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ON EXTENSION OF ROTUND NORMS. II

K. JOHN AND VÁCLAV E. ZIZLER

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It is proved that if X is a Banach space, $Y \subset X$ with X/Y separable and $\|\cdot\|$ is an equivalent locally uniformly rotund norm on Y , then $\|\cdot\|$ can be extended to such a norm on X .

This generalizes [2] where it was shown that any locally uniformly rotund equivalent norm on a closed subspace of a separable Banach space X can be extended to such a norm on X .

By a subspace we mean a closed linear subspace, $\text{sp } L$ denotes the linear hull of L and $x \rightarrow \hat{x}$ stands for the quotient map $X \rightarrow X/Y$ if Y is a subspace of X .

Let us recall that a norm $\|\cdot\|$ on a Banach space X is locally uniformly rotund (LUR) if whenever $\lim 2(\|x\|^2 + \|x_j\|^2) - \|x + x_j\|^2 = 0$, $x, x_j \in X$, then $\lim \|x - x_j\| = 0$. $\|\cdot\|$ is rotund (R) if for any $x, y \in X$, $x \neq y$, $2(\|x\|^2 + \|y\|^2) - \|x + y\|^2 > 0$.

THEOREM 1. *Let X be a Banach space, $Y \subset X$ a subspace of X . Suppose X/Y is separable and Y admits an equivalent norm $\|\cdot\|$ which is LUR (R). Then $\|\cdot\|$ can be extended to an equivalent norm $\|\cdot\|$ on X which is LUR (R).*

Proof. Let us start with the case of LUR.

First extend the given LUR norm $\|\cdot\|$ on Y to an equivalent norm $\|\cdot\|$ on X : This can easily be done as follows: Take the closed unit ball B_Y^1 of Y with respect to $\|\cdot\|$ and the closed ball B of X such that $B \cap Y \subset B_Y^1$. Then, easily, the Minkowski functional of $\text{conv}(B \cup B_Y^1)$ is the desired norm on X (cf. e.g., [4], [2]).

Furthermore, let $\{\hat{a}_n\}_{n=1}^\infty \subset X/Y$, $\hat{a}_n \neq 0$ be a dense subset of X/Y . Let $S: X/Y \rightarrow X$ denote the Bartle-Graves continuous selection map ($S\hat{x} \in \hat{x}$) and $a_n = S\hat{a}_n$.

For $n \in N$ (N positive integers), choose $f_n \in X^*$, $f_n(a_n) = 1$, $\|f_n\| = \|\hat{a}_n\|^{-1}$, $f_n = 0$ on Y and denote by $P_n(x) = f_n(x)a_n$, $P'_n = I - P_n$ where I is the identity map on X .

Consider

$$\|x\|^2 = (1 - c)\|x\|^2 + \sum_{n=1}^{\infty} 2^{-n}(1 + \|P_n\|)^{-2} \cdot \|x - P_n x\|^2 + \|\hat{x}\|^2,$$

where $c = \sum_{n=1}^{\infty} (1 + \|P_n\|)^{-2} 2^{-n}$, $\|\hat{\cdot}\|$ is an equivalent LUR norm on X/Y ([3]).

Then (i) $\|\cdot\|$ is an equivalent norm on X which agree with $\|\cdot\|$

on Y ,

(ii) $||| \cdot |||$ is LUR.

(i) is easily seen.

To see (ii), assume there is an $\varepsilon > 0$ such that

$$(1) \quad \lim 2(|||x|||^2 + |||x_m|||^2) - |||x + x_m|||^2 = 0$$

and

$$(2) \quad |||x - x_m||| > \varepsilon$$

and find a contradiction.

From (1),

$$(3) \quad \lim 2(||\hat{x}||^2 + ||\hat{x}_m||^2) - ||\hat{x} + \hat{x}_m||^2 = 0,$$

$$(4) \quad \lim_m 2(||P'_n x||^2 + ||P'_n x_m||^2) - ||P'_n(x + x_m)||^2 = 0, \quad \text{for } n \in N$$

$$(5) \quad \lim 2(||x||^2 + ||x_m||^2) - ||x + x_m||^2 = 0,$$

$$(6) \quad K = \max(\sup ||x_n||, 1) < \infty.$$

If $x \in Y$, then $\hat{x} = 0$ and from (3), $\lim ||\hat{x}_m|| = 0$, so there is a sequence $x'_m \in Y$ with $\lim ||x_m - x'_m|| = 0$ and so, by (5), (6) $\lim 2(||x||^2 + ||x'_m||^2) - ||x + x'_m||^2 = 0$ and therefore by LUR of $|| \cdot ||$ on Y , $\lim ||x - x'_m|| = 0$ and thus $\lim ||x - x_m|| = 0$, a contradiction with (2).

If $x \notin Y$, write $x = y_0 + a_0$, $a_0 = S\hat{x}$, $y_0 \in Y$. From LUR of $|| \cdot ||$ on Y , there is $\delta \in (0, 1/2)$ such that whenever

$$(7) \quad y \in Y, ||y - y_0|| \leq \delta, z \in Y, \text{ and } 2(||y||^2 + ||z||^2) - ||y + z||^2 \leq \delta,$$

then, $||y - z|| \leq \varepsilon/2$. By (3) and LUR of $|| \hat{\cdot} ||$,

$$(8) \quad \lim ||\hat{x}_n - \hat{x}|| = 0$$

and thus,

$$(9) \quad \lim S\hat{x}_m = S\hat{x} = a_0.$$

Let

$$(10) \quad \hat{a}_n \in \{\hat{a}_n\}, \lim \hat{a}_n = \hat{a}_0 = \hat{x} \text{ (and thus } \lim a_n = a_0 \text{)}.$$

Furthermore,

$$(11) \quad \lim ||P_n|| = ||a_0|| \cdot ||\hat{a}_0||^{-1}.$$

Let $\delta_1 = \min \{[1 + (5(||a_0|| \cdot ||\hat{a}_0||^{-1} + 2))^2(K + 1)]^{-1}\delta, \varepsilon/8\}$ (δ from (7)). Choose $n_0 \in N$ so that

$$(a) \quad ||P_{n_0}|| \leq ||a_0|| \cdot ||\hat{a}_0||^{-1} + 1$$

(b) $\|a_n - a_0\| < \delta_1$ for each $n \geq n_0$

(c) $\|\hat{x}_m - \hat{x}\| < \delta_1$ for each $m \geq n_0$.

Keeping this n_0 fixed, choose $n_1 \geq n_0$ so that

(d) $2(\|P'_{n_0}(x)\|^2 + \|P'_{n_0}(x_m)\|^2) - \|P'_{n_0}(x + x_m)\|^2 < \delta_1$ for each $m \geq n_1$.

Choose $z_{n_0} \in \hat{a}_{n_0}$ such that

$$(12) \quad \|z_{n_0} - x\| < \delta_1$$

and $x'_{n_0} \in \hat{a}_{n_0}$ such that

$$(13) \quad \|x'_{n_0} - x_{n_1}\| < 2\delta_1.$$

Since $x'_{n_0} = a_{n_0} + u_{n_0}$, $z_{n_0} = a_{n_0} + v_{n_0}$ for some $u_{n_0}, v_{n_0} \in Y$,

$$(14) \quad P'_{n_0}(x'_{n_0}) = x'_{n_0} - P_{n_0}(x'_{n_0}) = u_{n_0} \in Y \text{ and } P'_{n_0}(z_{n_0}) = v_{n_0} \in Y.$$

Furthermore, by (d), (a), (12), (13),

$$\begin{aligned} & 2(\|P'_{n_0}(z_{n_0})\|^2 + \|P'_{n_0}(x'_{n_0})\|^2) - \|P'_{n_0}(z_{n_0} + x'_{n_0})\|^2 \leq 2(\|P'_{n_0}(x)\|^2 + \|P'_{n_0}(x_{n_1})\|^2 \\ & \quad - P'_{n_0}\|(x + x_{n_1})\|^2 + 2\|P'_{n_0}(z_{n_0} - x)\|(\|P'_{n_0}(z_{n_0})\| + \|P'_{n_0}(x)\|) \\ & \quad + 2\|P'_{n_0}(x'_{n_0} - x_{n_1})\|(\|P'_{n_0}(x'_{n_0})\| + \|P'_{n_0}(x_{n_1})\|) \\ & \quad + (\|P'_{n_0}(z_{n_0} - x)\| + \|P'_{n_0}(x'_{n_0} - x_{n_1})\|) \\ & \quad \times (\|P'_{n_0}(z_{n_0})\| + \|P'_{n_0}(x)\| + \|P'_{n_0}(x'_{n_0})\| + \|P'_{n_0}(x_{n_1})\|) \\ & \leq \delta_1(1 + (5(\|\alpha_0\| \cdot \|\hat{a}_0\|^{-1} + 2))^2(K + 1)) \leq \delta. \end{aligned}$$

Thus, by (7) and (14),

$$\varepsilon/2 \geq \|P'_{n_0}(x'_{n_0}) - P'_{n_0}(z_{n_0})\| = \|x'_{n_0} - z_{n_0}\|.$$

So, $\|x_{n_1} - x\| \leq \|x'_{n_0} - z_{n_0}\| + \|x_{n_1} - x'_{n_0}\| + \|z_{n_0} - x\| \leq (7/8)\varepsilon < \varepsilon$, a contradiction.

For the case of rotund norms we define the norm $\|\cdot\|$ by the same formula as above.

Again, suppose

$$(1') \quad 2(\|x\|^2 + \|y\|^2) - \|x + y\|^2 = 0$$

and

$$(2') \quad \|x - y\| > \varepsilon > 0.$$

From (1'),

$$(3') \quad 2(\|\hat{x}\|^2 + \|\hat{y}\|^2) - \|\hat{x} + \hat{y}\|^2 = 0$$

$$(4') \quad 2(\|P'_n(x)\|^2 + \|P'_n(y)\|^2) - \|P'_n(x + y)\|^2 = 0 \quad \text{for } n \in N$$

$$(5') \quad 2(\|x\|^2 + \|y\|^2) - \|x + y\|^2 = 0.$$

If $x \in Y$, $\hat{x} = 0$ and from (3'), $\hat{y} = 0$, so $y \in Y$ and from R of $\|\cdot\|$ on Y and (5'), $x = y$.

If $x \notin Y$, then by R of $|\hat{\cdot}|$ and by (3'), $\hat{x} = \hat{y}$. So, write $x = a_0 + y_0$, $y = a_0 + z_0$, $y_0, z_0 \in Y$, $a_0 = S\hat{x}$. By R of $\|\cdot\|$ on Y , there is a $(1/2) > \delta > 0$ such that whenever

$$(6') \quad \begin{aligned} y \in Y, \quad z \in Y, \quad \|y - y_0\| \leq \delta, \quad \|z - z_0\| \leq \delta, \\ 2(\|y\|^2 + \|z\|^2) - \|y + z\|^2 \leq \delta, \end{aligned}$$

then

$$\|y - z\| \leq \varepsilon/2.$$

Denote by $\delta_1 = \min \{[1 + (5(\|a_0\| \cdot \|\hat{a}_0\|^{-1} + 2))^2(K + 1)]^{-1}\delta, \varepsilon/8\}$, where $K = \max(\|x\| = \|y\|, 1)$. Let $\hat{a}_n \in \{\hat{a}_n\}$, $\lim \hat{a}_n = \hat{a}_0 = \hat{x}$, $a_n = S\hat{a}_n$. Then $\lim a_n = a_0$, $\lim \|P_n\| = \|a_0\| \cdot \|\hat{a}_0\|^{-1}$.

Thus we can choose $n_0 \in N$ so that $\|P_{n_0}\| \leq \|a_0\| \cdot \|\hat{a}_0\|^{-1} + 1$, $\|a_{n_0} - a_0\| < \delta_1$. Choose $y_{n_0}, z_{n_0} \in \hat{a}_{n_0}$ such that $\|z_{n_0} - x\| < \delta_1$, $\|y_{n_0} - y\| < \delta_1$. Since

$$(7') \quad \begin{aligned} y_{n_0} &= a_{n_0} + z_{n_0}, \quad z_{n_0} = a_{n_0} + v_{n_0}, \quad u_{n_0}, v_{n_0} \in Y, \quad P'_{n_0}(y_{n_0}) \\ &= y_{n_0} - P_{n_0}(y_{n_0}) = u_{n_0} \in Y. \end{aligned}$$

Furthermore,

$$\begin{aligned} 2(\|P'_{n_0}(z_{n_0})\|^2 + \|P'_{n_0}(y_{n_0})\|^2) - \|P'_{n_0}(y_{n_0} + z_{n_0})\|^2 &\leq 2(\|P'_{n_0}x\|^2 \\ &+ \|P_{n_0}(y)\|^2) - \|P_{n_0}(x + y)\|^2 \\ &+ 2 \cdot \|P'_{n_0}(z_{n_0} - x)\|(\|P'_{n_0}(z_{n_0})\| + \|P'_{n_0}(x)\|) \\ &+ 2\|P'_{n_0}(y_{n_0} - y)\| \cdot (\|P'_{n_0}(y_{n_0})\| + \|P'_{n_0}(y)\|) \\ &+ (\|P'_{n_0}(y_{n_0} - y)\| + \|P'_{n_0}(z_{n_0} - x)\|) \\ &\times (\|P'_{n_0}\| \cdot [\|y_{n_0}\| + \|z_{n_0}\| + \|x\| + \|y\|]) \\ &\leq \delta_1(1 + (5(\|a_0\| \cdot \|\hat{a}_0\|^{-1} + 2))^2(K + 1)) \leq \delta. \end{aligned}$$

Thus, by (6'), (7'), $\varepsilon/2 \geq \|P'_{n_0}(y_{n_0}) - P'_{n_0} - (z_{n_0})\| = \|y_{n_0} - z_{n_0}\|$. So, $\|x - y\| \leq \|x - z_{n_0}\| + \|y_{n_0} - z_{n_0}\| + \|y_{n_0} + y\| \leq (3/4)\varepsilon < \varepsilon$, a contradiction.

We finish the note with the following

Question. Can Theorem 1 be generalized for the case of weakly compactly generated X/Y ?

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