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Mauricio A. Gutiérrez

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CONCORDANCE AND HOMOTOPY, I: FUNDAMENTAL GROUP

M. A. GUTIERREZ

We study the effect of a concordance on the fundamental group of the manifolds involved.

DEFINITION (A). Two submanifolds X, Y of M^n are said to be concordant if there is an embedding $c: X\times I \to M\times I (I = [0, 1])$ which is transversal on $M \times \partial I$ and $c^{-1}(M^n \times \partial I) = X\times\partial I$, $c(X\times 0) = X = 0$, $c(X\times 1) \approx Y\times 1$.

In [11], a similar definition—that of I-equivalence—is given for subcomplexes X, Y of a complex M by simply dropping all smoothness hypotheses from definition (A) and replacing them with cellularity hypotheses.

Let now G_1 be a group and G_i its lower central series (cf. § 1). Define $G_\infty = \cap G_i$ and $G = G_1/G_\infty$ ("group G_1 made residually nilpotent"). Observe $\{G_1/G_i, p_i\}$ is an inverse system where $p_i: G_1/G_{i+1} \to G_1/G_i$ is the obvious map. Let $\tilde{G}$ be its limit (nilpotent completion) which is, in general, uncountable. There is a natural inclusion $G \to \tilde{G}$. In particular, if S is a space, define $\pi(S) = \pi_1(S)/[\pi_1(S)]_\infty$ and $\tilde{\pi}(S) = [\pi_1(S)]^\sim$.

DEFINITION (B). Two (finitely generated) groups are *I-equivalent* if their nilpotent completions are isomorphic.

Let now X, Y be subcomplexes of M. If we have some sort of Alexander duality (v. gr. M a manifold), so that we can prove $H_q(M - X) \approx H_q(M - Y)$, then [11], *If X and Y are I-equivalent so are $\pi(M - X)$ and $\pi(M - Y)$.* The moral here is that we might as well work with residually nilpotent groups. *This we shall assume hereafter* so that we have no need of writing "G_1" for a group G. We have in mind extending the above results to concordances: let be the free group in letters $x_1, \cdots, x_r$. Define $G(x_1, \cdots, x_r)$ (or $G(x)$) as the free product $G*\Phi$. Let $\partial_i: G(x) \to Z$ be the map defined by $\partial_i | G = 0$, $\partial_i(x_j) = \delta_{ij}$. Let now $W = \{w_1, \cdots, w_r\}$ be an r-element subset of $G(x)$, and let NW be the smallest normal subgroup of $G(x)$ containing W. Assume the integral matrix $||\partial_i w_j||$ satisfies

$$(1) \qquad \det ||\partial_i w_j|| = \pm 1 \, .$$

Define $G(\xi_1, \cdots, \xi_r)_1$ (or $G(\xi)_1$) as the quotient $G(x)/NW$. Let $G(\xi) = G(\xi)_1/G(\xi)_\infty$, a residually nilpotent group. If $i: G \to G(\xi)$ is the map $G \to G(x) \to G(x)/NW \to G(\xi)_1/G(\xi)_\infty$, we prove i is monic and

$G(\xi) \subseteqq \tilde{G}$. If we identify G with $i(G)$, $G(\xi)$ is generated in $\tilde{G}$ by G and the $\xi_j = x_j \cdot NW$. By analogy we say $G(\xi)$ is an algebraic extension of G. For residually nilpotent groups, the Artin approach, as outlined in [6; VI, §§ 1, 2] shows the existence of an algebraic closure $\bar{G}$ which, as in field theory, is countable if G is. It is easy to see that since $G(\xi) \subseteqq \tilde{G}$, for all systems $\{w_i\}$ satisfying (1), then $\bar{G} \subseteqq \tilde{G}$. This process resembles the algebraic and analytic closures of the field Q if we write $\bar{Q}$ = algebraic closure of Q (countable) and $\tilde{Q} = C$, the complex numbers (uncountable).

It turns out that if we use the following,

Definition (C). Two (finitely presented) groups are *concordant* if their algebraic closures are isomorphic, then we can prove

If X and Y are concordant submanifolds of M, then $\pi(M - X)$ and $\pi(M - Y)$ are also concordant.

Actually something stronger holds: let $\bar{\mu}_1, \cdots, \bar{\mu}_n$ be generators of $H_1(M - X)$ and let $\mu_i \in \pi(M - X)$ project on $\bar{\mu}_i (1 \leqq i \leqq n)$. Then $G = \pi(M - X)$ and $H = \pi(M - Y)$ have a common finite algebraic extension K, obtained from G (or H) by adding roots to equations of the form $v^{-1}\mu_{i_j}v = x_j$ ($v \in G(x_1, \cdots, x_r)$, $j = 1, \cdots, r$). We do not know if all algebraic extensions are of this form, so we make

Definition (D). Two groups G and H are *simply concordant* if we can choose $\mu_1, \cdots, \mu_r \in G$, $\nu_1, \cdots, \nu_r \in H$ so that $G/G_2 \approx H/H_2$ is generated by the μ_j (or the ν_j) and if G and H have a common algebraic extension $G(\xi_1, \cdots, \xi_s) \approx H(\eta_1, \cdots, \eta_t)$ where the ξ and η are roots to equations

$$x_j = v_j \mu_{i_j} v_j^{-1} \text{ and } y_k = w_k v_{i_k} k_h^{-1} ,$$

$1 \leqq j \leqq s$, $1 \leqq k \leqq t$, $1 \leqq i_j, i_k \leqq r$, $v_j \in G(x)$, $w_k \in H(y)$.

For PL concordances of submanifolds X, Y the groups $\pi(M - X)$ and $\pi(M - Y)$ are simply concordant.

We describe this in § 1 and § 2; the generators and relations added to G correspond to the minima and saddle points of the concordance. Those generators and relations added to H (or removed from $G(\xi)$) correspond to singularities of index n and $n - 1$ (maxima and saddle points). In § 3 we study the case G = free and we give an application to links in § 4. In [4] we study a generalization of the algebraic problem.

We use the following conventions: $\hat{x}$ means x is to be deleted, $\sharp$ is the connected sum of manifolds, $+$ their disjoint union. All homology is integral.

This is the first of two articles; in part II we hope to study the homotopy system of $M - X$ in the sense of [13].

1. Algebraic extensions. Let G_1 be a group and let A and B be subsets of it. Define NA (or $N_G A$ if the context is not clear) to be the intersection of all normal subgroups of G_1 containing A. The group NA is called the *normal closure* of A. If C is the subset $\{[a, b]=aba^{-1}b^{-1}: a \in A, b \in B\}$, we write $[A, B]$ for NC. Notice $[A, B]$ is a normal subgroup even if A or B are not groups.

Inductively the lower central series of G_1 is defined by $G_i = [G_1, G_{i-1}]$ and $G_\infty = \cap G_i$. We say G_1 is *residually nilpotent* if $G_\infty = 1$. We work with residually nilpotent groups. For any group we write $G = G_1/G_\infty$ and G is *always* residually nilpotent. If G_1 has a presentation $\langle x_i: r_j\rangle$, we write $\langle x_i: r_j\rangle_\rho$ for a presentation of G of the form $\langle x_i: r_j, s_k\rangle$, where $G_\infty = N_{G_1}\{s_k\}$. If S is an arc connected CW-complex, $\pi(S)$ is $\pi_1(S)/[\pi_1(S)]_\infty$.

The inclusions $G_{i+1} \subseteqq G_i$ induce maps $p_i: G_1/G_{i+1} \to G_1/G_i$ and $\{G_1/G_i; p_i\}$ is an inverse system. Let $\tilde{G}$ be its limit. We justify this notation: a typical element of $\tilde{G}$ is a sequence $(g_i G_i)_{i \geqq 2}$ of cosets $\bar{g}_i = g_i G_i \in G_1/G_i$ subject to the condition that $p_i(\bar{g}_{i+1}) = \bar{g}_i$. Let $J_n = \{(\bar{g}_i) \in \tilde{G}: g_i \in G_{n-1}$ for $i \geqq 2\}$. Then J_n is a central series (i.e., $\tilde{G} = J_1$, $[J_1, J_{n-1}] \subseteqq J_n$) and so $\tilde{G}_n \subseteqq J_n$ because $\{\tilde{G}_n\}$ is the *lower* central series [7; Ch. 5]. Since $\cap J_n = 1$. $\tilde{G}$ is residually nilpotent. Notice $g \to (\bar{g}, \bar{g}, \cdots)$ defines a homomorphism $G_1 \to \tilde{G}$ with kernel G_∞ so $G \subseteqq \tilde{G}$. Further, if G is finitely generated then $(\tilde{G})_n = (G_n)^\sim = J_n$ and $G_1/G_n \approx \tilde{G}/\tilde{G}_n$. A proof can be found in [O].

In our applications we deal with fundamental groups of compact complexes and so, unless otherwise specified, all our groups are finitely generated. In particular $\tilde{G}$ need not be finitely generated if G is, in fact $\tilde{G}$ tends to be of a cardinality bigger than that of G.

If A_1 is any group and if we have a family $\alpha_i: G_1/G_n \to A_1/A_n$ of isomorphisms which commute with the p_i then the α_i define an embedding $\alpha: A \to \tilde{G}$ with $G \subseteqq \alpha(A)$. The converse however, is not true.

If $A \subseteqq \tilde{G}$ is a subgroup and $\Sigma \subseteqq \tilde{G}$ is a subset, let $A\{\Sigma\}$ be the subgroup generated by A and Σ.

LEMMA 1. *Let G be a residually nilpotent group and let $W = \{w_1, \cdots, w_r\} \subseteqq G(x_1, \cdots, x_r)$ satisfy* (1). *Then the map $G \to G(\xi_1, \cdots, \xi_r)$ is monic.*

Proof. Sequence $1 \to NW \to G(x) \xrightarrow{j} G(\xi)_1 \to 1$ gives rise to a homology sequence

$$(2) \qquad H_2G(x) \xrightarrow{j''} H_2G(\xi)_1 \longrightarrow NW/[W, G(x)] \xrightarrow{\varepsilon} H_1G(x) \xrightarrow{j'} H_1G(\xi)_1 \longrightarrow 0$$

(j', j'' are induced maps) (cf. [11; (2.1)]). Let $i: G \to G(\xi)_1$ be the map $j|G$. Since $H_2G(x) = H_2G$ and $H_1G(x) = H_1G \oplus Z^r$, and $j'' = i''$, $j' = i' + 0(0: Z^r \to 0)$, sequence (2) becomes

$$H_2G \xrightarrow{i''} H_2G(\xi)_1 \longrightarrow NW/[W, G(x)] \xrightarrow{\varepsilon} Z^r \longrightarrow 0 .$$

If $v \in NW$, by definition we may write $v = \prod_k v_k w_{l_k}^{\eta_k} v_k^{-1}$. Then, if $\varepsilon_l = \sum_{l_k=l} \eta_k$, $v \equiv w_1^{\varepsilon_1} \cdots w_r^{\varepsilon_r} \bmod [W, G(x)]$ and then $\varepsilon(v) = (\varepsilon_1, \cdots, \varepsilon_r)$ is an isomorphism. Thus, by (2), $i'': H_2G \to H_2G(\xi)_1$ is onto. On the other hand, by (1), $i': H_1G \to H_1G(\xi)_1$ is an isomorphism. In that fashion $l: G \to G(\xi)$ satisfies the hypothesis of [11; Thm. (3.1)] and so $G \subsetneqq G(\xi)_1/[G(\xi)]_\infty$ and so $G \subsetneqq G(\xi)$.

Hereafter we deal with $G(\xi)$ only.

We say the $\xi_1, \cdots, \xi_1$ are solutions of the system of equations $w_1 = 1, \cdots, w_r = 1$. If $\xi \in \tilde{G}$ we use isomorphism $\tilde{G}/\tilde{G}_k \approx G/G_k$ to define $\xi^{(k)}$ as $\xi\tilde{G}_k \in G/G_k$. If $\xi, \eta \in \tilde{G}$, $\xi = \eta$ iff $\xi^{(k)} = \eta^{(k)}$ for all k.

Consider $W = \{w_i\}$, $W' = \{w_i'\}$ $(i = 1, \cdots, r)$ subsets of $G(x_1, \cdots, x_r)$ satisfying (1). We say that W is equivalent to W' if we can get from one to the other by finitely many steps involving

(i) A permutation of the x_i,

(ii) A permutation of the w_j,

(iii) Replace x_i by $x_i x_j^\varepsilon$ $(i \neq j,\ \varepsilon = \pm 1)$,

(iv) Replace w_i by $w_i w_j^\varepsilon$ $(i \neq j,\ \varepsilon = \pm 1)$,

or their inverses. (cf. [7; § 3.3].)

Clearly the operations described above establish a Nielsen transformation (loc. cit.) of $G(\xi)$ into $G(\xi')$ which is, in particular, an isomorphism. Matrix $||\partial_i w_j||$ is a permutation matrix times a product of elementary matrices. By an operation of type (ii) we change $||\partial_i w_j||$ to a product of elementary matrices and operations of type (iv) finally reduce it to the identity, that is we may assume our system is equivalent to one of the form $\{x_i v_i^{-1}\}$, where $\partial_k v_i = 0$ for all i, k. Generally we write this as

$$(3) \qquad w_i: x_i = v_i(x_\alpha) \qquad (i, \alpha = 1, \cdots, r) .$$

If $\partial_k v_i = 0$, we may write v_i as $\prod_j M_{ji}(g_\nu, x_\mu) \cdot g_i$, where the M_{ji} are commutators involving one of the x_μ and $g_i \in G$ (see [7; p. 352 eq. (7) Thm. 5.14]).

LEMMA 3. *Let* $\{\xi_1, \cdots, \xi_r\}$ *and* $\{\eta_1, \cdots, \eta_r\}$ *be elements of* $\tilde{G}$ *which are solutions to* (3). *Then* $\xi_i = \eta_i (i = 1, \cdots, r)$.

Proof. For any $k \geqq 1$, consider $\xi_i^{(k)}$ and $\eta_i^{(k)}$ in $\tilde{G}/\tilde{G}_k$. Elements $\xi_i^{(k)}$ and $\eta_i^{(k)}$ are solutions to (3) in G/G_k too; as a result

$$\xi_i^{(k)} = \prod_j M_{ji}(g_\nu, v_\mu(\xi_\alpha^{(k)}))g_i \bmod G_k .$$

Now, each $M_{ji}(g_\nu, v_\mu)$ is a product $N_{ji}^{(1)}(g_\sigma)\prod_\beta M'_{ji\beta}(g_\nu, x_\mu)$, where $N_{ji}^1 \in G_2$ and $M'_{ji\beta} \in G(x)_3$.

By repeating this procedure k times, we obtain

$$\xi_i^{(k)} = \prod_j N_{ji}^{(1)} \cdots \prod_l N_{li}^{(k-1)} \bmod G_k , \tag{4}$$

where $N_{ij}^{(\alpha)} \in G_{\alpha+1}$ and so $\xi_i^{(k)}$ equals (4) in G/G_k. A similar reasoning applied to η_i shows $\eta_i^{(k)}$ is also given by expression (4) and so $\xi_i^{(k)} = \eta_i^{(k)}$ for all k.

We now repeat some of the results of [6; VII § 1] for algebraic extensions adapted to the present context. We say that $\xi \in \tilde{G}$ is an algebraic element if we can find a system (3) with solutions $\xi_1, \cdots, \xi_r$ and $\xi = \xi_1$. Let $G \subseteqq A \subseteqq \breve{G}$; we say that A is an algebraic extension of G if there exists a set Σ of algebraic elements such that $A = G\{\Sigma\}$. If Σ is finite we say A is a finite extension. If A and B are algebraic extensions of G, any homomorphism $A \to B$ leaving G fixed is said to be *over* G.

REMARKS (1). Let G be a finitely generated residually nilpotent group and let $A \subseteqq \breve{G}$ contain G. Then the following are equivalent

(ALG 1) A is an algebraic extension of G ,

(ALG 2) $N_A G = A$, and

(ALG 3) $\text{incl}_*\colon H_1(G; Z) \to H_1(A; Z)$ is an isomorphism.

Clearly (1) implies (2). Assume (2); let $a_i \in A$ be generators for A. Then there exist $g_{ij} \in G$, $\alpha_{ij} \in A$ (finite j for a fixed i) such that $a_i = \Pi_j \alpha_{ij} g_{ij} \alpha_{ij}^{-1}$. Let $\bar{a}_i = \Pi_j g_{ji} \in G$. Since $a_i = (a_i \bar{a}_i^{-1}) \cdot \bar{a}_i$, $a_i \equiv \bar{a}_i$ mod A_2 and $G/G_2 \to A/A_2$ is an isomorphism. Finally if (3) holds, let $a \in A$. Then $H_1(G) \to H_1(G\{a\})$ is an isomorphism. Let x be a letter and consider the epimorphism over G, $G(x) \to G\{a\}$ which sends x to a. Let K be its kernel. By hypothesis there exists $r \in K$ with $\partial_a r = 1$. Then $K = N_{G(x)}\{r, s_k\}$ for some $s_k \in G(x)$ with $\partial_a s_k = 0$. Consider $G(\xi) = [G(x)/N_{G(x)} r]_\rho \subseteqq \tilde{G}$. By Lemma (3) $G(\xi)$ is isomorphic to $G\{a\}$ ([11; (3.4)]). Thus a is algebraic.

(2) If we work with finitely generated A, conditions (ALG 2) and (ALG 3) are equivalent to "A is a finite extension of G".

(3) The above remarks show that a finite extension can be obtained from a finite sequence of simple extensions. We can also define algebraic elements intrinsically.

DEFINITION. We say $\xi \in \tilde{G}$ is *algebraic* if $\xi \in N_{G\{\xi\}}G$. If not we say ξ is transcendental.

(4) A residually nilpotent and finitely generated group A is a finite extension of G if and only if there exist isomorphisms $\theta_i: G/G_i \to A/A_i$ for all i which commute with the $p_i: X/X_{i+1} \to X/X_i$. In fact, if the θ_i exist they define and embedding $A \to \tilde{G}$ containing G and A satisfies (ALG 3). Conversely, if $G \subseteqq A \subseteqq \tilde{G}$ satisfies (ALG 3) [4; Lemma 7] implies that $H_2G \to H_2A$ is onto and [11, (3.4)] guarantees that inclusion induces isomorphisms $G/G_i \to A/A_i$ which, being canonical, commute with the p_i.

DEFINITION. A class $\mathscr{C}$ of extensions $A \subseteqq B (G \subseteqq A, B \subseteqq \tilde{G})$ is said to be *distinguished* if it satisfies the following conditions:

(i) Let $G \subseteqq A \subseteqq B$ be a tower of subgroups of $\tilde{G}$; extension $G \subseteqq B$ is in $\mathscr{C}$ if and only if $G \subseteqq A$ and $A \subseteqq B$ are in $\mathscr{C}$.

(ii) If $G \subseteqq A$ is in $\mathscr{C}$ and if B is any subgroup of $\tilde{G}$ containing G, then $B \subseteqq B\{A\}$ is in $\mathscr{C}$.

LEMMA 4. *The class $\mathscr{C}$ of algebraic extensions is distinguished.*

Proof. (i) If B is algebraic and $A \subseteqq B$, it follows from the above remarks that all the elements of A are algebraic. Conversely, if A is algebraic and B is algebraic over A, then $G/G_2 \to A/A_2 \to B/B_2$ is an isomorphism and B is algebraic over G.

(ii) If $a \in A$, then $a \in N_{G\{a\}}G \subseteqq N_{B\{a\}}B$ and a is algebraic over B.

Let $\bar{G}$ be the set of all algebraic elements of $\tilde{G}$.

PROPOSITION (5). *$\bar{G}$ is a subgroup of $\tilde{G}$ and both $\bar{G}$ and $\tilde{G}$ are algebraically closed.*

Proof. $\tilde{G}$ is closed by Lemma 1. This facilitates the proof of the closure of G (which is obviously a group): $\xi, \eta \in \bar{G}$ implies they are elements of a finite algebraic extension A of G. Thus $\xi\eta^{-1} \in A$. Any finite system $W \subseteqq \bar{G}(x)$ lies in some $G(\eta_1, \cdots, \eta_s)(x)$. By Lemma 4 the solutions ξ_i for it are algebraic over G.

Finally a curious note: Lemma 3 implies algebraic extensions are "purely inseparable" [6]. Thus it is not surprising that "primitive elements" [6, VII. 6] do not exist; that is, given $A = G(\xi, \eta)$ there does not necessarily exist a ζ such that $A = G(\zeta)$. The reason for this is topological in nature as can be seen in the proof of Theorem 6 below. On the other hand the fact that A is the top of a tower of simple extensions $G \subseteqq G(\xi) \subseteqq G(\xi, \eta)$ is explained topologically by the "handle exchange lemma". This is of course, just an analogy (but an interesting one).

2. Concordances. We now prove our main result:

THEOREM 6. *Let X, Y be two concordant submanifolds of S^n; then $\pi(S^n - X)$ and $\pi(S^n - Y)$ are simply concordant* ($n \geqq 5$).

Proof. If the codimension is not 2, there is nothing to prove. Notice we may prove the same result on any simply connected manifold instead of S^n. For the sake of simplicity we prove 6 only for S^n.

Secondly, as remarked by Giffen (6) holds for PL I-equivalences since any PL concordance fails to be locally flat at finitely many points where cell replacements in the sense of [10] take place. They do not affect $\pi(M - X)$ (ibid.) and so the algebra for the PL case is the same as that of the PL locally falt case where Morse theory can be defined.

Let now $c: X \times I \to S^n \times I$ be a concordance. A point $(x, t) \in S^n$ I is a *regular* point of c if either $(x, t) \notin \operatorname{Im} c$ or there is a neighborhood J of t in I, a manifold X' and a level preserving embedding $e: X' \times J \to S^n \times J$ onto a neighborhood of (x, t) in $\operatorname{Im} c$ [8; § 2]. A value $t \in I$ is *regular* if (x, t) is a regular point for all $x \in S^n$. If t is not regular we say t is a *critical* value.

If there are manifolds $X_0(t)$ and $X_1(t)$; neighborhoods J_0 and J_1 of t in $[0, t]$ and $[t, 1]$ respectively and isotopies $g: X_\varepsilon \times J_\varepsilon \to S^n \times J_\varepsilon$ ($\varepsilon = 0, 1$) such that $\operatorname{Im} c \cap \{S^n \times (J_0 \cup J_1)\} = \operatorname{Im} g_0 \cup \operatorname{Im} g_1 \cup h^p$, where $h^p \subset S^n \times \{t\}$, we say t is a *standard critical value of index* p ($0 \leqq p \leqq n - 1$) if we have a smooth (or PL locally flat) isomorphism

$$(h^p, h^p \cap \operatorname{Im} g_0, X \cap \operatorname{Im} g_1) \longrightarrow (D^p \times D^r, \partial D^p \times D^r, D^p \times \partial D^r)\ ,$$

where D^p is the p-disk, $\partial D^p = S^{p-1}$ and $p + r = n - 1$, and where the intersection $\operatorname{Im} g_0 \cap \operatorname{Im} g_1$ is the closure of $\operatorname{Im} g_\varepsilon - h$ for $\varepsilon = 0,1$. The result is a p handle attached to X_0 in $S^n \times \{t\}$ defining a surgery to X_1. See [8; pp. 433-434] especially Figure 1.

By [8; Lemma 2], we may assume there are finitely many criticaly values (all standard). Reordering, addition and cancellation of embedded handles are possible and the corresponding results are proved in [8; § 2] under the hypothesis $n \geqq 5$. Let $t_1 < \cdots < t_r$ be the critical values of c. Write $t_0 = 0$, $t_{r+1} = 1$ and let $0 < \varepsilon = 1/2 \min |t_i - t_{i-1}|$. Assume Index $t_i \leqq$ Index t_{i+1}. If $p: S^n \times I \to I$ is the natural projection, let $Z_t = S^n - (\operatorname{Im} c \cap p^{-1}(t))$. If $t, t' \in (t_i, t_{i+1})$, then Z_t and $Z_{t'}$ are diffeomorphic. Let $G_{(t)} = \pi(Z_t)$; then

$G_{(t)}$ changes only at the critical values of index 0, 1, $(n-3)$ or $n-2$. By duality (turning c upside down) it suffices to describe the effects of passing through values of index 0 and 1.

The contention is that these changes are algebraic in the sense of § 1. In fact we prove that for some $\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_n$ in $\tilde{G}$, $G_{(0)}\{\alpha_1, \cdots, \alpha_s\} \cong G_{(r+1)}\{\beta_1, \cdots, \beta_n\}$ is an algebraic extension of $G_{(0)}$ and $G_{(r+1)}$. As we shall see the equations for α and β are of the form described in Definition D.

If Index $t_i = 0$, let $H = G_{(t_i-\varepsilon)}$, $H' = G_{(t_i+\varepsilon)}$. At t_i we introduce a handle h^0 of index 0 or a minimum. This means $Z_{t_i+\varepsilon} = Z_{t_i-\varepsilon} - \sigma_i$, where σ_i is an unknotted sphere in codimension two in $Z_{t_i-\varepsilon}$. As a result $H' = H(y_i)$, where y_i is represented by a small loop in $Z_{t_i-\varepsilon}$ with linking number 1 with σ_i.

We assume Index $t_i \leqq$ Index t_{i+1}; let s be the largest subscript with Index $t_s = 0$. For convenience, let $Z = Z_{t_s+\varepsilon}$, $Z' = Z_{t_{s+1}+\varepsilon}$ and let K, K' be the corresponding groups. Then

$$K = G_0(y_1, \cdots, y_s) . \tag{5}$$

Let now v be the largest number such that Index $t_{s+v} = 1$. Clearly $v \geqq s$ since at least s handles of index 0 must be cancelled. Let X_i be the connected components of X, then $Z = S^n - [X_1 + \cdots + X_m + \sigma_1 + \cdots + \sigma_s\}$ (t is disjoint union). Z' then can be of two forms (# is connected sum)

$$Z' = S^n - \{X_1 + \cdots + (X_i \# \sigma_{j_i}) + \cdots + X_m + \sigma_1 + \cdots + \hat{\sigma}_{i_i} + \cdots + \sigma_s\} , \tag{a}$$

or

$$Z' = S^n - \{X_1 + \cdots + X_i' + \cdots + X_m + \sigma_1 + \cdots + \sigma_s\} , \tag{b}$$

where X_i' is obtained from X_i by removing two small disks and identifying the boundaries by a tube in Z. These are the two standard forms of a saddle point since possibilities $X_i \# X_j$ and $\sigma_k \# \sigma_l$ are excluded.

Now $\pi_1(Z') = \pi_1(Z)/R$, where R is the normal closure of an element of the form

$$w a_i w^{-1} y_j^{-1} \text{ if (a) is the case} \tag{6}$$

or

$$w a_i w^{-1} a_i^{-1} \text{ if (b) is the case,} \tag{7}$$

where $w \in \pi_1(Z)$ and a_i is represented by a fiber of the circle bundle of $X_i \subset S^n$ (appropriately based).

It follows from (5) and (6) that in case (a), K' is of the form

$G_{(0)}(y_1, \cdots, \hat{y}_{j_i}, \cdots, y_s)\{\eta_{j_i}\}$, where η_{j_i} is the root of equation (5) which is of form (3). For case (b) by Alexander duality $H_2(Z) = \Sigma H^{n-3}(X_j)$ and $H_2(Z') = \Sigma_{j\neq i} H^{n-3}(X_j) + H^{n-3}(X_i')$.

Recall X_i' is obtained from X_i by attaching a 1-handle, $X_i' = X_i \# S^{n-3} \times S^1$. Thus $H_1(X_i') = H^{n-3}(X_i) + Za$. Let $\bar{a} \in H_2 Z'$ be the element that corresponds to a via duality.

We know that the attached 2-handle must be cancelled by a 3-handle since X is diffeomorphic to Y and that this 3-handle must be attached by a map representing $\bar{a}$ which must, therefore, be a spherical element of $H_2 Z'$. It follows that $\pi_1(Z) \to \pi_1(Z')$ induces an isomorphism of abelianization and an epimorphism of second homologies because under the map $H_2 Z' \to H_2 \pi_i(Z')$ defined by Hopf, $\bar{a}$ goes to zero. By [11; (2.1)], $K = K'$ in case (b).

In conclusion $G_{(s+v+\varepsilon)} = G_{(0)}(\eta_1, \cdots, \eta_s)$ where the η_ι are roots of equations (6). Clearly then $\tilde{G}_0$ is isomorphic to $\tilde{G}_r$.

Corollary 7. *If $X, Y \subset S^n$ are concordant, $G = \pi(S^n - X)$, $G' = \pi(S^n - Y)$, then $G\{\eta_1, \cdots, \eta_s\} = G'\{\xi_1, \cdots, \xi_r\}$ where the η, ξ are roots of equations $y_i = w_i a_i w_i^{-1}$, $x_j = w_j' b_j (w_j')^{-1}$ and the a_i, b_j generate $H_1 G$ and $H_1 G'$ respectively. In other words G and G' are simply concordant.*

3. Algebraic extensions of free groups. Let Φ be the free group $\langle x_1, \cdots, x_m : \rangle$ and $h: \Phi \to G_1$ a homomorphism into a finitely presented group G_1 which induces isomorphisms $H_q(\Phi) \to H_q(G_1)$ for $q = 1, 2$. In particular G_1 has a preabelian presentation [7] of the form

$$(8) \quad \langle x_1', \cdots, x_m', b_1, \cdots, b_p : b_1 = B_1, \cdots, b_p = B_p, C_1 = 1, \cdots, C_q = 1\rangle$$

where the B_j, C_k are in the commutator of $\Psi = \Phi'(b_1, \cdots, b_p)$ ($\Phi' = \langle x_1', \cdots, x_m' : \rangle$) and where $h(x_i) = v_i x_i' v_i^{-1}$ for some $v_i \in \Psi$.

Also (8) defines a group $G_1^\circ = \langle x_i', b_j : b_j = B_j \rangle$ $(1 \leqq i \leqq m, 1 \leqq j \leqq p)$ which also satisfies the hypotheses above [5; p. 106]. Further, the natural epimorphism $e: G_1^\circ \to G_1$ induces [11; § 2] isomorphisms $G^\circ \to G$ and $\tilde{G}^\circ \to \tilde{G}$ and h induces (loc. cit.) an isomorphism $\tilde{\Phi} \to \tilde{G}$. Equations $b_j = B_j$ satisfy (1) and so G is an algebraic extension of Φ.

Let now $\mathscr{R}_i = N_G\{x_1', \cdots, \hat{x}_i', \cdots, x_m'\}$ and $J_i G = \mathscr{J}_i = G/\mathscr{R}_i$.

Lemma (8). *Let G be a group which is simply concordant to Φ and assume* (i) $\mathscr{J}_i = Zx_i'$ *for all i, and* (ii) $h_*: H_1(\Phi_2) \to H_1(G_2)$ *is an isomorphism. Then $x_i \to v_i x_i' v_i^{-1}$ defines an isomorphism $\Phi \to G$ for some $v_i \in G$.*

Proof. By hypothesis there exist letters $y_1, \cdots, y_r, z_1, \cdots, z_s$ and equations $y_k = w_k x'_{i_k} w_k^{-1}$ ($w_k \in G(y)$), $z_l = u_l x_{i_l} u_l^{-1}$ ($u_l \in \Phi(z)$) such that $G(\eta_1, \cdots, \eta_r) \approx \Phi(\zeta_1, \cdots, \zeta_s)$. In fact, G itself is of the form $\Phi(w_1, \cdots, \omega_t)$ for if $\mathscr{J}_i = Zx'_i$ then $B_j \in N_G\{x'_j : j \neq i\}$ for all i. As a result the B_j have the form

$$(9) \qquad B_j = \prod_l w_{l_j}[x_{i_l}, x_{k_l}]^{\varepsilon_l} w_{l_j}^{-1} .$$

In fact recall $\Psi = \Phi'(b_1, \cdots, b_p)$; let $K = \ker(\Psi \to G)$ the kernel of the map described by (8). Then, if R_j is the expression on the right hand side of (9) $B_j \equiv R_j \bmod K$, that is

$$B_j = R_j \cdot D$$

for some $D \in K$.

In D the words $b_l B_l^{-1} (l = 1, \cdots, p)$ must appear with zero exponent sum, that is, $D \in [K, \Psi]$ modulo the C_k, which in turn lie in $[\Psi, K] = K \cap \Psi_2$ since $H_2(G_1) = 0$.

If G'_1 is

$$\langle \Psi : b_l B_l^{-1}, C_1, \cdots, C_q, D \rangle$$

then G'_1 is an epimorphic image of G_1° and since $D \in [K, \Psi]$, $H_2(G_1^\circ) \to H_2(G'_1)$ is an epimorphism (both are zero) and $G^\circ \to G'$ is an isomorphism. Thus we may assume B_j has the desired from R_j since D is a relation in G°. Thus we may present G as $\langle x'_i, c_{l_j} : c_{l_j} = w'_{l_j}[x'_{i_l}, x'_{k_l}]^{\varepsilon_l} w'^{-1}_{l_j} \rangle_\rho$, where the w'_{lj} are obtained from the w_{lj} of (9) by substituting b_j by $\prod c_{lj}$. As in § 1, $\langle \cdots \rangle_\rho$ indicates that the presentation includes the relations $G_\infty = 1$.

Since $w'_{lj}[x'_{i_l}, x'_{k_l}](w'_{lj})^{-1} = [w'_{lj} x'_{i_l}, x'_{k_l}] \cdot [x'_{k_l}, w'_{l_j}]$, we may alter the above presentation to

$$(10) \qquad G = \langle x'_i, d_{l_j}, e_{l_j} : d_{lj} = [w''_{lj} x'_{i_l}, x'_{k_l}], e_{lj} = [x_{k_l}, w''_{lj}] \rangle_\rho$$

where the w'' are obtained from the w' by writing $c_{lj} = (d_{lj} e_{lj})^{\varepsilon_l}$.

Finally, let $d'_{lj} = d_{lj} x'_{k_l}$, $e'_{lj} = x_{k_l}^{-1} e_{lj}$. Then (10) has the form

$$(10a) \qquad \langle x'_i, d'_{lj}, e'_{lj} : d'_{lj} = w''_{lj} x_{i_l} (w'''_{lj})^{-1}, e'_{lj} = (w'''_{lj}) x_{k_l} (w'''_{lj})^{-1} \rangle_\rho$$

where transformation $d \to d' x^{-1}$, $e \to xe'$ throws w'' onto w'''.

Clearly (10a) has the desired form and G has a presentation (after relabeling everything)

$$\langle x'_1, \cdots, x'_m, z_1, \cdots, z_p : z_j = w_j x'_{i_j} w_j^{-1} , \quad 1 \leqq j \leqq p \rangle_\rho$$

or

$$G = \langle x'_i, z_j : z_j = w_j x'_{i_j} w_j^{-1}, r_l \rangle ,$$

where $r_l = r_l(x, z)$ are the remaining relations, $N_{G_1}\{r_l\} = G_\infty$. Clearly, if $\Psi = \Phi'(z)$, $r_l \in \Psi_2$.

Writing $w_j = u_j x_{i_j}^{\alpha}$, relation $z_j = w_j x'_{i_j} w_j^{-1}$ becomes $z_j = u_j x'_{i_j} u_j^{-1}$ that is, we may assume $\alpha = 0$ or $w_j \in \mathscr{R}_{i_j}$.

Transformations of the form $z_j \to x_k^{\alpha} z_j x_k^{-\alpha} (k \neq i_j)$ are Tietze transformations and by use of appropriate values of α we may assume w_j has zero exponent sum on x_k as well, that is $w_j \in \mathscr{R}_k$.

Our hypothesis (i) implies $G_2 = \cap \mathscr{R}_i$. Let $N = N_{G_1}\{\Phi_2\}$ then by (ii) $H_1\Phi_2 \to N/[N, N] \to G_2/[N, N] \to H_1 G_2$ is the identity and in particular $G_2 = N_{G_1}\{\Phi_2\}$.

Write $y_j = z_j(x'_{i_j})^{-1}$. By Tietze transformations, (10a) can be changed to

$$\langle x'_1, \cdots, x'_m, y_1, \cdots, y_p : y_p = [w'_j, x'_{i_j}], r' \rangle \tag{10b}$$

and $w'_j \in \cap \mathscr{R}_i$ so we may write $w_j = \Pi c_{ij} k_{ij} c_{ij}^{-1}$, $(k_{ij} \in \Phi_2)$. We may assume $c_{ij} \in N_{G_1}\{y_1, \cdots, y_p\}$, otherwise $c_{ij} = d \cdot e$, $d \in N\{y_j\}$, $e \in \Phi$. Redefine c_{ij} as d and k_{ij} as $ek_{ij}e^{-1}$. In fact we may assume the c_{ij} have zero exponent sum on the y. If not use Lemma 3 to change the equation $y_j = [w'_j, x'_{i_j}]$ to $y_j = [w''_j, x'_{i_j}]$ where w'' is obtained from w' by changing y_j to $[w'_j, x'_{i_j}]$. Since the solutions of the first set of equations are also solutions of the second set, Lemma 3 implies this change is allowable. Finally write $y'_j = y_j[\Pi k_{i_j}, x'_{i_j}]^{-1}$. Then (10b) becomes

$$\langle x'_1, \cdots, x'_m, y'_1, \cdots, y'_p : y'_j = [\bar{v}_j, x_{i_j}], r'''_l \rangle \tag{10c}$$

and the $\bar{v}_j = \Pi(\bar{c}_{ij} k_{ij} c_{ij}^{-1}) \cdot (\Pi k_{ij})^{-1}$ where the $\bar{c}$ are obtained from the c by the substitution $y_j = y'_j[\Pi k_{ij}, x'_{i_j}]$. Since the c have zero exponent sum on the y_j, the $\bar{c}$ have zero exponent number on the y; in particular $\bar{c}_{ij} \in N_{G_1}\{y_1, \cdots, y_p\}$.

Let now η_j be the image of y_j in G. We wish to prove that $\eta_j \in G_n$. By (10c) this is so for $n = 2$. If $\eta_k \in G_n$ then $\bar{c}_{ij} \in G_n$ and then $\bar{v}_j \in G_{n+1}$ so that $\eta'_j = [\bar{v}_j, x_{i_j}] \in G_{n+2}$. By induction $\eta_j \in G_\infty = 1$ and so h is an isomorphism. This argument is an adaptation of the proof found in [14; p. 152].

4. An example. Let X be the disjoint union ΣX_i of m copies of the n sphere $S^n (n \geqq 3)$. A link is an embedding $\mathscr{L}: X \to S^{n+2}$. The knots $\mathscr{L}_i = \mathscr{L} | X_i$ are called the components of $\mathscr{L}$ and $\mathscr{L}(X) = L$ is sometimes used instead of $\mathscr{L}$. The normal bundle of $L \subseteqq S^{n+2}$ is trivial and so we can extend $\mathscr{L}$ to an embedding $\bar{\mathscr{L}}: X \times D^2 \to S^{n+2}$. Let $U = U_L$ be the closure of $S^{n+2} - \bar{L}$ which is a compact manifold with boundary $X \times S^1$. As a result $\pi_1 = \pi_1(U)$ is finitely presented. Similarly, let U_i be the closure of $S^{n+2} - \bar{L}_i$ (the meaning of $\bar{L}_i$, $\bar{L}$ is, hopefully, clear); $U = \bigcap_{i=1}^m U_i$. We say U is the complement of $\mathscr{L}$.

Inclusion $X \times S^1 \to U$ induces a homomorphism $h: \Phi \to \pi_1$ of fundamental groups; let $p_i \in X_i$. The loops $\mu_i = \{p_i\} \times S^1$ (attached by simple arcs γ_i to a basepoint) are generators of a free group Φ' in π_1, and the image of h is in Φ'. Naturally h depends on the choice of the γ_i. At any rate h satisfies the hypothesis of [11; (3.1)] and by § 3, $\pi = \pi(U)$ is an algebraic extension of Φ; further

PROPOSITION 9. *The group π is simply concordant to Φ.*

Proof. (Kervaire [5]). By Theorem 3 of [5] it is possible to find a link $\mathscr{L}'$, concordant to the trivial link with $\pi = \pi(U_{L'})$. From Corollary 7 it follows that π is simply concordant to $\pi(U_{L_0}) = \Phi$, where $\mathscr{L}_0$ is the trivial link. Homomorphism $h: \Phi \to \pi$ defines a subgroup $h(\Phi)$ of Φ' and $\Phi'/h(\Phi)$ is a simple algebraic extension and so is π/Φ'.

COROLLARY 10. *Let $\mathscr{L}$ be a link, $\pi = \pi(U_L)$. If $\pi_1(U_i) = Z$ for $1 \leqq i \leqq m$, and if $h_*: H_1\Phi_2 \to H_1[\pi_1, \pi_1]$ is an isomorphism, then π is free generated by loops $\mu_i = \{p_i\} \times S^1$ attached to a basepoint by suitable γ_i.*

Proof. Immediate from Lemma 8 and Proposition 9.

LEMMA 11. *Let $\mathscr{L}$ be an arbitrary link; then $\mathscr{L}$ is concordant to a link satisfying the hypothesis of Corollary 10.*

Proof. We can write $\partial U = \Sigma X_i \times S^1$. Let A be the space obtained from ∂U by joining the $X_i \times S^1$ by means of arcs γ_i to a basepoint (cf. [3]). Consider diagram

$$\begin{array}{ccc} U & & \\ {\scriptstyle i}\uparrow & \searrow{\scriptstyle q} & \\ A & \xrightarrow{p} & S^1 \end{array} \tag{11}$$

where $p(X_j \times S^1)$ is the basepoint for $j \neq 1$ and $p | X_i \times S^1$ is the projection on the second coordinate. Triangle (11) can be completed if and only if

$$\begin{array}{ccc} \pi_1(U) & & \\ {\scriptstyle i_*}\uparrow & \searrow{\scriptstyle q_*} & \\ \Phi & \xrightarrow{p_*} & Z \end{array}$$

can be completed. The latter is obvious (by using $q_*: \pi_1(U) \to$

$H_1(U) \approx \sum_{k=1}^{m} Zx_k \to Zx_i)$ and so we can find $q\colon U \to S^1$ extending p. Let z_0 be a regular value of q; then $V_i = q^{-1}(z_0)$ is a compact framed $(n+1)$-submanifold of U with boundary $p^{-1}(Z_0) = X_i \times z_0 \subset \partial U$. Clearly V_i is not unique and $V_i \cap V_j \neq \phi$ (unless $\pi(U) = \Phi$ by Lemma 2 of [3]). We say the V_i are Seifert manifolds for $\mathscr{L}$.

For simplicity we assume $i = 1$, $m = 2$. Observe the surgeries performed below do not affect L_2 so if we reduce $\pi_1(U_1)$ to Z the same argument can then be used to reduce $\pi_1(U_2)$. The case $m \geqq 3$ is similar.

Let $\lambda\colon V_1 \to I$ be a smooth map with $\lambda^{-1}(0) = \partial V_1$. Define $W_1 = \{(v, t) \in V_1 \times I \; 0 \leqq t \leqq \lambda(t)\}$. W_1 is a manifold with boundary $V_1(0) \cup V_1(1)$, where $V_1(\varepsilon) = \{(v, t) \mid t = \varepsilon\lambda(v)\}$, $\varepsilon = 0, 1$. Also W_1 has a corner along $V_1(0) \cap V_1(1) = \partial V_1 = X_1$. Each $V_1(\varepsilon)$ is diffeomorphic to V_1. Let $\bar{V}_1(\varepsilon)$ be the complement in $V_1(\varepsilon)$ of an open smooth collar of $\partial V_1(\varepsilon)$. With the aid of the framing of V_1 we may embedd W_1 so that $V_1(0) = V_1$. Let T_1 be the closure of $U - W_1$ with boundary $\partial W_1 + (X_2 \times S^1)$. We define maps $\nu_1^{(\varepsilon)}\colon V_1 \to T_1(\varepsilon = 0, 1)$ by $V_1 \cong V_1(\varepsilon) \subset \partial T_1 \subset T_1$. Let $G_1 = \pi_1(T_1)$, $H_1 = \pi_1(V_1)$. Then we have induced homomorphisms $\nu_1^{(\varepsilon)}\colon H_1 \to G_1$.

Notice that if we identify $\bar{V}_1(0)$ to $\bar{V}_1(1)$ in T_1, we obtain U so $\pi_1 = \pi_1(U)$ is an HNN extension [9; § 5.1],

$$\langle G_1, x_1\colon x_1\nu_1^{(1)}(h)x_1^{-1} = \nu_1^{(0)}(h)\ , \quad h \in H_1\rangle\ .$$

Define $l\colon H_1 \to Z$ by $l(\alpha) = l(\alpha, X_2)$, the linking number of α and X_2 in S^{n+2}. If $l \equiv 0$, $h\colon \Phi \to \pi$ is an isomorphism, where $h(x_i) = \mu_i$ for some choice of arcs γ_i [3]. If $l \neq 0$, let $K = \ker l$, α a generator of H_1/K. Since H_1 is the semidirect product $K \times (H_1/K)$, we may assume $\alpha \in H_1$. Define $\delta(h) = \lambda_1^{(1)}(h)\lambda_1^{(0)}(h^{-1})$, and let $R = N_{\pi_1}\{i\nu_1^{(0)}(h)\colon h \in H_1\}$ where $i\colon T_1 \subset U$ induces $i\colon G_1 \to \pi_1$.

Since $[\nu_1^{(1)}(k), x_1^{-1}] = \delta(k)x_1^{-1}(x_1\nu_1^{(0)}(k)x_1^{-1}\nu_1^{(1)}(k))x_1$, it follows that $\delta(k) \in [R, \pi_1]\ (k \in K)$.

Every element of G_1 is a product of conjugates of $\delta(k)(k \in K)$, $\delta(\alpha)$ and $h(x_2)$. Also $i\delta(\alpha) = [i\nu_1^{(1)}(\alpha), x_1]$. In particular, $\nu_1^{(0)}(k)$ is a product of conjugates of the above elements. Since $l(k) = 0$, $h(x_2)$ and $\delta(\alpha)$ occur with zero exponent sum and so $i\nu_1^{(0)}(k) \in [R, \pi_1]$, $R/[R, \pi_1]$ is then 0 and by [11; (2.1)], $H_2(\pi_1/R) = 0$.

Consider $U \times I$; attach 2-handles to $U \times \{1\}$ a long representatives of the $i\nu_1^{(0)}(k)$. It is necessary to attach finitely many handles because K is the normal closure of finitely many elements in H_1. In fact, both H_1 and H_1/K are finitely presented [7; I]. The resulting space $M' = U \times I \cup \Sigma(h_i^2)$ has fundamental group π_1/R. Since $H_2(\pi_1/R) = 0$ all its second homology (which is free abelian generated by the handles (h_i^2)) is spherical [5; § 1] that is the generators of the 2nd homology can be represented by spheres which, since $n \geqq 3$, can be taken to be embedded.

Attach 3-handles along these embedded spheres to obtain $M'' = U \times I \cup \Sigma(h_k^2) \cap \Sigma(h_i^3)$ which by standard arguments is the complement in $S^{n+2} \times I$ of a concordance between L and a link L'' which admits Seifert manifolds V_1'', V_2'' and $\pi_1(V_1'') = Z$, $V_2'' \approx V_2$. Repeating this argument for V_2 we may assume L is concordant to a link L' with manifolds V_1', V_2' with infinite cyclic fundamental groups generated by α_1 and α_2 respectively. Represent α_i by a loop a_i: $S^1 \rightarrow V_i'$.

We may assume a_i extends to an embedding $D^2 \rightarrow U_i$ which misses V_i' althogh it will intersect $X_j (j \neq i)$, since $l(\alpha_i) \neq 0$.

If a_i does not intersect, let β be a loop in U with linking number $-l(\alpha_1, X_2)$ with X_2 and linking number zero with X_1. We may assume $\beta \cap V_1 = \emptyset$. Let τ be a tubular neighborhood of β, $\tau \cong \beta \times D^{n+1}$. We may alter V_1' to $V_1 \# \partial\tau$ the connected sum of V_1' and $\partial\tau$ along a tube that joins them and that is disjoint with X_2. Now $\pi_1(V_1 \# \partial\tau)$ is free in two generators α_1 and β and $\alpha_1\beta$ has zero linking number with X_2 so it can be eliminated as before.

The new link admits a Seifert manifold with infinite cyclic fundamental group generated by β. Since β bounds a disk in S^{n+2} disjoint from X_1 the assumption on a_i is possible. A similar process for V_2.

Clearly if a_i extends to a disk in U_i, $\pi_1(U_i) = Z$. To complete the proof we have to fulfill condition (ii) of Lemma 8. We work with $m = 2$ and use the technique of [5, p. 246] to construct an infinite cyclic cover $\mathfrak{U}_1$ of U by taking a quotient of the disjoint union of copies Y_n of U cut along V_1 with identifications $(V_1^+)_n = (V_1^-)_{n+1}$ (loc. cit.). This cover is associated to $\mathscr{R}_1$ and $H_1\mathfrak{U}_1 \approx Z[x_1, x_1^{-1}]$. Let $\bar{V}_2$ be the lift of V_2 to U_1. By cutting $\mathfrak{U}_1$ along $\bar{V}_2$ and repeating the construction we obtain a cover $\mathfrak{U}$ of U associated to $\mathscr{R}_1 \cap \mathscr{R}_2 = [\pi_1, \pi_1]$ and $H_1\mathfrak{U}$ contains $\Lambda = Z[x_1, x_2, x_1^{-1}, x_2^{-1}] \approx H_1\Phi_2$ as a direct summand. In fact $H_1\mathfrak{U} = H_1([\pi_1, \pi_1]) = H_1(\Phi_2) \oplus M$ for a certain Λ-module M. Since $H_2(U) = 0$ we obtain from the spectral sequence for $\mathfrak{U}$ that $M \otimes_\Lambda Z = 0$ and so every element of M is of the form $m = (x_1 - 1)m_1 + (x_2 - 1)m_2$. If y is a loop in U representing m then y is of the form $[x_1, y_1][x_2, y_2]$ for y_1, y_2 loops representing elements in M. Thus if we attach 2-handles to $U \times \{1\}$ to kill M we observe that the resulting homology is spherical since the new relations are products of commutators of themselves. This means our surgical argument can be repeated once more to insure that $h: H_1(\Phi_2) \rightarrow H_1([\pi_1, \pi_1])$ is an isomorphism.

In the remaining part of § 4 we assume $m = 2$ although similar results hold for all m.

COROLLARY 12. *Every link $\mathscr{L}$ is concordant to one $\mathscr{L}'$ which has mutually disjoint Seifert manifolds.*

Proof. We may assume that if $U' = U_L$, then $h: \Phi \to \pi(U')$ is an isomorphism, where $\mathscr{L}'$ is the link found by Lemma 11. Consider

(12)
$$\begin{array}{ccc} U' & & \\ {\scriptstyle i}\uparrow & \searrow {\scriptstyle g} & \\ \partial U & \xrightarrow{p} & S^1 \vee S^1 \end{array}$$

where $p|: X_j \times S^1 \to S^1 \vee S^1$ projects onto the jth circle, $j = 1, 2$. Triangle (12) extends if and only if the corresponding triangle of groups

$$\begin{array}{ccc} \pi_1(U') & & \\ {\scriptstyle h}\uparrow & \searrow {\scriptstyle q^*} & \\ \Phi & \xrightarrow{id.} & \Phi \end{array}$$

extends, that is, if there exists an epimorphism $q_*: \pi_1(U') \to \Phi$ which is a retract of h. By [11; (3.1)] q_* induces an isomorphism $q: \pi(U') \to \Phi$ which must be the inverse of h. Thus (12) extends iff $h: \Phi \to \pi(U')$ is an isomorphism. Using regular values of q: $U' \to S^1 \vee S^1$ we may construct disjoint Seifert manifolds by the same method used in the proof of Lemma 11.

We say a link $\mathscr{L}$ is *simple* if we can find Seifert manifolds $V_i^{n+1} \subseteqq U$ which are q-connected for $n = 2q$ or $n = 2q + 1$. If $n = 2q (q \geqq 2)$ the V_i are $(2q + 1)$-discs [5; III].

THEOREM 13. *Every link of dimension $n \geqq 3$ is concordant to a simple link.*

This is a consequence of the results of [3]. *Similar definitions and results for $m \geqq 3$. For $n = 2q$, Theorem* 13 *generalizes* [5; III. 6].

REMARKS. (1) In [1] it is shown that group $\rho = \langle x_1, x_2, b: b = [x_1^i, b][x_1^j, x_2]\rangle$ is residually nilpotent and not free if $ij \neq 0$. Map h: $\Phi \to \rho$ defined by $h(x_k) = x_k$ is monic and $\rho = N_\rho h(\Phi)$. By [5] $\rho = \pi_1(U_L)$ for some link $\mathscr{L}$. Observe $\mathscr{J}_1 = Z$ but $\mathscr{J}_2 = \langle x_1, b: b = [x_1^i, b]\rangle \neq Z$ if $i \neq 0$ as expected from Lemma 8.

(2) Similarly in [12] a link $\mathscr{L}: X \to S^4$ is found with $\pi_1(U) = \Phi$ but $h: \Phi \to \pi_1(U)$ is not onto. Again $\mathscr{J}_1 = Z$ but $\mathscr{J}_2 = \langle a, b: a^2 = b^3\rangle$ the trefoil knot group as expected from Corollary 10.

(3) Cappell has pointed out two facts; the first is a

PROPOSITION 14. *Let S_1, S_2 be spaces with the homotopy type of finite CW-complexes and let $f: S_1 \to S_2$ be a continuous map. If $f_*: H_q(S_1) \to H_q(S_2)$ is an isomorphism for $q = 1$ and an epimorphism for $q = 2$ then $\pi(S_1)$ is concordant to $\pi(S_2)$.*

The proof is based on the naturality of Hopf's sequence

$$\pi_2(S)_G \longrightarrow H_2(S) \longrightarrow H_2(G) \longrightarrow 0 \ ,$$

where $G = \pi_1(S)$ (cf [5; p. 106]) which is used to verify that the attached 2-cells that produce 2-homology actually generate spherical homology.

The second is more serious: Lemma 3 of [3] states more that it proves for odd dimensions. Theorem 13 is the best result possible. However Theorem 13 and the results of [2], yield a description of the concordance group for links.

(4) Unfortunately our remarks do not work for links in S^3; the presence of longitudes ruins everything. In the Notices of the Amer. Math. Soc. (24 (1977), announcement 77T-G15), J.A. Hilman exhibits a 2-link $\mathscr{L}$ with unknotted components, zero Alexander polynomial (so $\bar{\mu}(i_1, \cdots, i_r) = 0$) which satisfies $\Phi \neq \pi(U_L)$. If we try to imitate the construction algebraically we obtain the group

$$G_1 = \langle x_1, x_2, a, b \colon a = b^{-1}x_2x_1x_2^{-1}b,\ b = a^{-1}x_1x_2x_1^{-1}a \rangle \ .$$

Let $c = ax_1^{-1}$, $d = bx_2^{-1}$. Then G_1 has a presentation

$$\langle x_1, x_2, c, d \colon c = [x_1, x_2^{-1}dx_2],\ d = [x_2, x_1^{-1}cx_1] \rangle \ ,$$

eliminate d,

$$\begin{aligned} &\langle x_1, x_2, c \colon c = [x_1, x_2^{-1}[x_2, x_1^{-1}cx_1]x_2] \rangle \\ &\qquad = \langle x_1, x_2, c \colon c = [x_1, [x_1^{-1}cx_1, x_2^{-1}]] \rangle \end{aligned}$$

and $c \in G_\infty$ by the Green-Zeeman argument. Naturally G_1 is not $\pi(U)$ because of the extra relation $[x_1, \lambda] = 1$ (λ = longitude) which causes $H_2\pi_1(U_L)$ to be $\neq 0$, and $\pi_1(U_L)$ is not an algebraic extension of $\Phi = \langle x_1, x_2 \colon \rangle$. ($\mathscr{L}$ is however, nullconcordant). The key point is that the longitudes λ are nonzero in $H_1([\pi_1, \pi_1])$ and so hypothesis (ii) of Lemma 8 fails.

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TUFTS UNIVERSITY
MEDFORD, MA 02155

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