every right \( q \)-ring with no chain of length \( \geq 0 \) also a left \( q \)-ring? [Cf. 2, Remark 2.14.] With regard to this symmetry question one would like to know whether there is a local, right self-injective duo ring which is not left self-injective.

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THE UNIVERSITY OF NORTH CAROLINA
GREENSBORO, NC 27412
Werner Bäni, Subspaces of positive definite inner product spaces of countable dimension .................................................. 1
Marilyn Breen, The dimension of the kernel of a planar set ............... 15
Kenneth Alfred Byrd, Right self-injective rings whose essential right ideals are two-sided ................................................. 23
Patrick Cousot and Radhia Cousot, Constructive versions of Tarski’s fixed point theorems .......................................................... 43
Ralph S. Freese, William A. Lampe and Walter Fuller Taylor, Congruence lattices of algebras of fixed similarity type. I .......................... 59
Cameron Gordon and Richard A. Litherland, On a theorem of Murasugi ...... 69
Mauricio A. Gutiérrez, Concordance and homotopy. I. Fundamental group .................................................................................. 75
Richard I. Hartley, Metabelian representations of knot groups .......... 93
Ted Hurley, Intersections of terms of polycentral series of free groups and free Lie algebras ................................................................. 105
Roy Andrew Johnson, Some relationships between measures ............ 117
Oldřich Kowalski, On unitary automorphisms of solvable Lie algebras .... 133
Kee Yuen Lam, K O-equivalences and existence of nonsingular bilinear maps ................................................................................. 145
Ernest Paul Lane, PM-normality and the insertion of a continuous function .................................................................................... 155
Robert A. Messer and Alden H. Wright, Embedding open 3-manifolds in compact 3-manifolds ......................................................... 163
Gerald Ira Myerson, A combinatorial problem in finite fields. I ............ 179
James Nelson, Jr. and Mohan S. Putcha, Word equations in a band of paths ....................................................................................... 189
Baburao Govindrao Pachpatte and S. M. Singare, Discrete generalized Gronwall inequalities in three independent variables .................. 197
William Lindall Paschke and Norberto Salinas, C*-algebras associated with free products of groups ...................................................... 211
Bruce Reznick, Banach spaces with polynomial norms ......................... 223
David Rusin, What is the probability that two elements of a finite group commute? .............................................................................. 237
M. Shafii-Mousavi and Zbigniew Zielezny, On hypoelliptic differential operators of constant strength .................................................. 249
Joseph Gail Stampfli, On selfadjoint derivation ranges ......................... 257
Robert Charles Thompson, The case of equality in the matrix-valued triangle inequality ................................................................. 279
Marie Angela Vitulli, The obstruction of the formal moduli space in the negatively graded case .......................................................... 281