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GEORGE GOLIGHTLY

This paper extends a decomposition process of [Richard Arens, Operational calculus of linear relations, Pacific J. Math., 11 (1961), 9-23] for closed linear relations, T, on a Hilbert space to the setting in which T is a linear function from a dense linear subspace of a separable normed linear complete space S_1 to an innear product space S_2. This decomposition is used in showing that such a function contracted to a suitable dense linear subspace of its initial set is the contraction of a closed linear function from a dense linear subspace of S_1 to S_2. In particular, in case the initial set of T is S_1, it is shown that the contraction of T to a suitable dense linear subspace of S_1 is the contraction of a continuous linear function from S_1 to S_2.

It will be supposed in the following that (S_1, N_1) is a normed complex linear complete space, that (S_2, Q_2) is a complex complete inner product space with N_2 the norm for S_2 corresponding to Q_2, and that T is a linear function from the N_1-dense linear subspace S of S_1 to S_2.

DEFINITION. The set of all points x of S_1 such that there is a sequence z having N_1-limit x such that $T[z]$ has N_2-limit 0 will be denoted by $Z'(T)$ and the set of all points x of S_2 such that there is a sequence z having N_1-limit 0 such that $T[z]$ has N_2-limit x will be denoted by $Z''(T)$. Moreover, $P''(T)$ will denote the Q_2-orthogonal projection of S_2 onto the closed linear space $Z''(T)$. The statement that T is graph-dense means that T is dense in the subspace $S \times T(S)$ of the normed linear space $S_1 \times S_2$.

Note 1. One notes that $Z''(T)$ is $\{0\}$ only in case T is the contraction to S of a closed linear function from a dense linear subspace of S_1 to S_2. Moreover, in case $T(S)$ lies in $Z''(T)$, T is graph-dense. Indeed, suppose $T(S)$ lies in $Z''(T)$ and (x, y) is in $S \times T(S)$. Since $y - Tx$ is in $Z''(T)$, there is a sequence z having N_1-limit 0 such that $T[z]$ has N_2-limit $y - Tx$. Hence, x is the N_1-limit of $x + z$ and y the N_2-limit of $T[x + z]$.

Note 2. We note the following properties of $Z'(T)$ and $Z''(T)$.
(i) $T(S \cap Z'(T))$ lies in $Z''(T)$.
(ii) $T^{-1}(Z''(T))$ lies in $Z'(T)$.
(iii) In case B is a continuous linear function from S_1 to S_2,

$Z''(B + T)$ is $Z''(T)$.

(iv) In case B is a continuous linear function from S_2 to S_2, $Z'(T)$ lies in $Z'(BT)$ and the closure of $B(Z''(T))$ lies in $Z''(BT)$.

(v) In case B is reversibly continuous from S_1 onto S_1, $Z''(TB)$ is $Z''(T)$ and $Z'(TB)$ is $B^{-1}(Z'(T))$.

We shall establish (iv). Suppose x is in $Z'(T)$ and z is a sequence having N_1-limit x such that $T[z]$ has N_2-limit 0. Then $BT[z]$ has N_2-limit 0. Hence, x is in $Z'(BT)$.

Suppose y is in $Z''(T)$ and z is a sequence having N_1-limit 0 such that $T[z]$ has N_2-limit y. Then $BT[z]$ has N_2-limit By; hence, By is in $Z''(BT)$. Since $Z''(BT)$ is closed and includes $B(Z''(T))$, the closure of $B(Z''(T))$ lies in $Z''(BT)$.

THEOREM 1. *The linear function* $(1 - P''(T))T$ *is the contraction to* S *of a closed linear function from a dense linear subspace of* S_1 *to* S_2 *and* $P''(T)T$ *is graph-dense.*

Proof. Suppose x is in $P''(T)T(S)$. There is a sequence z having N_1-limit 0 such that $T[z]$ has N_2-limit x. Hence, since $P''(T)x$ is x, $P''(T)T[z]$ has N_2-limit x. Then, since $P''(T)T(S)$ lies in $Z''(P''(T)T)$, $P''(T)T$ is graph-dense.

Suppose x is in $Z''((1 - P''(T))T)$. If j is a positive integer, there is a point z_j of S such that $N_1(z_j) < 1/j$ and $N_2(x - (1 - P''(T))Tz_j) < 1/j$. Since $P''(T)Tz_j$ is in $Z''(T)$, there is for each positive integer j a point w_j of S such that $N_1(w_j) < 1/j$ and $N_2(P''(T)Tz_j - Tw_j) < 1/j$. Hence, $N_2(x - T(z_j - w_j)) \leqq N_2(x - (1 - P''(T))Tz_j) + N_2(P''(T)Tz_j - Tw_j) < 2/j$ and $N_1(z_j - w_j) < 2/j$. Thus, x is in $Z''(T)$. Since x is $\lim (1 - P''(T))T[z]$, $P''(T)x$ is 0. Hence, x is 0. Thus, by Note 1, $(1 - P''(T))T$ is the contraction to S of a closed linear function from a dense linear subspace of S_1 to S_2.

DEFINITION. In the setting that N_1 arises from an inner product Q_1 for S_1 we note that the closure in $S_1 \times S_2$ of $(1 - P''(T))T$ is the common part of the closure of T in $S_1 \times S_2$ and the orthogonal complement in $S_1 \times S_2$ of $\{0\} \times Z''(T)$, which is referred to in [2] as "the operator part" of the closure of T. We refer to $(1-P''(T))T$ as the closed part, and to $P''(T)T$ as the graph-dense part, of T. In particular, in case S is S_1, $(1 - P''(T))T$ is referred to as the continuous part of T.

Note 3. In analogy with Lemma 5.2 of [1], we have that, if T^* is the set to which (r, s) belongs only in case r is in S_2 and $Q_2(r, T\cdot)$ is continuous from S to the plane and s is $Q_2(r, T\cdot)$, i.e., T^* is the adjoint of T, and C is the closed part of T, then T^* is the contraction of C^* to the orthogonal complement of $Z''(T)$ in S_2. Indeed, if $P''(T)r$ is not 0, there is a sequence z having N_1-limit 0 such that $T[z]$ has N_2-limit $P''(T)r$; hence $\lim Q_2(r, T[z])$ is $N_2(P''(T)r)^2$. Hence, r is not in the initial set of T^*. If r is in the orthogonal complement in S_2 of $Z''(T)$, $Q_2(r, T\cdot) = Q_2(r, C\cdot)$; hence, (r, s) belongs to T^* only in case (r, s) belongs to C^*.

In particular, it should be noted that if the initial set of T is S_1, T^* is continuous. Suppose, now, that L is a continuous linear function from S_2 to S_2. Then in order that LT be continuous it is necessary and sufficient that the final set of L^* lie in the initial set of T^*. Suppose LT is continuous and y is in S_2. Then the function f on S such that $f(x)$ is $Q_2(Tx, L^*y)$ is continuous and L^*y is in the initial set of T^*. Conversely, suppose $L^*(S_2)$ lies in the initial set of T^*. Then the initial set of $(LT)^*$ is S_2. Hence, LT is continuous.

It may be seen that in case C is a closed linear function, D is a graph-dense linear function, T is $C + D$, and the final set of D is orthogonal in S_2 to the final set of C, then $Z''(D)$ includes $Z''(T)$. Indeed, if x is in $Z''(T)$ and z is a sequence having N_1-limit 0 such that $T[z]$ has N_2-limit x

$$(1 - P''(D))x = \lim ((1 - P''(D))T[z]) = \lim C[z] .$$

Hence, $(1 - P''(D))x$ is in $Z''(C)$. Since C is closed, $Z''(C)$ is $\{0\}$. Thus, x is in $Z''(D)$. In particular, in case S is S_1, $Z''(D)$ is $Z''(T)$, D is the graph-dense, and C the continuous, part of T.

In case S_2 is the plane and T is not continuous, then the kernel of T is dense. We note that in this setting $Z''(T)$ is the plane, so that T is graph-dense.

THEOREM 2. *Suppose S is S_1. Then $Z''(T)$ is finite dimensional only in case the contraction of T to a dense linear subspace W of S_1, having finite dimensional algebraic complement in S_1, is the contraction to W of a continuous linear function from S_1 to S_2.*

Proof. Suppose $Z''(T)$ is finite dimensional, C is the continuous, and D the graph-dense, part of T, and W is the kernel of D with closure $\bar{W}$ in S_1. Then T is C on S_1. Suppose $\bar{W}$ is not S_1. Suppose that U is an algebraic complement in S_1 of $\bar{W}$. Then U is finite dimensional. Suppose P_1 is the algebraic projection of S_1 onto $\boldsymbol{U}$ with respect to $\bar{W}$, so that P_1 is the identity on U and 0 on $\bar{W}$. We note that P_1 is continuous. Indeed, with the evident interpretation of $Z''(P_1)$, we have $Z''(P_1) = Z''(1 - P_1)$; hence, $Z''(P_1)$ lies in $U \cap \bar{W}$. Suppose that P_2 is the orthogonal projection of S_2 onto the orthogonal complement in S_2 of the finite dimensional subspace,

$D(\bar{W})$. Since $D(\bar{W}) \cap D(U)$ is $\{0\}$, $D(\bar{W})$ is not $D(S_1)$. Hence, P_2D is not 0. Since P_1 is continuous with finite dimensional final set, DP_1 is continuous. We have the identity

$$P_2D = P_2DP_1 + P_2D(1 - P_1) = P_2DP_1 .$$

Hence, P_2D is continuous. We note that $Z''(P_2D)$ is $\{0\}$, $P_2(Z''(D))$ lies in $Z''(P_2D)$, and that $D(S_1)$ lies in $Z''(D)$. Hence, $P_2D(S_1)$ is $\{0\}$. This is a contradiction. Thus, $\bar{W}$ is S_1.

Suppose that W is a dense linear subspace of S_1, W has finite dimensional algebraic complement in S_1, and the contraction of T to W is the contraction to W of a continuous linear function A. Then $(T - A)(S_1)$ is finite dimensional. Since $Z''(T - A)$ lies in the closure of $(T - A)(S_1)$ and $(T-A)(S_1)$ is finite dimensional $Z''(T-A)$ is finite dimensional. Noting that $Z''(T - A)$ is $Z''(T)$, we have that $Z''(T)$ is finite dimensional. Hence, the proof is complete.

Suppose (S_1, N_1) is (S_2, N_2) and S is S_1. In case the continuous part of T is self-adjoint with respect to Q_2, then $Z''(T)$ lies in $Z'(T)$. Indeed, $Z'(T)$ is the kernel of the continuous part C of T, and $Z''(T)$ lies in the orthogonal complement of $C(S_1)$.

THEOREM 3. *Suppose S is S_1, C is the continuous part of T, and D is the graph-dense part of T. Then for x in S_1 $N_2(Tx) \geqq N_2(Cx)$. Moreover, in case (S_1, N_1) is (S_2, N_2) and $Z''(T)$ lies in $Z'(T)$, for each positive integer p $N_2(T^px) \geqq N_2(C^px)$.*

Proof. If x is in S_1, $N_2(Tx)^2 = N_2(Cx)^2 + N_2(Dx)^2 \geqq N_2(Cx)^2$. Suppose (S_1, N_1) is (S_2, N_2) and $Z''(T)$ lies in $Z'(T)$. Then CD is 0. Hence, for each positive integer p, $(C + D)^p = \sum_{j=0}^{p} D^jC^{p-j}$. If x is in S_1,

$$N_2(T^px)^2 = Q_2(\textstyle\sum_{j=0}^{p} D^jC^{p-j}x, \sum_{j=0}^{p} D^jC^{p-j}x) = N_2(C^px)^2 + N_2(\sum_{j=1}^{p} D^jC^{p-j}x)^2 \geqq N_2(C^px)^2 .$$

We note in the setting of Theorem 3 that in case A is a continuous linear function from S_1 to S_2 which agrees with T on a dense linear subspace of S_1, then for each x in S_1 $N_2(Ax) \geqq N_2(Cx)$.

The following lemma to Theorem 4 is proved by Kato [4, Lemma 411, p. 278].

LEMMA. *Suppose (B, N) is a normed complex linear complete space with dual space, (B^*, N^*), and that $(w_p, w'_p)_{p=1}^{\infty}$ is a sequence in $B \times B^*$ such that if p is a positive integer $N(w_p) = 1$, $N^*(w'_p) = 1$, $w'_p(w_p) = 1$, and for $j < p$, w_p is in the kernel of w'_j. Then if $(a_p)_1^m$ is a complex-sequence and $1 \leqq k \leqq m$, $|a_k| \leqq 2^{k-1}N(\sum_1^m a_pw_p)$.*

THEOREM 4. *Suppose that T is graph-dense and (S_1, N_1) is separable. Then for each continuous linear function C from S_1 to S_2 such that $C(S_1)$ lies in the closure of $T(S_1)$ and positive number ε, there is a dense linear subspace S' of S_1 such that for x in S' $N_2(Tx - Cx) \leqq \varepsilon N_1(x)$.*

Proof. Suppose that (S_1, N_1) is separable and infinite dimensional and e is an S_1-sequence, the term-set of e is linearly independent, E is the linear span of the term-set of e, and E is dense in S_1. Then, with (S_1^*, N_1^*) denoting the dual space of (S_1, N_1), there is a sequence $(w_p, w'_p)_1^\infty$ in S_1 satisfying the hypotheses of the lemma such that the linear span of the term-set of $(w_p)_1^\infty$ is E. Indeed, if $(w_p, w'_p)_1^n$ is an n-term sequence in $S_1 \times S_1^*$ such that if $1 \leqq j < p \leqq n$, $N_1(w_p) = 1$, $N_1^*(w'_p) = 1$, $w'_p(w_p) = 1$, $w'_j(w_p) = 0$, and the linear span of the term-set of $(w_p)_1^n$ is the linear span of the term-set of $(e_p)_1^n$, then since the common part of the set of kernels of terms of $(w'_p)_1^n$ has n-dimensional algebraic complement in S_1, there is a point w_{n+1} of norm 1 in the linear span of the term-set of $(e_p)_1^{n+1}$ and in the common part of the set of kernels of terms of $(w'_p)_1^n$. Moreover, there is a point w'_{n+1} of norm 1 in S_1^* such that $w'_{n+1}(w_{n+1}) = 1$. Note that w_{n+1} is not in the linear span of the term-set of $(w_p)_1^n$. Thus, $(w_p, w'_p)_1^{n+1}$ is a sequence in $S_1 \times S_1^*$ such that if $1 \leqq j < p \leqq n + 1$, $N_1(w_p) = 1$, $N_1^*(w'_p) = 1$, $w'_j(w_p) = 0$, and the linear span of the term-set of $(w_p)_1^{n+1}$ is the linear span of the term-set of $(e_p)_1^{n+1}$.

Suppose that T is graph-dense and $0 < \varepsilon < .01$. If j is a positive integer, there is a point x_j of S_1 such that $N_1(x_j - w_j) < (\varepsilon/10)^j$ and $N_2(Tx_j) < (\varepsilon/10)^j$. Suppose that A is the linear function on E such that $A(\sum_1^m a_j w_j) = \sum_1^m a_j x_j$. Then

$$N_1\Big(A\Big(\sum_1^m a_j w_j\Big) - \sum_1^m a_j w_j\Big) \leqq \sum_1^m |a_j| N_1(x_j - w_j)$$
$$\leqq \sum_1^m 2^{j-1} N_1\Big(\sum_1^m a_j w_j\Big)(\varepsilon/10)^j \leqq \Big(N_1\Big(\sum_1^m a_j w_j\Big)\Big/2\Big)\Big(\sum_1^m (\varepsilon/5)^j\Big)$$
$$\leqq (\varepsilon/5) N_1\Big(\sum_1^m a_j w_j\Big).$$

Hence, A has only one continuous linear extension B to S_1 and the operator norm of $B - 1$ does not exceed .01. Hence B is reversibly continuous with final set S_1. Since the linear span, X, of the term-set of x is $B(E)$, X is dense in S_1. Note that

$$N_2\Big(TA\Big(\sum_1^m a_j w_j\Big)\Big) = N_2\Big(\sum_1^m a_j T(x_j)\Big) \leqq \sum_1^m |a_j| N_2(Tx_j)$$

$$\leqq \sum_1^m 2^{j-1} N_1\Big(\sum_1^m a_j w_j\Big)(\varepsilon/10)^j \leqq (1/2)\Big(\sum_1^m (\varepsilon/5)^j\Big) N_1\Big(\sum_1^m a_j w_j\Big)$$
$$\leqq (\varepsilon/5) N_1\Big(\sum_1^m a_j w_j\Big) .$$

Suppose x is in X.

$$N_2(TAA^{-1}x) \leqq (\varepsilon/5)N_1(A^{-1}x) \leqq (\varepsilon/5)(1/(1-\varepsilon/5))N_1(x) \leqq \varepsilon N_1(x) .$$

Suppose that C is a continuous linear function from S_1 to S_2 such that $C(S_1)$ lies in the closure of $T(S_1)$ and $\varepsilon > 0$. Then $T-C$ is graph-dense. Indeed, $Z''(T-C)$ is $Z''(T)$, the closure of $T(S_1)$; moreover, $(T-C)(S_1)$ lies in the closure of $T(S_1)$. There is a dense linear subspace X of S_1 such that for x in X $N_2((T-C)x) \leqq \varepsilon N_1(x)$.

THEOREM 5. *Suppose that (S_1, N_1) is separable. There is a dense linear subspace X of S_1 such that the contraction of T to X is the contraction to X of a closed linear function from a dense linear subspace of S_1 to S_2. In case S is S_1, then there is a dense linear subspace X of S_1 such that the contraction of T to X is the contraction to X of a continuous linear function from S_1 to S_2.*

Proof. Suppose D is the graph-dense part of T. Then there is a dense linear subspace X of S_1 such that for x in S_1 $N_2(Dx) \leqq N_1(x)$. Suppose that B is the continuous linear extension to S_1 of the contraction of D to X. Then T is $B + (T - D)$ on X, the contraction to X of a closed linear function. In case S is S_1, then $T - D$ is the continuous part of T and T is continuous on X.

Note 4. In case S is S_1 and X is a dense linear subspace of S_1 such that the contraction of T to X is the contraction of the continuous linear function B from S_1 to S_2, then the initial set of the common part of T and B is a dense linear subspace of S_1 lying properly in no linear subspace Y of S_1 such that the contraction of T to Y is continuous.

Note 5. Suppose that (S', Q') is a Hilbert space, M is a closed linear subspace of S', and W is a dense algebraic complement in S' of M. Then if ϕ is the algebraic projection of S' onto W with respect to M and P is the orthogonal projection of S' onto M, $1-P$ is the continuous part of ϕ. We note that W is a dense linear subspace of S' such that the contraction of ϕ to W is continuous.

Such a dense proper linear subspace W of S' having closed algebraic complement in S' is one for which there is an inner product Q_w such that (W, Q_w) is complete and the identity function from (S', Q') to (W, Q_w) is continuous. Indeed, if M is a closed algebraic complement of W in S' and P is the orthogonal projection of S' onto M, we may take Q_w to be $Q'((1-P)\cdot, \cdot)$. Conversely, if W is a dense proper linear subspace of S' for which there is such an inner product Q_w, then the set M of all limits in S' of sequences z having limit 0 with respect to the norm N_w for W corresponding to Q_w is a closed algebraic complement in S' of W and, with P the orthogonal projection of S' onto M, Q_w is equivalent on W to $Q'((1-P)\cdot, \cdot)$.

Note that the formula $Q(x, y) = Q'(x, y) + Q'(\phi x, \phi y)$ defines an inner product Q for S' such that (i) the identity function from (S', Q) to (S', Q') is continuous and (ii) ϕ is continuous with respect to the norm N for S' corresponding to Q.

Of course, in this example, each power of ϕ is continuous on $\phi(S')$. More generally, it may be seen that, if L is a linear function and Q is an inner product for S' such that (i) the identity function from (S', Q) to (S', Q') is continuous and (ii) L is continuous with respect to the norm N for S' corresponding to Q, then there is an N'-dense linear subspace X of S' such that the contraction of each power of L to X is continuous from (S', Q') to (S', Q'). Indeed, the condition that there be an inner product Q for S' for which (i) and (ii) hold is shown in [3, Theorem 1, p. 2] to be equivalent to the existence of a positive number b such that for x in S' $\sum_{p=0}^{\infty} N'(L^p x)^2/b^p$ converges. In this case, with $l^2(S')$ the space of all S'-sequences y for which $\sum_{p=0}^{\infty} N'(y_p)^2$ converges, $\tilde{Q}$ the inner product for $l^2(S')$ given by $\tilde{Q}(x, y) = \sum_{p=0}^{\infty} Q'(x_p, y_p)$, and $\tilde{L}$ the linear function from S' to $l^2(S')$ given by $\tilde{L}(x)_p = L^p(x)/(b)^{p/2}$, an application of Theorem 5 to $\tilde{L}$ yields a nonnegative number c and an N'-dense linear subspace X of S' such that for x in X $\sum_{p=0}^{\infty} N'(L^p x)^2/b^p = \tilde{Q}(\tilde{L}x, \tilde{L}x) \leqq c\tilde{N}'(x)^2$.

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