A REMARK ON GENERALIZED HAAR SYSTEMS IN $L_p$, $1 < p < \infty$

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We show that any chain from a generalized Haar system in $L_p$ is equivalent to the unit vector basis in $l_p$. The constant of the equivalence depends only on $p$.

We answer a question raised in [1]. Specifically, we prove

**Theorem.** Let $1 < p < \infty$. There exists a constant $K$, depending only on $p$, such that whenever $(h_n)$ is a chain from a generalized Haar system in $L_p$, $(h_n)$ is $K$-equivalent to the unit vector basis in $l_p$.

Our notation and terminology is standard. If $A$ is a subset of a Banach space, $[A]$ denotes the closed linear span of $A$. The unit vector basis in $l_p$ is denoted by $(e_n)$, and $\mu$ denotes Lebesgue measure on $(0, 1)$.

A generalized Haar system [1] in $L_p$ is a sequence $(h_n)$ defined as follows. Let $\{A_n;i: n = 0, 1, \cdots; 0 \leq i < 2^n\}$ satisfy $A_{n,0} = (0, 1)$; $A_{n+1,i} \cup A_{n+1,2i+1} = A_{n,i}$; and $A_{n+1,2i} \cap A_{n+1,2i+1} = \emptyset$. Let

$$H_{n,i} = \frac{1}{\mu(A_{n+1,2i})} \chi_{A_{n+1,2i}} - \frac{1}{\mu(A_{n+1,2i+1})} \chi_{A_{n+1,2i+1}},$$

and define $h_0 = 1$, $h_{n+1,i} = H_{n,i} \|H_{n,i}\|^{-1}$.

A chain from $(h_n)$ is a subsequence $(h'_n)$ such that $\text{supp } h'_{n+1} \subset \text{supp } h'_n$.

In [1] it is proved that a generalized Haar system is a monotone, unconditional basic sequence in $L_p$, with unconditional constant $\lambda$ depending only on $p$.

The proof of the theorem is based on the following lemma (see [2] and [3]).

**Lemma.** Let $1 \leq \lambda < \infty$, $\delta > 0$, $1 \leq p \leq 2$, and $(x_n)$ be a normalized unconditional basic sequence in $L_p$ with unconditional constant $\lambda$. Then,

(a) $\| \sum a_n x_n \| \leq \lambda (\sum |a_n|^p)^{1/p}$, and

(b) If there exist disjoint sets $(B_n)$ with

$$\|x_n | B_n \| \geq \delta, \text{ then } \delta \lambda (\sum |a_j|^p)^{1/p} \leq \| \sum a_n x_n \|,$$

for any scalar sequence $(a_n)$. 317
Proof of theorem. We shall denote the chain by \((h_n)\), and let \(A_{n,1} = \text{supp } h_n^+, A_{n,2} = \text{supp } h_n^-\). We will assume \(\text{supp } h_{n+1} \subseteq A_{n,1}\).

Any chain in \(L_2\) is an orthonormal system, so a chain in \(L_2\) is isometrically equivalent to the unit vectors in \(l_2\).

We consider now the case \(1 < p < 2\). Let \(N_1 = \{n: ||h_n| A_{n,2}| \geq 2^{-1/p}\} , N_2 = \{n: ||h_n| A_{n,1}| > 2^{-1/p}\} \), and consider first the chain \((h_n)_{n \in N_1}\). Setting \(B_n = A_{n,2} \setminus A_{n,1} \) and \(\delta = 2^{-1/p}\), it follows from the lemma that for all sequences \((a_j)\),

\[
2^{-1/p} \left( \sum_{j \in N_1} |a_j|^p \right)^{1/p} \leq \left\| \sum_{j \not\in N_1} a_j h_j \right\|.
\]

As for the chain \((h_n)_{n \in N_2}\), note that for each \(n \in N_2\), we have \(\mu(A_{n,2}) > \mu(A_{n,1})\). Thus, if \(j\) is the successor (in \(N_2\)) of \(n\), \(\mu(A_{n,1} - A_{j,1}) > (1/2)\mu(A_{n,1})\). Setting \(B_n = A_{n,1} - A_{j,1}\) we have \(||h_n| B_n|| > 2^{-2/p}\), so that

\[
2^{-2/p} \left( \sum_{j \in N_2} |a_j|^p \right)^{1/p} \leq \left\| \sum_{j \not\in N_2} a_j h_j \right\|.
\]

Using (1), (2), part (a) of the lemma, and the unconditionality of \((h_n)\) we have

\[
\frac{2^{-3/p}}{\lambda^2} \left( \sum |a_j|^p \right)^{1/p} \leq \frac{2^{-3/p}}{\lambda^2} \left( \sum |a_j|^p \right)^{1/p} + \frac{2^{-1/p}}{\lambda^2} \left( \sum |a_j|^p \right)^{1/p} \leq \frac{1}{\lambda} \left\| \sum_{j \not\in N_2} a_j h_j \right\| + \frac{1}{\lambda} \left\| \sum_{j \in N_1} a_j h_j \right\| \leq 2 \left\| \sum a_j h_j \right\| \leq 2\lambda \left( \sum |a_j|^p \right)^{1/p},
\]

as desired.

Now suppose \((h_n)\) is a chain from \(L_p, 2 < p < \infty\). Then \([h_n]\) is isometric to \(l_p\), as we may regard \(h_n = e_n + b_n e_m - \sum_{j=n+1}^{\infty} b_j e_j\). The biorthogonal sequence \((h_n^*)\) is a chain from a generalized Haar system in \(L_q\), with \(1/q + 1/p = 1\). Since \(1 < q < 2\), \((h_n^*)\) is equivalent to \(e_n^*\). Letting \(T: l_q \to l_q\) be the isomorphism realizing this equivalence, we have that \(T^* e_n = h_n\) and \(T^*\) is an isomorphism. Hence \((h_n)\) is equivalent to \((e_n)\).

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Received January 30, 1978.

GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GA 30332
Pacific Journal of Mathematics
Vol. 82, No. 2 February, 1979

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