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A hypoelliptic differential operator $P(D)$ with constant coefficients has the following property: For every $u \in \mathscr{D}'(\Omega)$, if $P(D)u$ is in a Gevrey class in Ω then so is u (though the two Gevrey classes are not necessarily the same).

In this paper we prove that hypoelliptic differential operators with variable coefficients have locally the same property, if they are of constant strength and their coefficients are in a Gevrey class.

Let $P(x, D)$ be a differential operator with C^∞ coefficients and of constant strength in a neighborhood of a point $x^\circ \in R^n$. L. Hörmander ([1], Theorem 7.3.1) proved the following local existence theorem:

There exists a sufficiently small neighborhood Ω of x° and a linear mapping E of $\mathscr{E}'(R^n)$ into $\mathscr{E}'(R^n)$ with the properties

$$(1) \qquad P(x, D)Eu = u \text{ in } \Omega\,, \quad \text{if} \quad u \in \mathscr{E}'(R^n)\,,$$

$$(2) \qquad EP(x, D)v = v \text{ in } \Omega\,, \quad \text{if} \quad v \in \mathscr{E}'(\Omega)\,,$$

and

$$(3) \qquad \| Eu \|_{p,\tilde{P}_\circ k} \leqq C_k \| u \|_{p,k}\,, \quad \text{if} \quad u \in \mathscr{E}'(R^n) \cap \mathscr{B}_{p,k}\,, \quad k \in \mathscr{K}\,.$$

If, in addition, the operator $P_\circ(D) = P(x^\circ, D)$ is hypoelliptic, then every solution $u \in \mathscr{D}'(\Omega)$ of the equation

$$(4) \qquad P(x, D)u = v$$

is in $C^\infty(\Omega)$, whenever $v \in C^\infty(\Omega)$ (see [1], Theorem 7.4.1).

But $P_\circ(D)$, being hypoelliptic, is d-hypoelliptic for some $d \geqq 1$, i.e., every solution $u \in \mathscr{D}'(\Omega)$ of the equation

$$P_\circ(D)u = v$$

is in the Gevrey class $\Gamma^{\rho'}(\Omega)$, when $v \in \Gamma^\rho(\Omega)$ and $\rho' = \max\{d, \rho\}$. The question arises whether the same is true for equation (4), if the coefficients of $P(x, D)$ are in $\Gamma^\rho(\Omega)$.

In this paper we prove a theorem on Gevrey regularity for a very wide class of operators—not necessarily differential operators. Next we apply this general theorem to a differential operator $P(x, D)$ of constant strength in a neighborhood of $x^\circ \in R^n$. We choose Ω as in the existence theorem mentioned above. If the coefficients of $P(x, D)$ are in $\Gamma^\rho(\Omega)$ and $P_\circ(D)$ is d-hypoelliptic, we show that every solution

$u \in \mathscr{E}'(\Omega)$ of equation (4) is in $\Gamma^{d\rho'}(\Omega)$, when $v \in \Gamma^{\rho}(\Omega) \cap \mathscr{E}'(\Omega)$ and $\rho' = \max\{d, \rho\}$.

Throughout the paper we use the notation of [1] and [3].

1. The general theorem. We recall that, if Ω is an open subset of R^n and $\rho > 0$, the ρth Gevrey class $\Gamma^{\rho}(\Omega)$ can be defined as the set of all functions $u \in C^{\infty}(\Omega)$ with the property: To every compact set $K \subset \Omega$ there exists a constant A (depending on K and u) such that

$$\| D^{\alpha} u \|_K \leqq A^{|\alpha|+1}(|\alpha|!)^{\rho} ,$$

where $\| \ \|_K$ is the L^2-norm over K and α is any multi-index.

Similarly, if P is a polynomial in R^n of degree m, we denote by $\Gamma_P^{\rho}(\Omega)$ the set of all functions $u \in C^{\infty}(\Omega)$ with the property: To every compact set $K \subset \Omega$ there exists a constant B (depending on P, K and u) such that

$$\| P^j(D) u \|_K \leqq B^{j+1}(j!)^{m\rho} , \qquad j = 0, 1, \cdots ,$$

where $P^j(D)$ is the jth power of $P(D)$.

The polynomial P is d-hypoelliptic if and only if there is a constant C such that, for every α,

$$|\xi|^{|\alpha|/d} |D^{\alpha}P(\xi)| \leqq C(1 + |P(\xi)|) , \qquad \xi \in R^n ,$$

(see [4], p. 440). Hence it follows that

$$(5) \qquad |\xi|^2 \leqq C'(1 + |P(\xi)|^2)^{d/m}$$

and

$$(6) \qquad |D^{\alpha}P(\xi)| \leqq C''(1 + |P(\xi)|)^{1-|\alpha|/md} , \qquad \xi \in R^n ,$$

where m is the degree of P and C', C'' are constants.

The following theorem is a refined version of a theorem stated in [3]; we include the proof for the sake of completeness.

THEOREM 1. *If P is a d-hypoelliptic polynomial then, for any open set $\Omega \subset R^n$ and any $\rho > 0$, we have*

$$\Gamma_P^{\rho}(\Omega) \subset \Gamma^{d\rho'}(\Omega) ,$$

where $\rho' = \max\{d, \rho\}$.

Proof. We may assume that d is the smallest number for which the estimate (5) holds. Then d is a rational number and we can write $d/m = \mu/\nu$, where μ and ν are integers > 0. It follows that

$$(7) \qquad |\xi^{\alpha}|^2 \leqq C_1(1 + |P^{\mu}(\xi)|^2) , \qquad \xi \in R^n ,$$

for some constant C_1 and every α with $|\alpha| \leqq \nu$. Furthermore, the polynomial P^μ is d-hypoelliptic, and so

$$(8) \qquad |D^\alpha P^\mu(\xi)| \leqq C_2(1 + |P^\mu(\xi)|)^{1-|\alpha|/m\mu d}, \qquad \xi \in R^n,$$

by (6).

Let Ω be a bounded open subset of R^n and Ω_δ, $\delta > 0$, the set of points $x \in \Omega$ whose distance from $C\Omega$ is greater than δ. By a theorem of Hörmander ([2], Theorem 4.2), conditions (7) and (8) imply that, for every $s \geqq 0$ and $t > 0$,

$$\sup_{0<\tau\leqq t} \tau^\lambda \| D^\alpha v \|_{\Omega_{s+\tau}} \leqq C\left\{ \sup_{0<\tau\leqq t} \tau^\lambda \| P^\mu(D) v \|_{\Omega_{s+\tau}} + \| v \|_{\Omega_s} \right\}, \qquad v \in C^\infty(\Omega_s),$$

where $|\alpha| \leqq \nu$, $\lambda = m\mu d$ and C is a constant depending only on P, μ, ν, and the diameter of Ω. Hence

$$(9) \qquad \| D^\alpha v \|_{\Omega_{s+t}} \leqq C\{\| P^\mu(D) v \|_{\Omega_s} + t^{-\lambda} \| v \|_{\Omega_s}\}, \qquad v \in C^\infty(\Omega_s).$$

Given an integer $k \geqq 1$ and a $\delta > 0$, we apply (9) with $s = (1 - 1/k)^\delta$, $t = \delta/k$ and $v = D^\beta u$, where $|\beta| = (k-1)\nu$. Next (if $k > 1$) we apply again (9) to both terms on the right-hand side with $s = (1 - 2/k)\delta$, $t = \delta/k$ and either $v = D^\gamma P^\mu(D)$ or $v = D^\gamma u$, where $|\gamma| = (k-2)\nu$. After k such steps we obtain

$$(10) \quad \max_{|\alpha|=k\nu} \| D^\alpha u \|_{\Omega_\delta} \leqq C^k \sum_{i=0}^{k} \binom{k}{i} \left(\frac{k}{\delta}\right)^{\lambda i} \| P^{(k-i)\mu}(D) u \|_\Omega, \qquad u \in C^\infty(\Omega).$$

On the other hand, for each multi-index α with $|\alpha| \leqq \nu - 1$ there is a multi-index β with $|\beta| = \nu$ such that

$$|\xi^\alpha|^2 \leqq 1 + |\xi^\beta|^2, \qquad \xi \in R^n.$$

Therefore, by the same theorem of Hörmander, there is a constant C' such that

$$(11) \qquad \| D^\alpha u \|_{\Omega_{2\delta}} \leqq C'\left\{\max_{|\beta|=\nu} \| D^\beta u \|_{\Omega_\delta} + \| u \|_{\Omega_\delta}\right\}, \qquad u \in C^\infty(\Omega_\delta),$$

where $|\alpha| \leqq \nu - 1$; the constant C' may depend on δ.

If α is an arbitrary multi-index and $k\nu \leqq |\alpha| < (k+1)\nu$, we choose another multi-index β so that $|\alpha - \beta| \leqq \nu - 1$ and $|\beta| = k\nu$. Then condition (11) yields

$$\| D^\alpha u \|_{\Omega_{2\delta}} \leqq C'\left\{ \max_{|\beta|=(\nu+1)k} \| D^\beta u \|_{\Omega_\delta} + \max_{|\beta|=\nu k} \| D^\beta u \|_{\Omega_\delta}\right\}, \qquad u \in C^\infty(\Omega_\delta).$$

Combining this estimate with (10), we obtain

$$
(12)\qquad \begin{aligned} \| D^\alpha u \|_{\Omega_{2\delta}} \leqq C'C^{k+1}\sum_{i=0}^{k+1}\binom{k+1}{i}\left(\frac{k+1}{\delta}\right)^{\lambda i} \| P^{(k+1-i)\mu}(D)u \|_\Omega \\ + C'C^k \sum_{i=0}^{k}\binom{k}{i}\left(\frac{k}{\delta}\right)^{\lambda i} \| P^{(k-i)\mu}(D)u \|_\Omega \,. \end{aligned}
$$

Suppose now that $u \in \Gamma_P^\rho(\Omega')$, where Ω' is an arbitrary open set in R^n. If K is a compact subset of Ω', there exists a bounded open set Ω and a $\delta > 0$ such that $K \subset \Omega_{2\delta} \subset \Omega \subset\subset \Omega'$. It follows that, for some constant B,

$$
\| P^i(D)u \|_\Omega \leqq B^{i+1} i^{im\rho}\,, \qquad i = 0, 1, \cdots .
$$

Note that in the definition of $\Gamma_P^\rho(\Omega)$ the terms $(j!)^{m\rho}$ can be replaced by $j^{jm\rho}$, by the Stirling formula. This leads to

$$
(13)\qquad \begin{aligned} k^{i\lambda} \| P^{(k-i)}P(D)u \|_\Omega &\leqq [(B+1)\mu^{m\rho}]^{k+1} k^{km\mu\rho + i(\lambda - m\mu\rho)} \\ &\leqq [(B+1)\mu^{m\rho}]^{k+1} k^{km\mu\rho'} \end{aligned}
$$

$i = 0, 1, \cdots, k$, where $\rho' = \max\{d, \rho\}$. Applying the estimates (13) to the righ-hand side of (12) we can find two constants B_1 and B_2 such that

$$
\begin{aligned} \| D^\alpha u \|_{\Omega_{2\delta}} \leqq B_1^{k+1} k^{km\mu\rho'} &\leqq B_2^{|\alpha|+1} |\alpha|^{|\alpha| m\mu\rho'/\nu} \\ &= B_2^{|\alpha|+1} |\alpha|^{|\alpha| d\rho'} \,. \end{aligned}
$$

Thus $u \in \Gamma^{d\rho'}(\Omega')$ and the proof is complete.

We are now in a position to prove the general theorem. We denote by $\| \;\|$ the L^2-norm over R^n.

THEOREM 2. *Let Ω be an open subset of R^n and let T be a mapping of $C_0^\infty(\Omega)$ into $C_0^\infty(\Omega)$ whose powers T^j, $j = 1, 2, \cdots$, have the following properties:*

(t_1) *There exists a d-hypoelliptic polynomial P such that*

$$
\| P^j(D)u \| \leqq A^j \| T^j u \|\,, \qquad u \in C_0^\infty(\Omega)\,, \qquad j = 1, 2, \cdots,
$$

where A is a constant independent of j.

(t_2) *For some $\rho > 1$,*

$$
\| T^j v \| \leqq B^{j+1}(j!)^{m\rho}\,, \qquad v \in \Gamma^\rho(\Omega) \cap \mathscr{E}'(\Omega)\,, \qquad j = 1, 2, \cdots,
$$

where m is the degree of P and B is a constant independent of j.

Then every solution $u \in C_0^\infty(\Omega)$ of the equation

$$
(14)\qquad Tu = v
$$

is in $\Gamma^{d\rho'}(\Omega)$, when $v \in \Gamma^\rho(\Omega) \cap \mathscr{E}'(\Omega)$; here, as before, $\rho' = \max\{d, \rho\}$.

Proof. The theorem follows immediately from Theorem 1. In fact, let $u \in C_0^\infty(\Omega)$ be a solution of equation (14) with $v \in \Gamma^\rho(\Omega) \cap \mathscr{E}'(\Omega)$. Making use of conditions (t_1) and (t_2) we then have

$$\begin{aligned} \| P^j(D)u \| &\leqq A^j \| T^j u \| \\ &= A^j \| T^{j-1} v \| \leqq (AB)^j (j!)^{m\rho} , \qquad j = 1, 2, \cdots . \end{aligned}$$

This proves that $u \in \Gamma_P^\rho(\Omega)$. Since P is d-hypoelliptic, it follows that $u \in \Gamma^{d\rho'}(\Omega)$, by Theorem 1.

2. Differential operators of constant strength. The following lemma is essential for the proof of our next theorem.

LEMMA. *Let P_0 be a hypoelliptic polynomial of degree m and set $k = \tilde{P}_0$. Then*

$$k_\nu(\xi) = \sup_{|\eta| \leqq \nu} k(\xi + \eta) , \qquad \nu > 0 , \tag{15}$$

is a function in $\mathscr{K}$ with the properties

$$k(\xi) \leqq k_\nu(\xi) \leqq k(\xi)(1 + C\nu)^m , \qquad \xi \in R^n , \tag{16}$$

$$(k^j)_\nu = (k_\nu)^j , \qquad j = 1, 2, \cdots , \tag{17}$$

$$M_{k_\nu}(\xi) = \sup_{\xi'} \frac{k_\nu(\xi + \xi')}{k_\nu(\xi')} \leqq (1 + C|\xi|)^m , \qquad \xi \in R^n , \tag{18}$$

where C is a constant, and

(19) $M_{k_\nu} \to 1$ *uniformly on compact subsets of R^n as $\nu \to \infty$.*

Proof. The properties (16)—(18) follow immediately from (15) and the inequality

$$k(\xi + \eta) \leqq k(\xi)(1 + C|\eta|)^m ,$$

which is valid for $k = \tilde{P}_0$. We prove the convergence (19).

If $|\xi| \leqq \mu$, then

$$\frac{k_\nu(\xi + \xi')}{k_\nu(\xi')} \leqq \frac{\sup_{|\eta| \leqq \mu + \nu} k(\xi' + \eta)}{\sup_{|\eta| \leqq \nu} k(\xi' + \eta)} . \tag{20}$$

We claim that the right-hand side of (20), which is $\geqq 1$, converges to 1 as $\nu \to \infty$, uniformly for $\xi' \in R^n$.

Assume the contrary. Then there exist $\varepsilon > 0$, a sequence $\{\nu_j\} \subset R$, $\nu_j \to \infty$, and sequences $\{\xi'_j\}$, $\{\eta_j\}$, $\{\eta'_j\} \subset R^n$ such that

$$k(\xi'_j + \eta_j) = \sup_{|\eta| \leqq \mu + \nu_j} k(\xi'_j + \eta) , \qquad k(\xi'_j + \eta'_j) = \sup_{|\eta| \leqq \nu_j} k(\xi'_j + \eta) , \tag{21}$$

and

$$\frac{k^2(\xi'_j + \eta_j)}{k^2(\xi'_j + \eta'_j)} \geqq 1 + \varepsilon , \qquad j = 1, 2, \cdots . \tag{22}$$

Obviously we have

$$\nu_j < |\eta_j| \leqq \mu + \nu_j \tag{23}$$

and

$$k(\xi'_j + \eta_j) \geqq k(\xi'_j - \eta_j) , \qquad j = 1, 2, \cdots . \tag{24}$$

But, for each j, one of the inequalities

$$|\xi'_j + \eta_j| > \nu_j \quad \text{or} \quad |\xi'_j - \eta_j| > \nu_j \tag{25}$$

must be satisfied. For, otherwise we could write

$$2|\eta_j| = |(\xi'_j + \eta_j) - (\xi'_j - \eta_j)| \leqq |\xi'_j + \eta_j| + |\xi'_j - \eta_j| \leqq 2\nu_j ,$$

which contradicts the first inequality in (23). Since $P_\circ$ is a hypoelliptic polynomial, it follows from (24) and (25) that $|\xi'_j + \eta_j| \to \infty$.

On the other hand, if $\nu_j > \mu/2$, then

$$\frac{k^2(\xi'_j + \eta_j)}{k^2\left(\xi'_j + \eta_j - \mu\dfrac{\eta_j}{|\eta_j|}\right)} \geqq 1 + \varepsilon , \tag{26}$$

in view of (21), (22) and (23). Furthermore, since k^2 is a polynomial of degree $2m$,

$$k^2\left(\xi'_j + \eta_j - \mu\frac{\eta_j}{|\eta_j|}\right) \geqq k^2(\xi'_j + \eta_j) - \sum_{\alpha \neq 0} |D^\alpha k^2(\xi'_j + \eta_j)| \frac{(1 + \mu)^{2m}}{\alpha!} , \tag{27}$$

by Taylor's formula. But we assumed that the polynomial $P_\circ$ is hypoelliptic and therefore

$$\frac{D^\alpha P_\circ(\xi)}{P_\circ(\xi)} \longrightarrow 0 , \quad \text{if } \alpha \neq 0 \text{ and } |\xi| \longrightarrow \infty$$

(see [1], p. 99, Theorem 4.1.3). Hence it follows that

$$\frac{D^\alpha k^2(\xi)}{k^2(\xi)} \longrightarrow 0 , \quad \text{if } \alpha \neq 0 \text{ and } |\xi| \longrightarrow \infty .$$

If now $|\xi'_j + \eta_j|$ is sufficiently large, we obtain from (27) the estimates

$$k^2\left(\xi'_j + \eta_j - \mu\frac{\eta_j}{|\eta_j|}\right) \geqq k^2(\xi'_j + \eta_j)\left(1 - \frac{\varepsilon}{2}\right) ,$$

which contradict the inequalities (26), when $\varepsilon < 1$.

The contradiction proves the desired convergence of the right-hand side of (20). The property (19) is an immediate consequence of that fact.

We remark that the weight function k_δ defined in [1] (see p. 35, Theorem 2.1.2) cannot be used for our purpose.

THEOREM 3. *Let $P(x, D)$ be a differential operator of constant strength in a neighborhood of x° and let Ω be a sufficiently small open neighborhood of x° for which the local existence theorem is valid. If the coefficients of $P(x, D)$ are in $\Gamma^\rho(\Omega)$, $\rho > 1$, and if the operator $P_\circ(D) = P(x^\circ, D)$ is d-hypoelliptic, then every solution $u \in \mathscr{E}'(\Omega)$ of equation* (4) *is in $\Gamma^{d\rho'}(\Omega)$, whenever $\nu \in \Gamma^\rho(\Omega) \cap \mathscr{E}'(\Omega)$.*

Proof. Since $P(x, D)$ is of constant strength in Ω and $P_\circ(D)$ is hypoelliptic, every solution $u \in \mathscr{E}'(\Omega)$ of equation (4) is in $C_0^\infty(\Omega)$, whenever $v \in C_0^\infty(\Omega)$ (see [1], p. 176, Theorem 7.4.1). For the proof of Theorem 3 it now suffices to show that $P(x, D)$ satisfies the conditions imposed on T in Theorem 2.

We first observe that, by the lemma, we can replace in the proof of the local existence theorem the function k_δ by our function k_ν. Next we recall that condition (3) is derived from the estimate

$$\| Ef \|_{p,\tilde{P}_\circ k_\delta} \leqq C_\circ \|f\|_{p,k_\delta} \qquad f \in \mathscr{E}'(R^n) \cap \mathscr{B}_{p,k} ,$$

where $C_\circ$ is a constant independent of f, k and δ (see [1], p. 176). Hence, if $p = 2$, $k = \tilde{P}_\circ$ and k_ν is as in our lemma, we obtain

$$\| E^j u \|_{2,k(k^{j-1})_\nu} \leqq A \| E^{j-1} u \|_{2,(k^{j-1})_\nu} , \qquad u \in C_\circ^\infty(\Omega) , \tag{28}$$

where A is a constant independent of j and u. But, in view of (16) and (17), there is a constant B independent of j and u such that

$$\| E^{j-1} u \|_{2,(k^{j-1})_\nu} \leqq B \| E^{j-1} u \|_{2,k(k^{j-2})_\nu} , \qquad u \in C_\circ^\infty(\Omega) . \tag{29}$$

Condition (16) and repeated application of (28) and (29) gives

$$\| E^j u \|_{2,k^j} \leqq \| E^j u \|_{2,k(k^{j-1})_\nu} \leqq (AB)^j \| u \|, \qquad u \in C_\circ^\infty(\Omega) . \tag{30}$$

Substituting now $P(x, D)u$ in place of u in (30) and observing that

$$\| P_\circ^j(D) f \| \leqq \|f\|_{2,k^j} , \qquad f \in C_\circ^\infty(\Omega) ,$$

we obtain

$$\| P_\circ^j(D) E^j P^j(x, D) u \| \leqq (AB)^j \| P^j(x, D) u \| .$$

Hence we conclude that

$$\| P_\circ^j(D) u \| \leqq (AB)^j \| P^j(x, D) u \| , \qquad u \in C_\circ^\infty(\Omega) , \tag{31}$$

because of (2), where $P_\circ$ is a d-hypoellptic polynomial.

On the other hand, if $v \in \Gamma^\rho(\Omega) \cap \mathscr{E}'(\Omega)$, we have

$$\| P^j(x, D)v \| \leqq C^{j+1}(j!)^{m\rho}, \qquad j = 1, 2, \cdots, \tag{32}$$

since the coefficients of $P(x, D)$ are in $\Gamma^\rho(\Omega)$.

By Theorem 2, conditions (31) and (32) imply that every solution $u \in C_\circ^\infty(\Omega)$ of equation (4) is in $\Gamma^{d\rho'}(\Omega)$, when $v \in \Gamma^\rho(\Omega) \cap \mathscr{E}'(\Omega)$. The theorem is thus established.

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