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# ON EXTENSION OF ROTUND NORMS II

K. JOHN AND V. ZIZLER

**It is proved that if $X$ is a Banach space, $Y \subset X$ with $X/Y$ separable and $||\cdot||$ is an equivalent locally uniformly rotund norm on $Y$, then $||\cdot||$ can be extended to such a norm on $X$.**

This generalizes [2] where it was shown that any locally uniformly rotund equivalent norm on a closed subspace of a separable Banach space $X$ can be extended to such a norm on $X$.

By a subspace we mean a closed linear subspace, $\operatorname{sp} L$ denotes the linear hull of $L$ and $x \to \hat{x}$ stands for the quotient map $X \to X/Y$ if $Y$ is a subspace of $X$.

Let us recall that a norm $||\cdot||$ on a Banach space $X$ is locally uniformly rotund (LUR) if whenever $\lim 2(||x||^2 + ||x_j||^2) - ||x + x_j||^2 = 0$, $x, x_j \in X$, then $\lim ||x - x_j|| = 0$. $||\cdot||$ is rotund $(R)$ if for any $x, y \in X$, $x \neq y$, $2(||x||^2 + ||y||^2) - ||x + y||^2 > 0$.

THEOREM 1. *Let $X$ be a Banach space, $Y \subset X$ a subspace of $X$. Suppose $X/Y$ is separable and $Y$ admits an equivalent norm $||\cdot||$ which is LUR $(R)$. Then $||\cdot||$ can be extended to an equivalent norm $||\cdot||$ on $X$ which is LUR $(R)$.*

*Proof.* Let us start with the case of LUR.

First extend the given LUR norm $||\cdot||$ on $Y$ to an equivalent norm $||\cdot||$ on $X$: This can easily be done as follows: Take the closed unit ball $B_1^Y$ of $Y$ with respect to $||\cdot||$ and the closed ball $B$ of $X$ such that $B \cap Y \subset B_1^Y$. Then, easily, the Minkowski functional of $\operatorname{conv}(B \cup B_1^Y)$ is the desired norm on $X$(cf. e.g., [4], [2]).

Furthermore, let $\{\hat{a}_n\}_{n=1}^{\infty} \subset X/Y$, $\hat{a}_n \neq 0$ be a dense subset of $X/Y$. Let $S: X/Y \to X$ denote the Bartle-Graves continuous selection map $(S\hat{x} \in \hat{x})$ and $a_n = S\hat{a}_n$.

For $n \in N$ ($N$ positive integers), choose $f_n \in X^*$, $f_n(a_n) = 1$, $||f_n|| = ||\hat{a}_n||^{-1}$, $f_n = 0$ on $Y$ and denote by $P_n(x) = f_n(x)a_n$, $P_n' = I - P_n$ where $I$ is the identity map on $X$.

Consider

$$|||x|||^2 = (1 - c)||x||^2 + \sum_{n=1}^{\infty} 2^{-n}(1 + ||P_n||)^{-2} \cdot ||x - P_n x||^2 + ||\hat{x}||^2 ,$$

where $c = \sum_{n=1}^{\infty}(1 + ||P_n||)^{-2}2^{-n}$, $||\hat{\cdot}||$ is an equivalent LUR norm on $X/Y$([3]).

Then (i) $|||\cdot|||$ is an equivalent norm on $X$ which agree with $||\cdot||$

on $Y$,

(ii) $|||\cdot|||$ is LUR.

(i) is easily seen.

To see (ii), assume there is an $\varepsilon > 0$ such that

$$\lim 2(|||x|||^2 + |||x_m|||^2) - |||x + x_m|||^2 = 0 \tag{1}$$

and

$$|||x - x_m||| > \varepsilon \tag{2}$$

and find a contradiction.

From (1),

$$\lim 2(||\hat{x}||^2 + ||\hat{x}_m||^2) - ||\hat{x} + \hat{x}_m||^2 = 0\ , \tag{3}$$

$$\lim_m 2(||P_n'x||^2 + ||P_n'x_m||^2) - ||P_n'(x + x_m)||^2 = 0\ , \qquad \text{for} \quad n \in N \tag{4}$$

$$\lim 2(||x||^2 + ||x_m||^2) - ||x + x_m||^2 = 0\ , \tag{5}$$

$$K = \max(\sup ||x_n||, 1) < \infty\ . \tag{6}$$

If $x \in Y$, then $\hat{x} = 0$ and form (3), $\lim ||\hat{x}_m|| = 0$, so there is a sequence $x_m' \in Y$ with $\lim ||x_m - x_m'|| = 0$ and so, by (5), (6) $\lim 2(||x||^2 + ||x_m'||^2) - ||x + x_m'||^2 = 0$ and therefore by LUR of $||\cdot||$ on $Y$, $\lim ||x - x_m'|| = 0$ and thus $\lim ||x - x_m|| = 0$, a contradiction with (2).

If $x \notin Y$, write $x = y_0 + a_0$, $a_0 = S\hat{x}$, $y_0 \in Y$. From LUR of $||\cdot||$ on $Y$, there is $\delta \in (0, 1/2)$ such that whenever

$$y \in Y,\ ||y - y_0|| \leqq \delta,\ z \in Y, \text{ and } 2(||y||^2 + ||z||^2) - ||y + z||^2 \leqq \delta\ , \tag{7}$$

then, $||y - z|| \leqq \varepsilon/2$. By (3) and LUR of $||\hat{\cdot}||$,

$$\lim ||\hat{x}_n - \hat{x}|| = 0 \tag{8}$$

and thus,

$$\lim S\hat{x}_m = S\hat{x} = a_0\ . \tag{9}$$

Let

$$\hat{a}_n \in \{\hat{a}_n\},\ \lim \hat{a}_n = \hat{a}_0 = \hat{x} \text{ (and thus } \lim a_n = a_0)\ . \tag{10}$$

Furthermore,

$$\lim ||P_n|| = ||a_0|| \cdot ||\hat{a}_0||^{-1}\ . \tag{11}$$

Let $\delta_1 = \min\{[1 + (5(||a_0|| \cdot ||\hat{a}_0||^{-1} + 2))^2(K + 1)]^{-1}\delta, \varepsilon/8\}$ ($\delta$ from (7)). Choose $n_0 \in N$ so that

(a) $||P_{n_0}|| \leqq ||a_0|| \cdot ||\hat{a}_0||^{-1} + 1$

(b) $||a_n - a_0|| < \delta_1$ for each $n \geqq n_0$
(c) $||\hat{x}_m - \hat{x}|| < \delta_1$ for each $m \geqq n_0$.
Keeping this $n_0$ fixed, choose $n_1 \geqq n_0$ so that
(d) $2(||P'_{n_0}(x)||^2 + ||P'_{n_0}(x_m)||^2) - ||P'_{n_0}(x + x_m)||^2 < \delta_1$ for each $m \geqq n_1$.
Choose $z_{n_0} \in \hat{a}_{n_0}$ such that

$$(12) \qquad ||z_{n_0} - x|| < \delta_1$$

and $x'_{n_0} \in \hat{a}_{n_0}$ such that

$$(13) \qquad ||x'_{n_0} - x_{n_1}|| < 2\delta_1 .$$

Since $x'_{n_0} = a_{n_0} + u_{n_0}$, $z_{n_0} = a_{n_0} + v_{n_0}$ for some $u_{n_0}, v_{n_0} \in Y$,

$$(14) \qquad P'_{n_0}(x'_{n_0}) = x'_{n_0} - P_{n_0}(x'_{n_0}) = u_{n_0} \in Y \text{ and } P'_{n_0}(z_{n_0}) = v_{n_0} \in Y .$$

Furthermore, by (d), (a), (12), (13),

$$\begin{aligned}
2(||P'_{n_0}(z_{n_0})||^2 + ||P'_{n_0}(x'_{n_0})||^2) - ||P'_{n_0}(z_{n_0} + x'_{n_0})||^2 &\leqq 2(||P'_{n_0}(x)||^2 + ||P'_{n_0}(x_{n_1})||^2 \\
&\quad - P'_{n_0}||(x + x_{n_1})||^2 + 2||P'_{n_0}(z_{n_0} - x)||(||P'_{n_0}(z_{n_0})|| + ||P'_{n_0}(x)||) \\
&\quad + 2||P'_{n_0}(x'_{n_0} - x_{n_1})||(||P'_{n_0}(x'_{n_0})|| + ||P'_{n_0}(x_{n_1})||) \\
&\quad + (||P'_{n_0}(z_{n_0} - x)|| + ||P'_{n_0}(x'_{n_0} - x_{n_1})||) \\
&\quad \times (||P'_{n_0}(z_{n_0})|| + ||P'_{n_0}(x)|| + ||P'_{n_0}(x'_{n_0})|| + ||P'_{n_0}(x_{n_1})||) \\
&\leqq \delta_1(1 + (5(||\alpha_0||\cdot||\hat{a}_0||^{-1} + 2))^2(K + 1)) \leqq \delta .
\end{aligned}$$

Thus, by (7) and (14),

$$\varepsilon/2 \geqq ||P'_{n_0}(x'_{n_0}) - P'_{n_0}(z_{n_0})|| = ||x'_{n_0} - z_{n_0}|| .$$

So, $||x_{n_1} - x|| \leqq ||x'_{n_0} - z_{n_0}|| + ||x_{n_1} - x'_{n_0}|| + ||z_{n_0} - x|| \leqq (7/8)\varepsilon < \varepsilon$, a contradiction.

For the case of rotund norms we define the norm $|||\cdot|||$ by the same formula as above.
Again, suppose

$$(1') \qquad 2(|||x|||^2 + |||y|||^2) - |||x + y|||^2 = 0$$

and

$$(2') \qquad ||x - y|| > \varepsilon > 0 .$$

From (1'),

$$(3') \qquad 2(||\hat{x}||^2 + ||\hat{y}||^2) - ||\hat{x} + \hat{y}||^2 = 0$$

$$(4') \qquad 2(||P'_n(x)||^2 + ||P'_n(y)|^2) - ||P'_n(x + y)||^2 = 0 \qquad \text{for} \quad n \in N$$

$$(5') \qquad 2(||x||^2 + ||y||^2) - ||x + y||^2) = 0 .$$

If $x \in Y$, $\hat{x} = 0$ and from (3′), $\hat{y} = 0$, so $y \in Y$ and from $R$ of $\|\cdot\|$ on $Y$ and (5′), $x = y$.

If $x \notin Y$, then by $R$ of $|\hat{\cdot}|$ and by (3′), $\hat{x} = \hat{y}$. So, write $x = a_0 + y_0$, $y = a_0 + z_0$, $y_0, z_0 \in Y$, $a_0 = S\hat{x}$. By $R$ of $\|\cdot\|$ on $Y$, there is a $(1/2) > \delta > 0$ such that whenever

$$\begin{aligned} &y \in Y\,, \quad z \in Y\,, \quad \|y - y_0\| \leqq \delta\,, \quad \|z - z_0\| \leqq \delta\,, \\ &2(\|y\|^2 + \|z\|^2) - \|y + z\|^2 \leqq \delta\,, \end{aligned} \tag{6′}$$

then

$$\|y - z\| \leqq \varepsilon/2\,.$$

Denote by $\delta_1 = \min\{[1 + (5(\|a_0\| \cdot \|\hat{a}_0\|^{-1} + 2))^2(K + 1)]^{-1}\delta, \varepsilon/8\}$, where $K = \max(\|x\| = \|y\|, 1)$. Let $\hat{a}_n \in \{\hat{a}_n\}$, $\lim \hat{a}_n = \hat{a}_0 = \hat{x}$, $a_n = S\hat{a}_n$. Then $\lim a_n = a_0$, $\lim \|P_n\| = \|a_0\| \cdot \|\hat{a}_0\|^{-1}$.

Thus we can choose $n_0 \in N$ so that $\|P_{n_0}\| \leqq \|a_0\| \cdot \|\hat{a}_0\|^{-1} + 1$, $\|a_{n_0} - a_0\| < \delta_1$. Choose $y_{n_0}, z_{n_0} \in \hat{a}_{n_0}$ such that $\|z_{n_0} - x\| < \delta_1$, $\|y_{n_0} - y\| < \delta_1$. Since

$$\begin{aligned} y_{n_0} &= a_{n_0} + z_{n_0},\ z_{n_0} = a_{n_0} + v_{n_0},\ u_{n_0},\ v_{n_0} \in Y,\ P'_{n_0}(y_{n_0}) \\ &= y_{n_0} - P_{n_0}(y_{n_0}) = u_{n_0} \in Y\,. \end{aligned} \tag{7′}$$

Furthermore,

$$\begin{aligned} 2(\|P'_{n_0}(z_{n_0})\|^2 + \|P'_{n_0}(y_{n_0})\|^2) - \|P'_{n_0}(y_{n_0} + z_{n_0})\|^2 &\leqq 2(\|P'_{n_0}x\|^2 \\ &\quad + \|P_{n_0}(y)\|^2) - \|P_{n_0}(x + y)\|^2 \\ &\quad + 2\cdot\|P'_{n_0}(z_{n_0} - x)\|(\|P'_{n_0}(z_{n_0})\| + \|P'_{n_0}(x)\|) \\ &\quad + 2\|P'_{n_0}(y_{n_0} - y)\| \cdot (\|P'_{n_0}(y_{n_0})\| + \|P'_{n_0}(y)\|) \\ &\quad + (\|P'_{n_0}(y_{n_0} - y)\| + \|P'_{n_0}(z_{n_0} - x)\|) \\ &\quad \times (\|P'_{n_0}\| \cdot [\|y_{n_0}\| + \|z_{n_0}\| + \|x\| + \|y\|]) \\ &\leqq \delta_1(1 + (5(\|a_0\| \cdot \|a_0\|^{-1} + 2))^2(K + 1)) \leqq \delta\,. \end{aligned}$$

Thus, by (6′), (7′), $\varepsilon/2 \geqq \|P'_{n_0}(y_{n_0}) - P'_{n_0} - (z_{n_0})\| = \|y_{n_0} - z_{n_0}\|$. So, $\|x - y\| \leqq \|x - z_{n_0}\| + \|y_{n_0} - z_{n_0}\| + \|y_{n_0} + y\| \leqq (3/4)\varepsilon < \varepsilon$, a contradiction.

We finish the note with the following

*Question.* Can Theorem 1 be generalized for the case of weakly compactly generated $X/Y$?

## References

1. J. Diestel, *Geometry of Banach spaces,* Selected topics, Lecture Notes in Math. 485, Springer-Verlag, 1975.
2. K. John and V. Zizler, *On extension of rotund norms,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **24** (1976), 705–707.
3. M. I. Kadec, *Spaces isomorphic to a locally uniformly convex space,* Izv. vysch. Ucheb. Zaved. Matematika, **6** (1959), 51–57 and **6** (1961), 86–87.
4. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces,* Lecture Notes in Math. 338, Springer-Verlag, 1973.

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