ON EXTENSION OF ROTUND NORMS. II

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It is proved that if $X$ is a Banach space, $Y \subseteq X$ with $X/Y$ separable and $\| \cdot \|$ is an equivalent locally uniformly rotund norm on $Y$, then $\| \cdot \|$ can be extended to such a norm on $X$.

This generalizes [2] where it was shown that any locally uniformly rotund equivalent norm on a closed subspace of a separable Banach space $X$ can be extended to such a norm on $X$.

By a subspace we mean a closed linear subspace, $\text{sp} L$ denotes the linear hull of $L$ and $x \to \hat{x}$ stands for the quotient map $X \to X/Y$ if $Y$ is a subspace of $X$.

Let us recall that a norm $\| \cdot \|$ on a Banach space $X$ is locally uniformly rotund (LUR) if whenever $\lim 2(\|x\|^2 + \|x_j\|^2) - \|x + x_j\|^2 = 0$, $x, x_j \in X$, then $\lim \|x - x_j\| = 0$. $\| \cdot \|$ is rotund (R) if for any $x, y \in X$, $x \neq y$, $2(\|x\|^2 + \|y\|^2) - \|x + y\|^2 > 0$.

**Theorem 1.** Let $X$ be a Banach space, $Y \subseteq X$ a subspace of $X$. Suppose $X/Y$ is separable and $Y$ admits an equivalent norm $\| \cdot \|$ which is LUR (R). Then $\| \cdot \|$ can be extended to an equivalent norm $\| \cdot \|$ on $X$ which is LUR (R).

**Proof.** Let us start with the case of LUR.

First extend the given LUR norm $\| \cdot \|$ on $Y$ to an equivalent norm $\| \cdot \|$ on $X$: This can easily be done as follows: Take the closed unit ball $B_Y^*$ of $Y$ with respect to $\| \cdot \|$ and the closed ball $B_X$ of $X$ such that $B \cap Y \subseteq B_Y^*$. Then, easily, the Minkowski functional of $\text{conv}(B \cup B_Y^*)$ is the desired norm on $X$ (cf. e.g., [4], [2]).

Furthermore, let $\{\hat{a}_n\}_{n=1}^{\infty} \subseteq X/Y$, $\hat{a}_n \neq 0$ be a dense subset of $X/Y$. Let $S: X/Y \to X$ denote the Bartle-Graves continuous selection map $(S\hat{x} \in \hat{x})$ and $a_n = S\hat{a}_n$.

For $n \in N$ ($N$ positive integers), choose $f_n \in X^*$, $f_n(a_n) = 1$, $\|f_n\| = \|\hat{a}_n\|^{-1}$, $f_n = 0$ on $Y$ and denote by $P_n(x) = f_n(x)a_n$, $P_n' = I - P_n$ where $I$ is the identity map on $X$.

Consider

$$\|x\|^2 = (1 - c)\|x\|^2 + \sum_{n=1}^{\infty} 2^{-n}(1 + \|P_n\|)^{-2}\|x - P_nx\|^2 + \|\hat{x}\|^2,$$

where $c = \sum_{n=1}^{\infty} (1 + \|P_n\|)^{-2}2^{-n}$, $\| \cdot \|$ is an equivalent LUR norm on $X/Y$ ([3]).

Then (i) $\| \cdot \|$ is an equivalent norm on $X$ which agree with $\| \cdot \|$. 

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on $Y$,

(ii) $||\cdot||$ is LUR.

(i) is easily seen.

To see (ii), assume there is an $\varepsilon > 0$ such that

\[(i) \quad \lim 2(||x||^2 + ||x_m||^2) - ||x + x_m||^2 = 0\]

and

\[(ii) \quad ||x - x_m|| > \varepsilon\]

and find a contradiction.

From (1),

\[(3) \quad \lim 2(||\hat{x}||^2 + ||\hat{x}_m||^2) - ||\hat{x} + \hat{x}_m||^2 = 0,\]

\[(4) \quad \lim 2(||P_n'x||^2 + ||P_n'x_m||^2) - ||P_n'(x + x_m)||^2 = 0, \quad \text{for } n \in \mathbb{N}\]

\[(5) \quad \lim 2(||x||^2 + ||x_m||^2) - ||x + x_m||^2 = 0,\]

\[(6) \quad K = \max (\sup ||x_n||, 1) < \infty .\]

If $x \in Y$, then $\hat{x} = 0$ and form (3), $\lim ||\hat{x}_m|| = 0$, so there is a sequence $x_m \in Y$ with $\lim ||x_m - x'|| = 0$ and so, by (5), (6) $\lim 2(||x||^2 + ||x'||^2) - ||x + x'||^2 = 0$ and therefore by LUR of $||\cdot||$ on $Y$, $\lim ||x - x'|| = 0$ and thus $\lim ||x - x_m|| = 0$, a contradiction with (2).

If $x \in Y$, write $x = y_0 + a_0$, $a_0 = S\hat{x}^0$, $y_0 \in Y$. From LUR of $||\cdot||$ on $Y$, there is $\delta \in (0, 1/2)$ such that whenever

\[(7) \quad y \in Y, ||y - y_0|| \leq \delta, z \in Y, \text{ and } 2(||y||^2 + ||z||^2) - ||y + z||^2 \leq \delta ,\]

then, $||y - z|| \leq \varepsilon/2$. By (3) and LUR of $||\cdot||$,

\[(8) \quad \lim ||\hat{x}_n - \hat{x}|| = 0\]

and thus,

\[(9) \quad \lim S\hat{x}_m = S\hat{x} = a_0 .\]

Let

\[(10) \quad \hat{a}_n \in \{\hat{a}_m\}, \text{ lim } \hat{a}_n = \hat{a}_0 = \hat{x} \text{ (and thus lim } a_n = a_0) .\]

Furthermore,

\[(11) \quad \lim ||P_n|| = ||a_0|| \cdot ||\hat{a}_0||^{-1} .\]

Let $\delta_1 = \min \{[1 + (5(||a_0|| \cdot ||\hat{a}_0||^{-1} + 2))^{2}(K + 1)]^{-1}\delta, \varepsilon/8\}$ (\delta$ from (7)).$

Choose $n_0 \in \mathbb{N}$ so that

\[(a) \quad ||P_{n_0}|| \leq ||a_0|| \cdot ||\hat{a}_0||^{-1} + 1\]
(b) \(|a_n - a_0| < \delta_1\) for each \(n \geq n_0\)

(c) \(|\hat{x}_m - \hat{x}| < \delta_1\) for each \(m \geq n_0\).

Keeping this \(n_0\) fixed, choose \(n_1 \geq n_0\) so that

(d) \(2(||P_n(x)||^2 + ||P_n'(x_m)||^2) - ||P_n'(x + x_m)||^2 < \delta_1\) for each \(m \geq n_1\).

Choose \(z_{n_0} \in \hat{a}_{n_0}\) such that

\[(12) \quad ||z_{n_0} - x|| < \delta_1\]

and \(x'_{n_0} \in \hat{a}_{n_0}\) such that

\[(13) \quad ||x'_{n_0} - x_{n_1}|| < 2\delta_1.\]

Since \(x'_{n_0} = a_{n_0} + u_{n_0}, z_{n_0} = a_{n_0} + v_{n_0}\) for some \(u_{n_0}, v_{n_0} \in Y,\)

\[(14) \quad P_{n_0}(x_{n_0}) = x'_{n_0} - P_{n_0}(x'_{n_0}) = u_{n_0} \in Y\text{ and } P_{n_0}'(z_{n_0}) = v_{n_0} \in Y.\]

Furthermore, by (d), (a), (12), (13),

\[
2(||P_{n_0}'(z_{n_0})||^2 + ||P_{n_0}'(x_{n_0}')||^2) - ||P_{n_0}'(z_{n_0} + x_{n_0}')||^2 \leq 2(||P_{n_0}'(x)||^2 + ||P_{n_0}'(x_{n_1})||^2

- P_{n_0}'||(x + x_{n_1})||^2 + 2||P_{n_0}'(z_{n_0} - x)||(||P_{n_0}'(z_{n_0})|| + ||P_{n_0}'(x)||)

+ 2||P_{n_0}'(x_{n_0} - x_{n_1})||(||P_{n_0}'(z_{n_0})|| + ||P_{n_0}'(x_{n_1})||)

+ (||P_{n_0}'(z_{n_0} - x)|| + ||P_{n_0}'(x_{n_0} - x_{n_1})||)

\times (||P_{n_0}'(z_{n_0})|| + ||P_{n_0}'(x)|| + ||P_{n_0}'(x_{n_0})|| + ||P_{n_0}'(x_{n_1})||)

\leq \delta_1(1 + (5(||a_0|| \cdot ||\hat{a}_0||^{-1} + 2))2(K + 1)) \leq \delta.\]

Thus, by (7) and (14),

\[
\varepsilon/2 \geq ||P_{n_0}'(x_{n_0}') - P_{n_0}'(z_{n_0})|| = ||z_{n_0}' - z_{n_0}||.\]

So, \(||x_{n_1} - x|| \leq ||x_{n_0}' - z_{n_0}|| + ||x_{n_1} - x_{n_0}'|| + ||z_{n_0} - x|| \leq (7/8)\varepsilon < \varepsilon,\]
a contradiction.

For the case of rotund norms we define the norm \(\|\cdot\|\) by the same formula as above.

Again, suppose

\[(1') \quad 2(||x||^2 + ||y||^2) - ||x + y||^2 = 0\]

and

\[(2') \quad ||x - y|| > \varepsilon > 0.\]

From \((1')\),

\[(3') \quad 2(||\hat{x}||^2 + ||\hat{y}||^2) - ||\hat{x} + \hat{y}||^2 = 0\]

\[(4') \quad 2(||P_n(x)||^2 + ||P_n'(y)||^2) - ||P_n'(x + y)||^2 = 0 \quad \text{for } n \in N\]

\[(5') \quad 2(||x||^2 + ||y||^2) - ||x + y||^2 = 0.\]
If $x \in Y$, $\hat{x} = 0$ and from (3'), $\hat{y} = 0$, so $y \in Y$ and from $R$ of $\| \cdot \|$ on $Y$ and (5'), $x = y$.

If $x \in Y$, then by $R$ of $| \cdot |$ and by (3'), $\hat{x} = \hat{y}$. So, write $x = a_o + y_o$, $y = a_o + z_o$, $y_o, z_o \in Y$, $a_o = S\hat{x}$. By $R$ of $\| \cdot \|$ on $Y$, there is a $(1/2) > \delta > 0$ such that whenever

\[
(6') \quad y \in Y, \quad z \in Y, \quad \| y - y_o \| \leq \delta, \quad \| z - z_o \| \leq \delta,
\]

\[
2(\| y \|^2 + \| z \|^2) - \| y + z \|^2 \leq \delta,
\]

then

\[
\| y - z \| \leq \varepsilon/2.
\]

Denote by $\delta_1 = \min \{ (1 + (5(\| a_o \| \cdot \| \hat{a}_o \|^{-1} + 2))^2(K + 1))^{-1}/2, \varepsilon/8 \}$, where $K = \max(\| x \| = \| y \|, 1)$. Let $\hat{a}_n \in \{ \hat{a}_n \}$, $\lim \hat{a}_n = \hat{a}_0 = \hat{x}$, $a_n = S\hat{a}_n$. Then $\lim a_n = a_o$, $\lim \| P_n \| = \| a_o \| \cdot \| \hat{a}_0 \|^{-1}$.

Thus we can choose $n_0 \in \mathbb{N}$ so that $\| P_{n_0} \| \leq \| a_o \| \cdot \| \hat{a}_0 \|^{-1} + 1$, $\| a_{n_0} - a_o \| < \delta_1$. Choose $y_{n_0}, z_{n_0} \in \hat{a}_{n_0}$ such that $\| z_{n_0} - x \| < \delta_1$, $\| y_{n_0} - y \| < \delta_1$. Since

\[
(7') \quad y_{n_0} = a_{n_0} + z_{n_0}, \quad z_{n_0} = a_{n_0} + v_{n_0}, \quad u_{n_0} = v_{n_0} \in Y, \quad P_{n_0}(y_{n_0}) = y_{n_0} - P_{n_0}(y_{n_0}) = u_{n_0} \in Y.
\]

Furthermore,

\[
2(\| P'_{n_0}(z_{n_0}) \|^2 + \| P'_{n_0}(y_{n_0}) \|^2) - \| P'_{n_0}(y_{n_0} + z_{n_0}) \|^2 \leq 2(\| P'_{n_0} x \|^2
\]

\[
+ \| P_{n_0}(y) \|^2 - \| P_{n_0}(x + y) \|^2
\]

\[
+ 2(\| P'_{n_0}(z_{n_0} - x) \| \cdot (\| P'_{n_0}(z_{n_0}) \| + \| P'_{n_0}(x) \|)
\]

\[
+ 2 \| P'_{n_0}(y_{n_0} - y) \| \cdot (\| P'_{n_0}(y_{n_0}) \| + \| P'_{n_0}(y) \|)
\]

\[
+ \| P'_{n_0}(y_{n_0} - y) \| \cdot \| P'_{n_0}(z_{n_0} - x) \|
\]

\[
\times (\| P'_{n_0} \| \cdot (\| y_{n_0} \| + \| z_{n_0} \| + \| x \| + \| y \|))
\]

\[
\leq \delta_1(1 + (5(\| a_o \| \cdot \| a_o \|^{-1} + 2))^2(K + 1)) \leq \delta.
\]

Thus, by (6'), (7'), $\varepsilon/2 \geq \| P'_{n_0}(y_{n_0}) - P'_{n_0} - (z_{n_0}) \| = \| y_{n_0} - z_{n_0} \|$. So, $\| x - y \| \leq \| x - z_{n_0} \| + \| y_{n_0} - z_{n_0} \| + \| y_{n_0} + y \| \leq (3/4)\varepsilon < \varepsilon$, a contradiction.

We finish the note with the following

**Question.** Can Theorem 1 be generalized for the case of weakly compactly generated $X/Y$?
REFERENCES


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