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THE CASE OF EQUALITY IN THE MATRIX-VALUED TRIANGLE INEQUALITY

ROBERT CHARLES THOMPSON

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This paper presents an analysis of the case of equality in the matrix-valued triangle inequality. There is complete analogy with the case of equality in the usual scalar triangle inequality.

In order to describe our assertion more precisely, let A and B be n -square complex matrices, and by $|A|$ denote the positive semidefinite Hermitian matrix

$$|A| = (AA^*)^{1/2},$$

where A^* is the adjoint of A . It has been speculated several times in the literature that this inequality should "naturally" hold:

$$|A + B| \leq |A| + |B|,$$

where the inequality sign signifies that the right hand side minus the left hand side is positive semidefinite. This inequality is false, however, as easy 2×2 examples show. Nevertheless, there is a valid matrix valued triangle inequality. It was discovered in [1], and takes the form

$$(1) \quad |A + B| \leq U|A|U^* + V|B|V^*$$

for appropriately chosen unitary matrices U and V (dependent upon A and B). However, no analysis of a "case of equality" for (1) was given in [1], and the purpose of this note is to supply such an analysis. Specifically, we have:

THEOREM 1. *The inequality sign in (1) must be equality if A and B have polar decompositions with a common unitary factor.*

THEOREM 2. *Suppose A and B are such that inequality (1) can hold only with the equality sign. Then A and B have polar factorizations with a common unitary factor.*

Proof of Theorem 1. We have $A = WH$ and $B = WK$, where W is unitary and H, K are positive semidefinite Hermitian. From (1) we easily deduce that

$$H + K \leq U_1 H U_1^* + V_1 K V_1^*,$$

where U_1, V_1 are unitary. Thus the matrix $U_1 H U_1^* + V_1 K V_1^* - (H + K)$ is positive semidefinite; but its trace is zero, so it can only be zero.

Proof of Theorem 2. We have to refer to the proof of the matrix triangle inequality in [1]. Let $C = A + B$. After multiplying C, A , and B by a unitary factor to make C positive semidefinite, and renaming the resulting matrices as C, A, B , again, the proof considers the expression

$$C = \frac{1}{2}(A + A^*) + \frac{1}{2}(B + B^*),$$

then uses $1/2(A + A^*) \leq U|A|U^*$ for an appropriate unitary U , and a similar fact for B . The hypothesis in the theorem implies that we must have $1/2(A + A^*) = U|A|U^*$ (so that $1/2(A + A^*)$ is necessarily positive semidefinite). Squaring and taking traces, we get

$$\operatorname{tr}\left(\frac{A + A^*}{2}\right)^2 = \operatorname{tr} AA^* = \frac{\operatorname{tr} AA^* + \operatorname{tr} A^* A}{2}.$$

Hence

$$0 = \operatorname{tr}(A - A^*)(A^* - A),$$

so that $\|A - A^*\|^2 = 0$. Therefore A is Hermitian. Since $1/2(A + A^*)$ is semidefinite, A is semidefinite Hermitian. Similarly, so is B . That is to say: after multiplying the original A, B, C by a unitary matrix to make C semidefinite, A and B then also become semidefinite. This completes the proof.

REFERENCE

1. R. C. Thompson, *Convex and concave functions of singular values of matrix sums*, Pacific J. Math., **16** (1976), 285-290.

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UNIVERSITY OF CALIFORNIA,
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Werner Bäni, <i>Subspaces of positive definite inner product spaces of countable dimension</i>	1
Marilyn Breen, <i>The dimension of the kernel of a planar set</i>	15
Kenneth Alfred Byrd, <i>Right self-injective rings whose essential right ideals are two-sided</i>	23
Patrick Cousot and Radhia Cousot, <i>Constructive versions of Tarski's fixed point theorems</i>	43
Ralph S. Freese, William A. Lampe and Walter Fuller Taylor, <i>Congruence lattices of algebras of fixed similarity type. I</i>	59
Cameron Gordon and Richard A. Litherland, <i>On a theorem of Murasugi</i>	69
Mauricio A. Gutiérrez, <i>Concordance and homotopy. I. Fundamental group</i>	75
Richard I. Hartley, <i>Metabelian representations of knot groups</i>	93
Ted Hurley, <i>Intersections of terms of polycentral series of free groups and free Lie algebras</i>	105
Roy Andrew Johnson, <i>Some relationships between measures</i>	117
Oldřich Kowalski, <i>On unitary automorphisms of solvable Lie algebras</i>	133
Kee Yuen Lam, <i>K O-equivalences and existence of nonsingular bilinear maps</i>	145
Ernest Paul Lane, <i>PM-normality and the insertion of a continuous function</i>	155
Robert A. Messer and Alden H. Wright, <i>Embedding open 3-manifolds in compact 3-manifolds</i>	163
Gerald Ira Myerson, <i>A combinatorial problem in finite fields. I</i>	179
James Nelson, Jr. and Mohan S. Putcha, <i>Word equations in a band of paths</i>	189
Baburao Govindrao Pachpatte and S. M. Singare, <i>Discrete generalized Gronwall inequalities in three independent variables</i>	197
William Lindall Paschke and Norberto Salinas, <i>C*-algebras associated with free products of groups</i>	211
Bruce Reznick, <i>Banach spaces with polynomial norms</i>	223
David Rusin, <i>What is the probability that two elements of a finite group commute?</i>	237
M. Shafii-Mousavi and Zbigniew Zielezny, <i>On hypoelliptic differential operators of constant strength</i>	249
Joseph Gail Stampfli, <i>On selfadjoint derivation ranges</i>	257
Robert Charles Thompson, <i>The case of equality in the matrix-valued triangle inequality</i>	279
Marie Angela Vitulli, <i>The obstruction of the formal moduli space in the negatively graded case</i>	281