

Pacific Journal of Mathematics

**A REMARK ON GENERALIZED HAAR SYSTEMS IN L_p ,
 $1 < p < \infty$**

ALFRED DAVID ANDREW

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We show that any chain from a generalized Haar system in L_p is equivalent to the unit vector basis in l_p . The constant of the equivalence depends only on p .

We answer a question raised in [1]. Specifically, we prove

THEOREM. *Let $1 < p < \infty$. There exists a constant K , depending only on p , such that whenever (h_n) is a chain from a generalized Haar system in L_p , (h_n) is K -equivalent to the unit vector basis in l_p .*

Our notation and terminology is standard. If A is a subset of a Banach space, $[A]$ denotes the closed linear span of A . The unit vector basis in l_p is denoted by (e_n) , and μ denotes Lebesgue measure on $(0, 1)$.

A generalized Haar system [1] in L_p is a sequence (h_n) defined as follows. Let $\{A_{n,i}: n = 0, 1, \dots; 0 \leq i < 2^n\}$ satisfy $A_{0,0} = (0, 1)$; $A_{n+1,2i} \cup A_{n+1,2i+1} = A_{n,i}$; and $A_{n+1,2i} \cap A_{n+1,2i+1} = \emptyset$. Let

$$H_{n,i} = \frac{1}{\mu(A_{n+1,2i})} \chi_{A_{n+1,2i}} - \frac{1}{\mu(A_{n+1,2i+1})} \chi_{A_{n+1,2i+1}},$$

and define $h_0 \equiv 1$, $h_{2^n+i} = H_{n,i}/\|H_{n,i}\|$.

A chain from (h_n) is a subsequence (h'_n) such that $\text{supp } h'_{n+1} \subset \text{supp } h'_n$.

In [1] it is proved that a generalized Haar system is a monotone, unconditional basic sequence in L_p , with unconditional constant λ depending only on p .

The proof of the theorem is based on the following lemma (see [2] and [3]).

LEMMA. *Let $1 \leq \lambda < \infty$, $\delta > 0$, $1 \leq p \leq 2$, and (x_n) be a normalized unconditional basic sequence in L_p with unconditional constant $\leq \lambda$. Then,*

- (a) $\|\sum a_n x_n\| \leq \lambda(\sum |a_n|^p)^{1/p}$, and
- (b) *If there exist disjoint sets (B_n) with*

$$\|x_n | B_n\| \geq \delta, \text{ then } \frac{\delta}{\lambda} (\sum |a_j|^p)^{1/p} \leq \|\sum a_n x_n\|,$$

for any scalar sequence (a_n) .

Proof of theorem. We shall denote the chain by (h_n) , and let $A_{n,1} = \text{supp } h_n^+$, $A_{n,2} = \text{supp } h_n^-$. We will assume $\text{supp } h_{n+1} \subset A_{n,1}$.

Any chain in L_2 is an orthonormal system, so a chain in L_2 is isometrically equivalent to the unit vectors in l_2 .

We consider now the case $1 < p < 2$. Let $N_1 = \{n: \|h_n | A_{n,2}\| \geq 2^{-1/p}\}$, $N_2 = \{n: \|h_n | A_{n,1}\| > 2^{-1/p}\}$, and consider first the chain $(h_n)_{n \in N_1}$. Setting $B_n = A_{n,2}$ and $\delta = 2^{-1/p}$, it follows from the lemma that for all sequences (a_j) ,

$$(1) \quad \frac{2^{-1/p}}{\lambda} \left(\sum_{j \in N_1} |a_j|^p \right)^{1/p} \leq \left\| \sum_{j \in N_1} a_j h_j \right\|.$$

As for the chain $(h_n)_{n \in N_2}$, note that for each $n \in N_2$, we have $\mu(A_{n,2}) > \mu(A_{n,1})$. Thus, if j is the successor (in N_2) of n , $\mu(A_{n,1} - A_{j,1}) > (1/2)\mu(A_{n,1})$. Setting $B_n = A_{n,1} - A_{j,1}$ we have $\|h_n | B_n\| > 2^{-2/p}$, so that

$$(2) \quad \frac{2^{-2/p}}{\lambda} \left(\sum_{j \in N_2} |a_j|^p \right)^{1/p} \leq \left\| \sum_{j \in N_2} a_j h_j \right\|.$$

Using (1), (2), part (a) of the lemma, and the unconditionality of (h_n) we have

$$\begin{aligned} \frac{2^{-2/p}}{\lambda^2} \left(\sum |a_j|^p \right)^{1/p} &\leq \frac{2^{-2/p}}{\lambda^2} \left(\sum_{j \in N_2} |a_j|^p \right)^{1/p} + \frac{2^{-1/p}}{\lambda^2} \left(\sum_{j \in N_1} |a_j|^p \right)^{1/p} \\ &\leq \frac{1}{\lambda} \left\| \sum_{j \in N_2} a_j h_j \right\| + \frac{1}{\lambda} \left\| \sum_{j \in N_1} a_j h_j \right\| \\ &\leq 2 \left\| \sum a_j h_j \right\| \leq 2\lambda \left(\sum |a_j|^p \right)^{1/p}, \end{aligned}$$

as desired.

Now suppose (h_n) is a chain from L_p , $2 < p < \infty$. Then $[(h_n)]$ is isometric to l_p , as we may regard $h_n = c_n e_1 + b_n e_n - \sum_{j=n+1}^{\infty} b_j e_j$. The biorthogonal sequence (h_n^*) is a chain from a generalized Haar system in L_q , with $1/q + 1/p = 1$. Since $1 < q < 2$, (h_n^*) is equivalent to e_n^* . Letting $T: l_q \rightarrow l_q$ be the isomorphism realizing this equivalence, we have that $T^* e_n = h_n$ and T^* is an isomorphism. Hence (h_n) is equivalent to (e_n) .

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