

# Pacific Journal of Mathematics

**EXISTENCE OF A STRONG LIFTING COMMUTING WITH A  
COMPACT GROUP OF TRANSFORMATIONS. II**

RUSSELL ALLAN JOHNSON

## EXISTENCE OF A STRONG LIFTING COMMUTING WITH A COMPACT GROUP OF TRANSFORMATIONS II

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Let  $G$  be a locally compact group with left Haar measure  $\gamma$ . The well-known "Theorem LCG" of A. and C. Ionescu-Tulcea states that there is a strong lifting of  $M^\infty(G, \gamma)$  commuting with left translations. Our purpose here is to prove a generalization of this theorem in case  $G$  is compact. Thus let  $(G, X)$  be a free left transformation group with  $X$  and  $G$  compact. Let  $\nu_0$  be a Radon measure on  $Y=X/G$ , and let  $\mu$  be the Haar lift of  $\nu_0$ . Let  $\rho_0$  be a strong lifting of  $M^\infty(Y, \nu_0)$ . We will show that  $M^\infty(X, \mu)$  admits a strong lifting  $\rho$  which extends  $\rho_0$  and commutes with  $G$ .

In [6], the result just stated was proved when  $G$  and  $X$  satisfied certain restrictions. The following theorem, which may be of independent interest, enables us to remove the conditions imposed in [6]: Let  $H$  be a closed normal Lie subgroup of a compact group  $G$ ; then there is a  $D'$  sequence (see 1.2 and [1] in  $H$ , consisting of compact neighborhoods  $V_n (n \geq 1)$  of the identity, such that  $g^{-1}V_n g = V_n$  for all  $g \in G$ .

### 1.

NOTATION 1.1. Let  $G$  be a compact topological group,  $H$  a closed, normal, real Lie subgroup. Let  $\gamma$  be normalized Haar measure on  $G$ , and let  $\lambda$  be normalized Haar measure on  $H$ . For each  $g \in G$ , define  $\alpha_g: H \rightarrow H: h \rightarrow g^{-1}hg$ . Let  $\mathfrak{H}$  be the Lie algebra of  $H$ ; let  $\exp: \mathfrak{H} \rightarrow H$  be the exponential map.

DEFINITION 1.2. ([1]). A  $D'$ -sequence in  $H$  is a sequence  $(W_n)_{n=1}^\infty$  of  $\lambda$ -measurable subsets of  $H$  such that (i)  $W_n \supset W_{n+1} (n \geq 1)$ ; (ii)  $0 < \lambda(W_n W_n^{-1}) < C \cdot \lambda(W_n)$  for some  $C > 0$  and all  $n$ ; (iii) every neighborhood of idy ( $\equiv$  identity)  $\in H$  contains some  $W_n$ .

PROPOSITION 1.3. *There is a  $D'$ -sequence  $(V_n)_{n=1}^\infty$  in  $H$ , consisting of compact neighborhoods of idy, such that  $g^{-1}V_n g = V_n (n \geq 1, g \in G)$ .*

*Proof.* Let  $W$  be a neighborhood of 0 in  $\mathfrak{H}$  such that  $\exp|_W$  is a diffeomorphism onto  $\exp(W) \subset H_0$ , the identity component of  $H$ . Define  $\log$  to be the inverse of  $\exp|_W$ . There is a neighborhood  $N \subset \exp(W)$  of idy such that  $g^{-1}Ng \subset W (g \in G)$ . Let  $\varphi_g(x) = \log \circ \alpha_g \circ$

$\exp(x) = \log(g^{-1} \cdot \exp(x) \cdot g)$  for all  $x \in W_1 = \log(N)$ . Then  $\varphi_g: W_1 \rightarrow W$ , and  $\varphi_g(0) = 0(g \in G)$ .

Each map  $\alpha_g$  is a continuous isomorphism of  $H$ , hence is analytic ([9], Theorem 5.22). Let  $\text{Ad}_g: \mathfrak{G} \rightarrow \mathfrak{G}$  be the derivative at  $\text{id}_y \in H$  of  $\alpha_g$ . Then  $\text{Ad}_g(x) = D\varphi_g(0) \cdot x(x \in \mathfrak{G})$ . The map  $g \rightarrow \text{Ad}_g$  is a homomorphism of  $G$  into  $GL(\mathfrak{G})$ . We show that it is continuous. Let  $G_0 = \{g \in G \mid g^{-1}hg = h \text{ for all } h \in H_0\}$ . Then  $G_0$  is a closed normal subgroup of  $G$ . The group  $G/G_0$  acts effectively on  $H_0$  via the map  $\eta: G/G_0 \times H_0 \rightarrow H_0: (gG_0, h) \rightarrow g^{-1}hg$ . Therefore  $G/G_0$  is a Lie group, and the map  $\eta$  is analytic ([8], pp. 208, 212, 213). It follows that  $g \rightarrow \text{Ad}_g$  is continuous.

Let  $\langle , \rangle_1$  be an inner product on  $\mathfrak{G}$ . Define an inner product  $\langle , \rangle$ , invariant under each  $\text{Ad}_g$ , by

$$\langle x, y \rangle = \int_G \langle \text{Ad}_g(x), \text{Ad}_g(y) \rangle_1 d\gamma(g)(x, y \in \mathfrak{G}).$$

Observe that, if  $B_r = \{x \in \mathfrak{G} \mid \|x\| \leq r, \text{ where } \|x\|^2 = \langle x, x \rangle\}$ , then  $\text{Ad}_g(B_r) = B_r(g \in G)$ . Also observe that, if  $m$  is a Lebesgue measure on  $\mathfrak{G}$ , then there is a constant  $\beta$  such that  $m(B_r) = \beta r^k$ , where  $k = \dim H$ .

Consider the measure  $\lambda|_{\exp W}$ . By ([7], Corollary 2, p. 106), there is a Lebesgue measure  $m$  on  $\mathfrak{G}$  and an analytic function  $\rho: W \rightarrow \mathbb{R}$ , satisfying  $\rho(0) = 1$ , such that  $\lambda(\exp B) = \int_B \rho(x) dm(x)$  for each Borel set  $B \subset W$ . Let  $W_2$  be a neighborhood of  $0 \in \mathfrak{G}$  such that  $1/2 \leq \rho(x) \leq 2(x \in W_2)$ .

Now let  $0 < \varepsilon < 1$  satisfy  $(1-\varepsilon)^k > 1/2(k = \dim H)$ . Recall that  $\varphi_g(0) = 0$  for all  $g \in G$ , that  $\text{Ad}_g(x) = D\varphi_g(0) \cdot x$ , that  $G$  is compact, and that  $(gG_0, x) \rightarrow \varphi_g(x): G/G_0 \times W_2 \rightarrow W$  is analytic. We can therefore find  $r' > 0$  such that

(\*)  $\|\varphi_g(x) - \text{Ad}_g(x)\| < \varepsilon\|x\|$  for all  $g \in G$  if  $\|x\| \leq r'$  (recall  $\|x\|^2 = \langle x, x \rangle$ ). Choose  $r_0 \leq r'$  such that  $B_{3r_0} \subset W_2$  and  $\exp(B_r) \cdot \exp(B_r) \subset \exp B_{3r}$  if  $r \leq r_0$ . Let  $r_n = r_0/n$ . Define  $C_n = \bigcap_{g \in G} \varphi_g(B_{r_n})$ , and let  $V_n = \exp(C_n)$ . By (\*),  $B_{(1-\varepsilon)r_n} \subset C_n$  for each  $n$ . Hence  $V_n$  is a compact neighborhood of  $\text{id}_y$  for each  $n(n \geq 1)$ .

We show that  $(V_n)_{n=1}^\infty$  is the desired  $D'$ -sequence in  $H$ . First note that  $g^{-1}V_n g = \alpha_g \circ \exp(C_n) = \exp \circ \varphi_g(C_n) = \exp C_n = V_n$  for all  $g \in G$ . Next, observe that  $V_n V_n^{-1} = \exp(C_n) \cdot \exp(-C_n) \subset \exp(B_{r_n}) \cdot \exp(B_{r_n}) \subset \exp B_{3r_n}$ . So  $\exp(B_{(1-\varepsilon)r_n}) \subset V_n \subset V_n V_n^{-1} \subset \exp B_{3r_n}$ . So, on the one hand,  $\lambda(V_n V_n^{-1}) \leq \lambda(\exp B_{3r_n}) = \int_{B_{3r_n}} \rho(x) dm(x) \leq 2 \cdot \beta \cdot 3^k \cdot (r_n)^k$ , while on the other hand,

$$\lambda(V_n) \geq \int_{B_{(1-\varepsilon)r_n}} \rho(x) dm(x) \geq 1/2\beta(1-\varepsilon)^k (r_n)^k > 1/4\beta(r_n)^k.$$

Hence  $\lambda(V_n V_n^{-1}) \leq 8 \cdot 3^k \lambda(V_n)$ , so (ii) of 1.2 is satisfied with  $C = 8 \cdot 3^k$ .

It is easy to see that  $(V_n)_{n=1}^\infty$  satisfies (i) and (iii) of 1.2. This completes the proof of 1.3.

REMARK 1.4. The sequence  $(V_n)_{n=1}^\infty$  is also a  $D'$ -sequence ([1]); that is, each  $V_n$  contains a subset  $U_n$  such that  $U_n \cup U_n U_n^{-1} \subset V_n$ , and  $\lambda(V_n) < C'\lambda(U_n)$  for some constant  $C'$  ( $n \geq 1$ ). To see this, let  $s_n = (1 - \varepsilon)r_n/3$ , and let  $U_n = \exp B_{s_n}$ . Then  $U_n \cdot U_n^{-1} \subset \exp B_{(1-\varepsilon)r_n} \subset V_n$ , and it is easy to see that we may choose  $C' = 8 \cdot 3^k$ .

2. The reader is warned that much of the terminology of this section was discussed in ([6]); that discussion will not be repeated in all detail.

NOTATION 2.1. Let  $X$  be a compact Hausdorff space, and let  $G$  be a compact Hausdorff topological group. Suppose  $(G, X)$  is a (left) transformation group (thus there is a continuous map  $\Phi: G \times X \rightarrow X: (g, x) \rightarrow g \cdot x$  satisfying (i)  $\text{id}_y \cdot x = x$ ; (ii)  $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$  ( $x \in X; g, g_1, g_2 \in G$ )). Suppose also that  $G$  acts *freely* (thus  $g \cdot x = x \Rightarrow g = \text{id}_y$  ( $g \in G, x \in X$ )). Let  $Y = X/G$  be the space of  $G$ -orbits, with the quotient topology; let  $\pi_0: X \rightarrow Y$  be the canonical projection. Let  $\gamma$  be normalized Haar measure on  $G$ , and fix a Radon measure  $\nu_0$  on  $Y$ . Let  $M^\infty(Y, \nu_0)$  be the algebra of all bounded  $\nu_0$ -measurable complex functions on  $Y$ , and let  $L^\infty(Y, \nu_0)$  be the (usual) space of equivalence classes in  $M^\infty(Y, \nu_0)$ .

DEFINITION 2.2. The *Haar lift*  $\mu$  of  $\nu_0$  is defined as follows:  $\mu(f) = \int_Y \left( \int_G f(g \cdot x) d\gamma(g) \right) d\nu_0(y)$  for each  $f \in C(X)$ .

DEFINITION 2.3. Let  $\rho_0$  be a fixed strong lifting ([6], 1.4; see the references given there) of  $M^\infty(Y, \nu_0)$ . Let  $\rho$  be a linear lifting of  $M^\infty(X, \mu)$ . Note that  $M^\infty(Y, \nu_0)$  may be embedded in  $M^\infty(X, \mu)$  via  $f \rightarrow f \circ \pi$ . Say  $\rho$  *extends*  $\rho_0$  if  $\rho|_{M^\infty(Y, \nu_0)} = \rho_0$ . Say  $\rho$  *commutes with*  $G$  if

$$\rho(f \cdot g)(x) = \rho(f)(g \cdot x)(g \in G, x \in X, f \in M^\infty(X, \mu));$$

here  $(f \cdot g)(x) \equiv f(g \cdot x)$ .

The following theorem was proved in ([6]) subject to various additional assumptions. We prove it here in full generality.

THEOREM 2.4. *Suppose  $(G, X)$  is a free left transformation group. Let  $\rho_0$  be a strong lifting of  $M^\infty(Y, \nu_0)$ . Then there exists a strong lifting  $\rho$  of  $M^\infty(X, \mu)$  which extends  $\rho_0$  and commutes with*

$G$ , where  $\mu$  is the Haar lift of  $\nu_0$ .

More notation is necessary before we can discuss the proof of 2.4.

NOTATION 2.5. Let  $H$  be a closed, normal, real Lie subgroup of  $G$ . Let  $Z = X/H$ , and let  $\pi: X \rightarrow Z$  be the projection. Note  $(G/H, Z)$  is a free left transformation group. Write  $g \cdot z$  for  $(gH) \cdot z (g \in G, z \in Z)$ . Define a Radon measure  $\nu$  on  $Z$  by  $\nu = \pi(\mu)$ . Let  $\lambda$  be normalized Haar measure on  $H$ . For each  $z \in Z$ , let  $\lambda_z$  be the Radon measure on  $X$  defined by  $\lambda_z(f) = \int_H f(h \cdot x) d\lambda(h)$  for one (hence all)  $x \in \pi^{-1}(z)$ . Then  $\mu(f) = \int_Z \lambda_z(f) d\nu(z)$  for all  $f \in C(X)$ .

It can be shown that 2.4 follows from 2.6 below. See the paragraphs under "Proof of 2.2, using 2.7" in ([6]), and the reference given there. See also the proofs of Theorems 2 and 3 in ([5], Chpt. IV).

THEOREM 2.6. Let  $H, Z, \nu, \pi$  be as in 2.5, and suppose there is a strong lifting  $\delta$  of  $M^\infty(Z, \nu)$  which commutes with  $G/H$ . Then there is a strong lifting  $\rho$  of  $M^\infty(X, \mu)$  which extends  $\delta$  and commutes with  $G$ .

To prove 2.6, we need only revise the proof of Proposition 3.11 in ([6]). For each  $z_0 \in Z$  and  $f \in M^\infty(X, \mu)$ , define  $R^f(z_0)$  as in ([6], 3.3-3.5). Thus  $R^f(z_0)$  is an element of  $L^\infty(X, \lambda_{z_0})$ . Abusing notation, we think of  $R^f(z_0)$  as a function on  $\pi^{-1}(z_0)$ . We repeat Proposition 3.9 of ([6]):

PROPOSITION 2.7.  $R^{f \cdot g}(z_0)(h \cdot x_0) = R^f(g \cdot z_0)(ghg^{-1} \cdot gz_0)(x_0 \in X, z_0 = \pi(x_0), h \in H, g \in G)$ .

DEFINITION 2.8. Let  $(V_n)_{n=1}^\infty$  be the  $D'$ -sequence of §1. Let  $x_0 \in X, z_0 = \pi(x_0)$ . As in ([6], 3.10, Case I), define

$$\begin{aligned} T_n^f(x_0) &= \frac{1}{\lambda(V_n)} \int_x R^f(z_0)(\bar{x}) \psi_{V_n \cdot x_0}(\bar{x}) d\lambda_{z_0}(\bar{x}) \\ &= \frac{1}{\lambda(V_n)} \int_H R^f(z_0)(hx_0) \psi_{V_n}(h) d\lambda(h) \end{aligned}$$

(here  $\psi$  denotes characteristic function).

PROPOSITION 2.9.  $T_n^{f \cdot g}(x_0) = T_n^f(g \cdot x_0)(g \in G, x_0 \in X)$ .

*Proof.*

$$\begin{aligned}
 T_n^{f \cdot g}(x_0) &= \frac{1}{\lambda(V_n)} \int_H R^{f \cdot g}(z_0)(h \cdot x_0) \psi_{V_n}(h) d\lambda(h) \\
 &= (\text{by 2.7 above}) \frac{1}{\lambda(V_n)} \int_H R^f(g \cdot z_0)(ghg^{-1} \cdot gx_0) \psi_{V_n}(h) d\lambda(h) \\
 &= (\text{by ([2], 28.72e)}) \frac{1}{\lambda(V_n)} \int_H R^f(g \cdot z_0)(h \cdot gx_0) \psi_{gV_n g^{-1}}(h) d\lambda(h) \\
 &= T_n^f(g \cdot x_0).
 \end{aligned}$$

2.10. *Proof of 2.6.* Combine the following: (i) the just-proved 2.9; (ii) the reasoning of the Case I portions of ([6], 3.12, 3.13, and 3.14); (iii) ([6], 3.15).

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