Pacific Journal of Mathematics

# EXISTENCE OF A STRONG LIFTING COMMUTING WITH A COMPACT GROUP OF TRANSFORMATIONS. II

Russell Allan Johnson

Vol. 82, No. 2 February 1979

PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 2, 1979

# EXISTENCE OF A STRONG LIFTING COMMUTING WITH A COMPACT GROUP OF TRANSFORMATIONS II

RUSSELL A. JOHNSON

**Let $G$ be a locally compact group with left Haar measure $\gamma$. The well-known "Theorem LCG" of A. and C. Ionescu-Tulcea states that there is a strong lifting of $M^\infty(G,\gamma)$ commuting with left translations. Our purpose here is to prove a generalization of this theorem in case $G$ is compact. Thus let $(G, X)$ be a free left transformation group with $X$ and $G$ compact. Let $\nu_0$ be a Radon measure on $Y=X/G$, and let $\mu$ be the Haar lift of $\nu_0$. Let $\rho_0$ be a strong lifting of $M^\infty(Y,\nu_0)$. We will show that $M^\infty(X,\mu)$ admits a strong lifting $\rho$ which extends $\rho_0$ and commutes with $G$.**

In [6], the result just stated was proved when $G$ and $X$ satisfied certain restrictions. The following theorem, which may be of independent interest, enables us to remove the conditions imposed in [6]: Let $H$ be a closed normal Lie subgroup of a compact group $G$: then there is a $D'$ sequence (see 1.2 and [1] in $H$, consisting of compact neighborhoods $V_n(n \geqq 1)$ of the identity, such that $g^{-1}V_n g = V_n$ for all $g \in G$.

## 1.

NOTATION 1.1. Let $G$ be a compact topological group, $H$ a closed, normal, real Lie subgroup. Let $\gamma$ be normalized Haar measure on $G$, and let $\lambda$ be normalized Haar measure on $H$. For each $g \in G$, define $\alpha_g: H \to H: h \to g^{-1}hg$. Let $\mathfrak{H}$ be the Lie algebra of $H$; let $\exp: \mathfrak{H} \to H$ be the exponential map.

DEFINITION 1.2. ([1]). A *$D'$-sequence in $H$* is a sequence $(W_n)_{n=1}^\infty$ of $\lambda$-measurable subsets of $H$ such that (i) $W_n \supset W_{n+1}(n \geqq 1)$; (ii) $0 < \lambda(W_n W_n^{-1}) < C\cdot\lambda(W_n)$ for some $C > 0$ and all $n$; (iii) every neighborhood of idy ($\equiv$identity) $\in H$ contains some $W_n$.

PROPOSITION 1.3. *There is a $D'$-sequence $(V_n)_{n=1}^\infty$ in $H$, consisting of compact neighborhoods of* idy, *such that $g^{-1}V_n g = V_n(n \geqq 1, g \in G)$.*

*Proof.* Let $W$ be a neighborhood of 0 in $\mathfrak{H}$ such that $\exp|_W$ is a diffeomorphism onto $\exp(W) \subset H_0$, the identity component of $H$. Define log to be the inverse of $\exp|_W$. There is a neighborhood $N \subset \exp(W)$ of idy such that $g^{-1}Ng \subset W(g \in G)$. Let $\varphi_g(x) = \log\circ\alpha_g\circ$

$\exp(x) = \log(g^{-1}\cdot\exp(x)\cdot g)$ for all $x \in W_1 = \log(N)$. Then $\varphi_g\colon W_1 \to W$, and $\varphi_g(0) = 0 (g \in G)$.

Each map $\alpha_g$ is a continuous isomorphism of $H$, hence is analytic ([9], Theorem 5.22). Let $\mathrm{Ad}_g\colon \mathfrak{H} \to \mathfrak{H}$ be the derivative at $\mathrm{idy} \in H$ of $\alpha_g$. Then $\mathrm{Ad}_g(x) = D\varphi_g(0)\cdot x (x \in \mathfrak{H})$. The map $g \to \mathrm{Ad}_g$ is a homomorphism of $G$ into $GL(\mathfrak{H})$. We show that it is continuous. Let $G_0 = \{g \in G \mid g^{-1}hg = h \text{ for all } h \in H_0\}$. Then $G_0$ is a closed normal subgroup of $G$. The group $G/G_0$ acts effectively on $H_0$ via the map $\eta\colon G/G_0 \times H_0 \to H_0\colon (gG_0, h) \to g^{-1}hg$. Therefore $G/G_0$ is a Lie group, and the map $\eta$ is analytic ([8], pp. 208, 212, 213). It follows that $g \to \mathrm{Ad}_g$ is continuous.

Let $\langle\ ,\ \rangle_1$ be an inner product on $\mathfrak{H}$. Define an inner product $\langle\ ,\ \rangle$, invariant under each $\mathrm{Ad}_g$, by

$$\langle x, y\rangle = \int_G \langle \mathrm{Ad}_g(x), \mathrm{Ad}_g(y)\rangle_1 \, d\gamma(g) (x, y \in \mathfrak{H}) .$$

Observe that, if $B_r = \{x \in \mathfrak{H} \mid \|x\| \leqq r, \text{ where } \|x\|^2 = \langle x, x\rangle\}$, then $\mathrm{Ad}_g(B_r) = B_r (g \in G)$. Also observe that, if $m$ is a Lebesgue measure on $\mathfrak{H}$, then there is a constant $\beta$ such that $m(B_r) = \beta r^k$, where $k = \dim H$.

Consider the measure $\lambda|_{\exp W}$. By ([7], Corollary 2, p. 106), there is a Lebesgue measure $m$ on $\mathfrak{H}$ and an analytic function $\rho\colon W \to \boldsymbol{R}$, satisfying $\rho(0) = 1$, such that $\lambda(\exp B) = \int_B \rho(x)dm(x)$ for each Borel set $B \subset W$. Let $W_2$ be a neighborhood of $0 \in \mathfrak{H}$ such that $1/2 \leqq \rho(x) \leqq 2 (x \in W_2)$.

Now let $0 < \varepsilon < 1$ satisfy $(1-\varepsilon)^k > 1/2 (k = \dim H)$. Recall that $\varphi_g(0) = 0$ for all $g \in G$, that $\mathrm{Ad}_g(x) = D\varphi_g(0)\cdot x$, that $G$ is compact, and that $(gG_0, x) \to \varphi_g(x)\colon G/G_0 \times W_2 \to W$ is analytic. We can therefore find $r' > 0$ such that

(*) $\|\varphi_g(x) - \mathrm{Ad}_g(x)\| < \varepsilon\|x\|$ for all $g \in G$ if $\|x\| \leqq r'$ (recall $\|x\|^2 = \langle x, x\rangle$). Choose $r_0 \leqq r'$ such that $B_{3r} \subset W_2$ and $\exp(B_r)\cdot\exp(B_r) \subset \exp B_{3r}$ if $r \leqq r_0$. Let $r_n = r_0/n$. Define $C_n = \bigcap_{g \in G} \varphi_g(B_{r_n})$, and let $V_n = \exp(C_n)$. By (*), $B_{(1-\varepsilon)r_n} \subset C_n$ for each $n$. Hence $V_n$ is a compact neighborhood of idy for each $n (n \geqq 1)$.

We show that $(V_n)_{n=1}^\infty$ is the desired $D'$-sequence in $H$. First note that $g^{-1}V_n g = \alpha_g \circ \exp(C_n) = \exp\circ\varphi_g(C_n) = \exp C_n = V_n$ for all $g \in G$. Next, observe that $V_n V_n^{-1} = \exp(C_n)\cdot\exp(-C_n) \subset \exp(B_{r_n})\cdot\exp(B_{r_n}) \subset \exp B_{3r_n}$. So $\exp(B_{(1-\varepsilon)r_n}) \subset V_n \subset V_n V_n^{-1} \subset \exp B_{3r_n}$. So, on the one hand, $\lambda(V_n V_n^{-1}) \leqq \lambda(\exp B_{3r_n}) = \int_{B_{3r_n}} \rho(x)dm(x) \leqq 2\cdot\beta\cdot 3^k\cdot(r_n)^k$, while on the other hand,

$$\lambda(V_n) \geqq \int_{B_{(1-\varepsilon)r_n}} \rho(x)dm(x) \geqq 1/2\beta(1-\varepsilon)^k(r_n)^k > 1/4\beta(r_n)^k .$$

Hence $\lambda(V_n V_n^{-1}) \leqq 8\cdot 3^k \lambda(V_n)$, so (ii) of 1.2 is satisfied with $C = 8\cdot 3^k$.

It is easy to see that $(V_n)_{n=1}^{\infty}$ satisfies (i) and (iii) of 1.2. This completes the proof of 1.3.

REMARK 1.4. The sequence $(V_n)_{n=1}^{\infty}$ is also a $D''$-sequence ([1]); that is, each $V_n$ contains a subset $U_n$ such that $U_n \cup U_n U_n^{-1} \subset V_n$, and $\lambda(V_n) < C'\lambda(U_n)$ for some constant $C'(n \geqq 1)$. To see this, let $s_n = (1-\varepsilon)r_n/3$, and let $U_n = \exp B_{s_n}$. Then $U_n \cdot U_n^{-1} \subset \exp B_{(1-\varepsilon)r_n} \subset V_n$, and it is easy to see that we may choose $C' = 8 \cdot 3^k$.

2. The reader is warned that much of the terminology of this section was discussed in ([6]); that discussion will not be repeated in all detail.

NOTATION 2.1. Let $X$ be a compact Hausdorff space, and let $G$ be a compact Hausdorff topological group. Suppose $(G, X)$ is a (left) transformation group (thus there is a continuous map $\Phi: G \times X \to X: (g, x) \to g \cdot x$ satisfying (i) $\mathrm{idy} \cdot x = x$; (ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x (x \in X; g, g_1, g_2 \in G)$). Suppose also that $G$ acts *freely* (thus $g \cdot x = x \Rightarrow g = \mathrm{idy}\, (g \in G, x \in X)$). Let $Y = X/G$ be the space of $G$-orbits, with the quotient topology; let $\pi_0: X \to Y$ be the canonical projection. Let $\gamma$ be normalized Haar measure on $G$, and fix a Radon measure $\nu_0$ on $Y$. Let $M^\infty(Y, \nu_0)$ be the algebra of all bounded $\nu_0$-measurable complex functions on $Y$, and let $L^\infty(Y, \nu_0)$ be the (usual) space of equivalence classes in $M^\infty(Y, \nu_0)$.

DEFINITION 2.2. The *Haar lift* $\mu$ of $\nu_0$ is defined as follows: $\mu(f) = \int_Y \left( \int_G f(g \cdot x) d\gamma(g) \right) d\nu_0(y)$ for each $f \in C(X)$.

DEFINITION 2.3. Let $\rho_0$ be a fixed strong lifting ([6], 1.4; see the references given there) of $M^\infty(Y, \nu_0)$. Let $\rho$ be a linear lifting of $M^\infty(X, \mu)$. Note that $M^\infty(Y, \nu_0)$ may be embedded in $M^\infty(X, \mu)$ via $f \to f \circ \pi$. Say $\rho$ *extends* $\rho_0$ if $\rho|_{M^\infty(Y, \nu_0)} = \rho_0$. Say $\rho$ *commutes with* $G$ if

$$\rho(f \cdot g)(x) = \rho(f)(g \cdot x)(g \in G, x \in X, f \in M^\infty(X, \mu)) ;$$

here $(f \cdot g)(x) \equiv f(g \cdot x)$.

The following theorem was proved in ([6]) subject to various additional assumptions. We prove it here in full generality.

THEOREM 2.4. *Suppose $(G, X)$ is a free left transformation group. Let $\rho_0$ be a strong lifting of $M^\infty(Y, \nu_0)$. Then there exists a strong lifting $\rho$ of $M^\infty(X, \mu)$ which extends $\rho_0$ and commutes with*

*G, where $\mu$ is the Haar lift of $\nu_0$.*

More notation is necessary before we can discuss the proof of 2.4.

NOTATION 2.5. Let $H$ be a closed, normal, real Lie subgroup of $G$. Let $Z = X/H$, and let $\pi: X \to Z$ be the projection. Note $(G/H, Z)$ is a free left transformation group. Write $g \cdot z$ for $(gH) \cdot z (g \in G, z \in Z)$. Define a Radon measure $\nu$ on $Z$ by $\nu = \pi(\mu)$. Let $\lambda$ be normalized Haar measure on $H$. For each $z \in Z$, let $\lambda_z$ be the Radon measure on $X$ defined by $\lambda_z(f) = \int_H f(h \cdot x) d\lambda(h)$ for one (hence all) $x \in \pi^{-1}(z)$. Then $\mu(f) = \int_Z \lambda_z(f) d\nu(z)$ for all $f \in C(X)$.

It can be shown that 2.4 follows from 2.6 below. See the paragraphs under "Proof of 2.2, using 2.7" in ([6]), and the reference given there. See also the proofs of Theorems 2 and 3 in ([5], Chpt. IV).

THEOREM 2.6. *Let $H, Z, \nu, \pi$ be as in 2.5, and suppose there is a strong lifting $\delta$ of $M^\infty(Z, \nu)$ which commutes with $G/H$. Then there is a strong lifting $\rho$ of $M^\infty(X, \mu)$ which extends $\delta$ and commutes with $G$.*

To prove 2.6, we need only revise the proof of Proposition 3.11 in ([6]). For each $z_0 \in Z$ and $f \in M^\infty(X, \mu)$, define $R^f(z_0)$ as in ([6], 3.3–3.5). Thus $R^f(z_0)$ is an element of $L^\infty(X, \lambda_{z_0})$. Abusing notation, we think of $R^f(z_0)$ as a function on $\pi^{-1}(z_0)$. We repeat Proposition 3.9 of ([6]):

PROPOSITION 2.7. $R^{f \cdot g}(z_0)(h \cdot x_0) = R^f(g \cdot z_0)(ghg^{-1} \cdot gz_0)(x_0 \in X, z_0 = \pi(x_0), h \in H, g \in G)$.

DEFINITION 2.8. Let $(V_n)_{n=1}^\infty$ be the $D'$-sequence of § 1. Let $x_0 \in X$, $z_0 = \pi(x_0)$. As in ([6], 3.10, Case I), define

$$T_n^f(x_0) = \frac{1}{\lambda(V_n)} \int_X R^f(z_0)(\bar{x}) \psi_{V_n \cdot x_0}(\bar{x}) d\lambda_{z_0}(\bar{x})$$
$$= \frac{1}{\lambda(V_n)} \int_H R^f(z_0)(hx_0) \psi_{V_n}(h) d\lambda(h)$$

(here $\psi$ denotes characteristic function).

PROPOSITION 2.9. $T_n^{f \cdot g}(x_0) = T_n^f(g \cdot x_0)(g \in G, x_0 \in X)$.

*Proof.*

$$T_n^{f\cdot g}(x_0) = \frac{1}{\lambda(V_n)}\int_H R^{f\cdot g}(z_0)(h\cdot x_0)\psi_{V_n}(h)d\lambda(h)$$

$$= \text{(by 2.7 above) } \frac{1}{\lambda(V_n)}\int_H R^f(g\cdot z_0)(ghg^{-1}\cdot gx_0)\psi_{V_n}(h)d\lambda(h)$$

$$= \text{(by ([2], 28.72e))} \frac{1}{\lambda(V_n)}\int_H R^f(g\cdot z_0)(h\cdot gx_0)\psi_{gV_ng^{-1}}(h)d\lambda(h)$$

$$= T_n^f(g\cdot x_0).$$

2.10. *Proof of* 2.6. Combine the following: (i) the just-proved 2.9; (ii) the reasoning of the Case I portions of ([6], 3.12, 3.13, and 3.14); (iii) ([6], 3.15).

## References


1. R. Ewards and E. Hewitt, *Pointwise limits for sequences of convolution operators*, Acta Math., 113 (1965), 181-218.
2. E. Hewitt and K. Ross, *Abstract Harmonic Analysis, Vol. II*, Springer-Verlag, New York, Heidelberg, Berlin, 1970.
3. A. Ionescu-Tulcea, *On the lifting property* (*V*), Annals of Math. Stat., **36** (1965), 819-828.
4. A. and C. Ionescu-Tulcea, *On the existence of a lifting . . . locally compact group*, Proc. Fifth Berk. Symp. Math. Stat. and Prob., Vol. 2, part 1, pp. 63-97.
5. ———, *Topics in the Theory of Lifting*, Springer-Verlag, New York, 1969.
6. R. Johnson, *Existence of a strong lifting commuting with a compact group of transformations*, to appear in Pacific J. Math.
7. A. A. Kirillov, *Elements of the Theory of Representations*, translated by Edwin Hewitt, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
8. D. Montgomery and L. Zippin, *Topological Transformation Groups*, Interscience, New York, 1955.
9. A. Sagle and R. Walde, *Introduction to Lie Groups and Lie Algebras*, Pure and Applied Mathematics Series, Academic Press, New York, 1973.


Received July 13, 1978 and in revised form January 11, 1979.

University of Southern California
Los Angeles, CA 90007

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $84.00 a year (6 Vols., 12 issues). Special rate: $42.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

# Pacific Journal of Mathematics

Vol. 82, No. 2 February, 1979