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## **ON HYPOELLIPTIC DIFFERENTIAL OPERATORS OF CONSTANT STRENGTH**

M. SHAFII-MOUSAVI AND ZBIGNIEW ZIELEZNY

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**A hypoelliptic differential operator  $P(D)$  with constant coefficients has the following property: For every  $u \in \mathcal{D}'(\Omega)$ , if  $P(D)u$  is in a Gevrey class in  $\Omega$  then so is  $u$  (though the two Gevrey classes are not necessarily the same).**

**In this paper we prove that hypoelliptic differential operators with variable coefficients have locally the same property, if they are of constant strength and their coefficients are in a Gevrey class.**

Let  $P(x, D)$  be a differential operator with  $C^\infty$  coefficients and of constant strength in a neighborhood of a point  $x^\circ \in R^n$ . L. Hörmander ([1], Theorem 7.3.1) proved the following local existence theorem:

There exists a sufficiently small neighborhood  $\Omega$  of  $x^\circ$  and a linear mapping  $E$  of  $\mathcal{E}'(R^n)$  into  $\mathcal{E}'(R^n)$  with the properties

$$(1) \quad P(x, D)Eu = u \text{ in } \Omega, \quad \text{if } u \in \mathcal{E}'(R^n),$$

$$(2) \quad EP(x, D)v = v \text{ in } \Omega, \quad \text{if } v \in \mathcal{E}'(\Omega),$$

and

$$(3) \quad \|Eu\|_{p, \tilde{P}, k} \leq C_k \|u\|_{p, k}, \quad \text{if } u \in \mathcal{E}'(R^n) \cap \mathcal{B}_{p, k}, \quad k \in \mathcal{H}.$$

If, in addition, the operator  $P_\circ(D) = P(x^\circ, D)$  is hypoelliptic, then every solution  $u \in \mathcal{D}'(\Omega)$  of the equation

$$(4) \quad P(x, D)u = v$$

is in  $C^\infty(\Omega)$ , whenever  $v \in C^\infty(\Omega)$  (see [1], Theorem 7.4.1).

But  $P_\circ(D)$ , being hypoelliptic, is  $d$ -hypoelliptic for some  $d \geq 1$ , i.e., every solution  $u \in \mathcal{D}'(\Omega)$  of the equation

$$P_\circ(D)u = v$$

is in the Gevrey class  $\Gamma^{\rho'}(\Omega)$ , when  $v \in \Gamma^\rho(\Omega)$  and  $\rho' = \max\{d, \rho\}$ . The question arises whether the same is true for equation (4), if the coefficients of  $P(x, D)$  are in  $\Gamma^\rho(\Omega)$ .

In this paper we prove a theorem on Gevrey regularity for a very wide class of operators—not necessarily differential operators. Next we apply this general theorem to a differential operator  $P(x, D)$  of constant strength in a neighborhood of  $x^\circ \in R^n$ . We choose  $\Omega$  as in the existence theorem mentioned above. If the coefficients of  $P(x, D)$  are in  $\Gamma^\rho(\Omega)$  and  $P_\circ(D)$  is  $d$ -hypoelliptic, we show that every solution

$u \in \mathcal{E}'(\Omega)$  of equation (4) is in  $\Gamma^{d\rho'}(\Omega)$ , when  $v \in \Gamma^{\rho}(\Omega) \cap \mathcal{E}'(\Omega)$  and  $\rho' = \max\{d, \rho\}$ .

Throughout the paper we use the notation of [1] and [3].

1. **The general theorem.** We recall that, if  $\Omega$  is an open subset of  $R^n$  and  $\rho > 0$ , the  $\rho$ th Gevrey class  $\Gamma^{\rho}(\Omega)$  can be defined as the set of all functions  $u \in C^{\infty}(\Omega)$  with the property: To every compact set  $K \subset \Omega$  there exists a constant  $A$  (depending on  $K$  and  $u$ ) such that

$$\|D^{\alpha}u\|_K \leq A^{|\alpha|+1}(|\alpha|!)^{\rho},$$

where  $\|\cdot\|_K$  is the  $L^2$ -norm over  $K$  and  $\alpha$  is any multi-index.

Similarly, if  $P$  is a polynomial in  $R^n$  of degree  $m$ , we denote by  $\Gamma_P^{\rho}(\Omega)$  the set of all functions  $u \in C^{\infty}(\Omega)$  with the property: To every compact set  $K \subset \Omega$  there exists a constant  $B$  (depending on  $P, K$  and  $u$ ) such that

$$\|P^j(D)u\|_K \leq B^{j+1}(j!)^{m\rho}, \quad j = 0, 1, \dots,$$

where  $P^j(D)$  is the  $j$ th power of  $P(D)$ .

The polynomial  $P$  is  $d$ -hypoelliptic if and only if there is a constant  $C$  such that, for every  $\alpha$ ,

$$|\xi|^{|\alpha|/d} |D^{\alpha}P(\xi)| \leq C(1 + |P(\xi)|), \quad \xi \in R^n,$$

(see [4], p. 440). Hence it follows that

$$(5) \quad |\xi|^2 \leq C'(1 + |P(\xi)|^2)^{d/m}$$

and

$$(6) \quad |D^{\alpha}P(\xi)| \leq C''(1 + |P(\xi)|)^{1-|\alpha|/md}, \quad \xi \in R^n,$$

where  $m$  is the degree of  $P$  and  $C', C''$  are constants.

The following theorem is a refined version of a theorem stated in [3]; we include the proof for the sake of completeness.

**THEOREM 1.** *If  $P$  is a  $d$ -hypoelliptic polynomial then, for any open set  $\Omega \subset R^n$  and any  $\rho > 0$ , we have*

$$\Gamma_P^{\rho}(\Omega) \subset \Gamma^{d\rho'}(\Omega),$$

where  $\rho' = \max\{d, \rho\}$ .

*Proof.* We may assume that  $d$  is the smallest number for which the estimate (5) holds. Then  $d$  is a rational number and we can write  $d/m = \mu/\nu$ , where  $\mu$  and  $\nu$  are integers  $> 0$ . It follows that

$$(7) \quad |\xi^{\alpha}|^2 \leq C_1(1 + |P^{\mu}(\xi)|^2), \quad \xi \in R^n,$$

for some constant  $C_1$  and every  $\alpha$  with  $|\alpha| \leq \nu$ . Furthermore, the polynomial  $P^\mu$  is  $d$ -hypoelliptic, and so

$$(8) \quad |D^\alpha P^\mu(\xi)| \leq C_2(1 + |P^\mu(\xi)|)^{1-|\alpha|/m\mu d}, \quad \xi \in R^n,$$

by (6).

Let  $\Omega$  be a bounded open subset of  $R^n$  and  $\Omega_\delta$ ,  $\delta > 0$ , the set of points  $x \in \Omega$  whose distance from  $C\Omega$  is greater than  $\delta$ . By a theorem of Hörmander ([2], Theorem 4.2), conditions (7) and (8) imply that, for every  $s \geq 0$  and  $t > 0$ ,

$$\sup_{0 < \tau \leq t} \tau^\lambda \|D^\alpha v\|_{\Omega_{s+\tau}} \leq C \left\{ \sup_{0 < \tau \leq t} \tau^\lambda \|P^\mu(D)v\|_{\Omega_{s+\tau}} + \|v\|_{\Omega_s} \right\}, \quad v \in C^\infty(\Omega_s),$$

where  $|\alpha| \leq \nu$ ,  $\lambda = m\mu d$  and  $C$  is a constant depending only on  $P$ ,  $\mu$ ,  $\nu$ , and the diameter of  $\Omega$ . Hence

$$(9) \quad \|D^\alpha v\|_{\Omega_{s+t}} \leq C \{ \|P^\mu(D)v\|_{\Omega_s} + t^{-\lambda} \|v\|_{\Omega_s} \}, \quad v \in C^\infty(\Omega_s).$$

Given an integer  $k \geq 1$  and a  $\delta > 0$ , we apply (9) with  $s = (1 - 1/k)^\delta$ ,  $t = \delta/k$  and  $v = D^\beta u$ , where  $|\beta| = (k - 1)\nu$ . Next (if  $k > 1$ ) we apply again (9) to both terms on the right-hand side with  $s = (1 - 2/k)\delta$ ,  $t = \delta/k$  and either  $v = D^\gamma P^\mu(D)u$  or  $v = D^\gamma u$ , where  $|\gamma| = (k - 2)\nu$ . After  $k$  such steps we obtain

$$(10) \quad \max_{|\alpha|=k\nu} \|D^\alpha u\|_{\Omega_\delta} \leq C^k \sum_{i=0}^k \binom{k}{i} \left(\frac{k}{\delta}\right)^{\lambda i} \|P^{(k-i)\mu}(D)u\|_{\Omega}, \quad u \in C^\infty(\Omega).$$

On the other hand, for each multi-index  $\alpha$  with  $|\alpha| \leq \nu - 1$  there is a multi-index  $\beta$  with  $|\beta| = \nu$  such that

$$|\xi^\alpha|^2 \leq 1 + |\xi^\beta|^2, \quad \xi \in R^n.$$

Therefore, by the same theorem of Hörmander, there is a constant  $C'$  such that

$$(11) \quad \|D^\alpha u\|_{\Omega_{2\delta}} \leq C' \left\{ \max_{|\beta|=\nu} \|D^\beta u\|_{\Omega_\delta} + \|u\|_{\Omega_\delta} \right\}, \quad u \in C^\infty(\Omega_\delta),$$

where  $|\alpha| \leq \nu - 1$ ; the constant  $C'$  may depend on  $\delta$ .

If  $\alpha$  is an arbitrary multi-index and  $k\nu \leq |\alpha| < (k + 1)\nu$ , we choose another multi-index  $\beta$  so that  $|\alpha - \beta| \leq \nu - 1$  and  $|\beta| = k\nu$ . Then condition (11) yields

$$\|D^\alpha u\|_{\Omega_{2\delta}} \leq C' \left\{ \max_{|\beta|=(\nu+1)k} \|D^\beta u\|_{\Omega_\delta} + \max_{|\beta|=k\nu} \|D^\beta u\|_{\Omega_\delta} \right\}, \quad u \in C^\infty(\Omega_\delta).$$

Combining this estimate with (10), we obtain

$$\begin{aligned}
 (12) \quad \| D^\alpha u \|_{\Omega_{2\delta}} &\leq C' C^{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} \left( \frac{k+1}{\delta} \right)^{\lambda i} \| P^{(k+1-i)\mu}(D)u \|_{\Omega} \\
 &+ C' C^k \sum_{i=0}^k \binom{k}{i} \left( \frac{k}{\delta} \right)^{\lambda i} \| P^{(k-i)\mu'}(D)u \|_{\Omega} .
 \end{aligned}$$

Suppose now that  $u \in I_P^\rho(\Omega')$ , where  $\Omega'$  is an arbitrary open set in  $R^n$ . If  $K$  is a compact subset of  $\Omega'$ , there exists a bounded open set  $\Omega$  and a  $\delta > 0$  such that  $K \subset \Omega_{2\delta} \subset \Omega \subset \subset \Omega'$ . It follows that, for some constant  $B$ ,

$$\| P^i(D)u \|_{\Omega} \leq B^{i+1} i^{m\rho} , \quad i = 0, 1, \dots .$$

Note that in the definition of  $I_P^\rho(\Omega)$  the terms  $(j!)^{m\rho}$  can be replaced by  $j^{j m\rho}$ , by the Stirling formula. This leads to

$$\begin{aligned}
 (13) \quad k^{i\lambda} \| P^{(k-i)} P(D)u \|_{\Omega} &\leq [(B+1)\mu^{m\rho}]^{k+1} k^{km\mu\rho+i(\lambda-m\rho)} \\
 &\leq [(B+1)\mu^{m\rho}]^{k+1} k^{km\mu\rho'}
 \end{aligned}$$

$i = 0, 1, \dots, k$ , where  $\rho' = \max\{d, \rho\}$ . Applying the estimates (13) to the right-hand side of (12) we can find two constants  $B_1$  and  $B_2$  such that

$$\begin{aligned}
 \| D^\alpha u \|_{\Omega_{2\delta}} &\leq B_1^{|\alpha|+1} k^{km\mu\rho'} \leq B_2^{|\alpha|+1} |\alpha|^{|\alpha| m\rho'/\nu} \\
 &= B_2^{|\alpha|+1} |\alpha|^{|\alpha| d\rho'} .
 \end{aligned}$$

Thus  $u \in I^{d\rho'}(\Omega')$  and the proof is complete.

We are now in a position to prove the general theorem. We denote by  $\| \|$  the  $L^2$ -norm over  $R^n$ .

**THEOREM 2.** *Let  $\Omega$  be an open subset of  $R^n$  and let  $T$  be a mapping of  $C^\infty(\Omega)$  into  $C^\infty(\Omega)$  whose powers  $T^j$ ,  $j = 1, 2, \dots$ , have the following properties:*

( $t_1$ ) *There exists a  $d$ -hypoelliptic polynomial  $P$  such that*

$$\| P^j(D)u \| \leq A^j \| T^j u \| , \quad u \in C^\infty(\Omega) , \quad j = 1, 2, \dots ,$$

where  $A$  is a constant independent of  $j$ .

( $t_2$ ) *For some  $\rho > 1$ ,*

$$\| T^j v \| \leq B^{j+1} (j!)^{m\rho} , \quad v \in I^\rho(\Omega) \cap \mathcal{E}'(\Omega) , \quad j = 1, 2, \dots ,$$

where  $m$  is the degree of  $P$  and  $B$  is a constant independent of  $j$ .

Then every solution  $u \in C^\infty(\Omega)$  of the equation

$$(14) \quad Tu = v$$

is in  $I^{d\rho'}(\Omega)$ , when  $v \in I^\rho(\Omega) \cap \mathcal{E}'(\Omega)$ ; here, as before,  $\rho' = \max\{d, \rho\}$ .

*Proof.* The theorem follows immediately from Theorem 1. In fact, let  $u \in C^\infty_0(\Omega)$  be a solution of equation (14) with  $v \in \Gamma^r(\Omega) \cap \mathcal{E}'(\Omega)$ . Making use of conditions  $(t_1)$  and  $(t_2)$  we then have

$$\begin{aligned} \|P^j(D)u\| &\leq A^j \|T^j u\| \\ &= A^j \|T^{j-1}v\| \leq (AB)^j (j!)^{m\rho}, \quad j = 1, 2, \dots \end{aligned}$$

This proves that  $u \in \Gamma^r_P(\Omega)$ . Since  $P$  is  $d$ -hypoelliptic, it follows that  $u \in \Gamma^{d\rho'}(\Omega)$ , by Theorem 1.

**2. Differential operators of constant strength.** The following lemma is essential for the proof of our next theorem.

**LEMMA.** *Let  $P_0$  be a hypoelliptic polynomial of degree  $m$  and set  $k = \tilde{P}_0$ . Then*

$$(15) \quad k_\nu(\xi) = \sup_{|\eta| \leq \nu} k(\xi + \eta), \quad \nu > 0,$$

is a function in  $\mathcal{K}$  with the properties

$$(16) \quad k(\xi) \leq k_\nu(\xi) \leq k(\xi)(1 + C\nu)^m, \quad \xi \in R^n,$$

$$(17) \quad (k^j)_\nu = (k_\nu)^j, \quad j = 1, 2, \dots,$$

$$(18) \quad M_{k_\nu}(\xi) = \sup_{\xi'} \frac{k_\nu(\xi + \xi')}{k_\nu(\xi')} \leq (1 + C|\xi|)^m, \quad \xi \in R^n,$$

where  $C$  is a constant, and

$$(19) \quad M_{k_\nu} \rightarrow 1 \text{ uniformly on compact subsets of } R^n \text{ as } \nu \rightarrow \infty.$$

*Proof.* The properties (16)–(18) follow immediately from (15) and the inequality

$$k(\xi + \eta) \leq k(\xi)(1 + C|\eta|)^m,$$

which is valid for  $k = \tilde{P}_0$ . We prove the convergence (19).

If  $|\xi| \leq \mu$ , then

$$(20) \quad \frac{k_\nu(\xi + \xi')}{k_\nu(\xi')} \leq \frac{\sup_{|\eta| \leq \mu + \nu} k(\xi' + \eta)}{\sup_{|\eta| \leq \nu} k(\xi' + \eta)}.$$

We claim that the right-hand side of (20), which is  $\geq 1$ , converges to 1 as  $\nu \rightarrow \infty$ , uniformly for  $\xi' \in R^n$ .

Assume the contrary. Then there exist  $\varepsilon > 0$ , a sequence  $\{\nu_j\} \subset R$ ,  $\nu_j \rightarrow \infty$ , and sequences  $\{\xi'_j\}$ ,  $\{\eta_j\}$ ,  $\{\eta'_j\} \subset R^n$  such that

$$(21) \quad k(\xi'_j + \eta_j) = \sup_{|\eta| \leq \mu + \nu_j} k(\xi'_j + \eta), \quad k(\xi'_j + \eta'_j) = \sup_{|\eta| \leq \nu_j} k(\xi'_j + \eta),$$

and

$$(22) \quad \frac{k^2(\xi'_j + \eta_j)}{k^2(\xi'_j + \eta'_j)} \geq 1 + \varepsilon, \quad j = 1, 2, \dots.$$

Obviously we have

$$(23) \quad \nu_j < |\eta_j| \leq \mu + \nu_j$$

and

$$(24) \quad k(\xi'_j + \eta_j) \geq k(\xi'_j - \eta_j), \quad j = 1, 2, \dots.$$

But, for each  $j$ , one of the inequalities

$$(25) \quad |\xi'_j + \eta_j| > \nu_j \quad \text{or} \quad |\xi'_j - \eta_j| > \nu_j$$

must be satisfied. For, otherwise we could write

$$2|\eta_j| = |(\xi'_j + \eta_j) - (\xi'_j - \eta_j)| \leq |\xi'_j + \eta_j| + |\xi'_j - \eta_j| \leq 2\nu_j,$$

which contradicts the first inequality in (23). Since  $P_o$  is a hypoelliptic polynomial, it follows from (24) and (25) that  $|\xi'_j + \eta_j| \rightarrow \infty$ .

On the other hand, if  $\nu_j > \mu/2$ , then

$$(26) \quad \frac{k^2(\xi'_j + \eta_j)}{k^2\left(\xi'_j + \eta_j - \mu \frac{\eta_j}{|\eta_j|}\right)} \geq 1 + \varepsilon,$$

in view of (21), (22) and (23). Furthermore, since  $k^2$  is a polynomial of degree  $2m$ ,

$$(27) \quad k^2\left(\xi'_j + \eta_j - \mu \frac{\eta_j}{|\eta_j|}\right) \geq k^2(\xi'_j + \eta_j) - \sum_{\alpha \neq 0} |D^\alpha k^2(\xi'_j + \eta_j)| \frac{(1 + \mu)^\alpha}{\alpha!},$$

by Taylor's formula. But we assumed that the polynomial  $P_o$  is hypoelliptic and therefore

$$\frac{D^\alpha P_o(\xi)}{P_o(\xi)} \longrightarrow 0, \quad \text{if } \alpha \neq 0 \text{ and } |\xi| \longrightarrow \infty$$

(see [1], p. 99, Theorem 4.1.3). Hence it follows that

$$\frac{D^\alpha k^2(\xi)}{k^2(\xi)} \longrightarrow 0, \quad \text{if } \alpha \neq 0 \text{ and } |\xi| \longrightarrow \infty.$$

If now  $|\xi'_j + \eta_j|$  is sufficiently large, we obtain from (27) the estimates

$$k^2\left(\xi'_j + \eta_j - \mu \frac{\eta_j}{|\eta_j|}\right) \geq k^2(\xi'_j + \eta_j) \left(1 - \frac{\varepsilon}{2}\right),$$

which contradict the inequalities (26), when  $\varepsilon < 1$ .

The contradiction proves the desired convergence of the right-hand side of (20). The property (19) is an immediate consequence of that fact.

We remark that the weight function  $k_\delta$  defined in [1] (see p. 35, Theorem 2.1.2) cannot be used for our purpose.

**THEOREM 3.** *Let  $P(x, D)$  be a differential operator of constant strength in a neighborhood of  $x^\circ$  and let  $\Omega$  be a sufficiently small open neighborhood of  $x^\circ$  for which the local existence theorem is valid. If the coefficients of  $P(x, D)$  are in  $\Gamma^\rho(\Omega)$ ,  $\rho > 1$ , and if the operator  $P_\circ(D) = P(x^\circ, D)$  is  $d$ -hypoelliptic, then every solution  $u \in \mathcal{E}'(\Omega)$  of equation (4) is in  $\Gamma^{d\rho'}(\Omega)$ , whenever  $\nu \in \Gamma^\rho(\Omega) \cap \mathcal{E}'(\Omega)$ .*

*Proof.* Since  $P(x, D)$  is of constant strength in  $\Omega$  and  $P_\circ(D)$  is hypoelliptic, every solution  $u \in \mathcal{E}'(\Omega)$  of equation (4) is in  $C_\circ^\infty(\Omega)$ , whenever  $v \in C_\circ^\infty(\Omega)$  (see [1], p. 176, Theorem 7.4.1). For the proof of Theorem 3 it now suffices to show that  $P(x, D)$  satisfies the conditions imposed on  $T$  in Theorem 2.

We first observe that, by the lemma, we can replace in the proof of the local existence theorem the function  $k_\delta$  by our function  $k_\nu$ . Next we recall that condition (3) is derived from the estimate

$$\|E^j f\|_{p, \tilde{P}_\circ, k_\delta} \leq C_\circ \|f\|_{p, k_\delta} \quad f \in \mathcal{E}'(\mathbb{R}^n) \cap \mathcal{B}_{p, k},$$

where  $C_\circ$  is a constant independent of  $f, k$  and  $\delta$  (see [1], p. 176). Hence, if  $p = 2$ ,  $k = \tilde{P}_\circ$  and  $k_\nu$  is as in our lemma, we obtain

$$(28) \quad \|E^j u\|_{2, k(k^{j-1})_\nu} \leq A \|E^{j-1} u\|_{2, (k^{j-1})_\nu}, \quad u \in C_\circ^\infty(\Omega),$$

where  $A$  is a constant independent of  $j$  and  $u$ . But, in view of (16) and (17), there is a constant  $B$  independent of  $j$  and  $u$  such that

$$(29) \quad \|E^{j-1} u\|_{2, (k^{j-1})_\nu} \leq B \|E^{j-1} u\|_{2, k(k^{j-2})_\nu}, \quad u \in C_\circ^\infty(\Omega).$$

Condition (16) and repeated application of (28) and (29) gives

$$(30) \quad \|E^j u\|_{2, k^j} \leq \|E^j u\|_{2, k(k^{j-1})_\nu} \leq (AB)^j \|u\|, \quad u \in C_\circ^\infty(\Omega).$$

Substituting now  $P(x, D)u$  in place of  $u$  in (30) and observing that

$$\|P_\circ^j(D)f\| \leq \|f\|_{2, k^j}, \quad f \in C_\circ^\infty(\Omega),$$

we obtain

$$\|P_\circ^j(D)E^j P^j(x, D)u\| \leq (AB)^j \|P^j(x, D)u\|.$$

Hence we conclude that

$$(31) \quad \|P_\circ^j(D)u\| \leq (AB)^j \|P^j(x, D)u\|, \quad u \in C_\circ^\infty(\Omega),$$



because of (2), where  $P_0$  is a  $d$ -hypoelliptic polynomial.

On the other hand, if  $v \in \Gamma^\rho(\Omega) \cap \mathcal{E}'(\Omega)$ , we have

$$(32) \quad \|P^j(x, D)v\| \leq C^{j+1}(j!)^{m_\rho}, \quad j = 1, 2, \dots,$$

since the coefficients of  $P(x, D)$  are in  $\Gamma^\rho(\Omega)$ .

By Theorem 2, conditions (31) and (32) imply that every solution  $u \in C_\infty^\rho(\Omega)$  of equation (4) is in  $\Gamma^{d\rho'}(\Omega)$ , when  $v \in \Gamma^\rho(\Omega) \cap \mathcal{E}'(\Omega)$ . The theorem is thus established.

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