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ON THE ORDER OF DIRICHLET L -FUNCTIONS

GRIGORI ABRAMOVICH KOLESNIK

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G. KOLESNIK

1. **Introduction.** Let $L(s, \chi)$ be a Dirichlet L -function, where χ is a nonprincipal character (mod q) and $s = \sigma + it$. A standard estimate for $L(s, \chi)$ based on bounds for $\zeta(s, w)$, is

$$(1) \quad |L(s, \chi)| \leq C_1(\varepsilon) \cdot \tau^{\varepsilon(1-\sigma)+\varepsilon} q^{1-\sigma}, \quad \frac{1}{2} \leq \sigma \leq 1,$$

where $\tau = |t| + 2$, $c = 1/6$ (see, for example, Prachar [5, (4.12)]), and in fact, c can be replaced by a constant $< 1/6$. An immediate application of Richert's work [6] gives

$$(2) \quad |L(s, \chi)| \leq C_1 \tau^{100(1-\sigma)^{3/2}} q^{1-\sigma} \log^{2/3} \tau, \quad \frac{1}{2} \leq \sigma \leq 1,$$

which is better than (1) if σ is near 1.

Another estimate can easily be obtained from $|L(1 + it, \chi)| \leq C_2 \log \tau q$ and the functional equation of $L(s, \chi)$ as follows. First,

$$\begin{aligned} |L(it, \chi)| &= 2 \cdot |(2\pi)^{it-1} q^{1/2-it} \\ &\times \cos \frac{1}{2} \pi \left(1 - it + \frac{1}{2} - \frac{1}{2} \bar{\chi}(-1) \right) \Gamma(1 - it) L(1 - it, \bar{\chi})| \\ &\leq C_3 \sqrt{\tau q} \log \tau q. \end{aligned}$$

Now the convexity principle yields for

$$(3) \quad |L(s, \chi)| \leq (C_3 \sqrt{\tau q} \log \tau q)^{1-\sigma} \cdot (C_2 \log \tau q)^\sigma \leq C_4 (\tau q)^{1/2(1-\sigma)} \times \log \tau q, \quad 0 \leq \sigma \leq 1.$$

Neglecting dependence on τ , Davenport [2], improved (3):

$$(4) \quad |L(s, \chi)| \leq C_2(\tau) q^{1/2(1-\sigma)}, \quad 0 \leq \sigma \leq 1.$$

Also, Burgess [1] improved (4) by establishing

$$|L(s, \chi)| \leq C_1(\varepsilon, \tau) q^{3/8(1-\sigma)+\varepsilon}, \quad \frac{1}{2} \leq \sigma \leq 1.$$

By examining Burgess' proof, it can be seen that the constant $C(\varepsilon, \tau)$ can be taken to be $C_2(\varepsilon) \pi^{2(1-\sigma)}$ and his result can be further sharpened to yield

$$(5) \quad |L(s, \chi)| \leq C_6 \tau^{2(1-\sigma)} q^{3/8(1-\sigma)} C^\omega \log \tau, \quad \frac{1}{2} \leq \sigma \leq 1,$$

where $\omega = \log q / \log \log q$. The estimates (3), (4), and (5) are better than (1) if q is large compared to τ .

For $\sigma = 1/2$, the previous estimates were improved by Fujii, Gallagher and Montgomery, [3], who showed that if P is a fixed set of primes and q is composed only of primes in P , then

$$(6) \quad \left| L\left(\frac{1}{2} + it, \chi\right) \right| \leq C(\varepsilon, P)(\tau q)^{1/6+\varepsilon}.$$

In this paper we prove two more estimates which imply (1), (4), and (5) and which are better than (2), (3), and (6) in some range of σ , τ , and q . We prove:

THEOREM 1. *Let χ be a nonprincipal character (mod q). Let $1/2 \leq \sigma \leq 1$, $\tau = |t| + 2$ and $\omega = \log q / \log \log q$. Then*

$$(7) \quad |L(s, \chi)| \ll \tau^{-\sigma} q^{3/8(1-\sigma)} C^\omega \log \tau,$$

where C is some absolute constant.

THEOREM 2. *Let χ be a character (mod q). Let $1/2 \leq \sigma \leq 1$ and $\tau = |t| + 2$. Then*

$$(8) \quad |L(s, \chi)| \ll \tau^{35/108(1-\sigma)} q^{1-\sigma} \log^3 \tau q.$$

In particular, (7) and (8) imply

$$\left| L\left(\frac{1}{2} + it, \chi\right) \right| \ll \sqrt{\tau} q^{3/16} C^\omega \log \tau$$

and

$$\left| L\left(\frac{1}{2} + it, \chi\right) \right| \ll \tau^{35/216} \sqrt{q} \log^3 \tau q.$$

The estimates of $L(s, \chi)$ for $\sigma \in [0, 1/2]$ can be obtained by using (7) or (8) and the functional equation of $L(s, \chi)$.

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2. Notation.

$$e(f(x)) = \exp(2\pi i f(x)).$$

$$\omega = \log q / \log \log q.$$

$$s = \sigma + it, \quad \frac{1}{2} \leq \sigma \leq 1.$$

$$\tau = |t| + 2.$$

C denotes some appropriate absolute constant, not always the same.

3. Application of the estimate of Burgess. In this section we will show that

$$|L(s, \chi)| \ll \pi^{1-\sigma} q^{3/8(1-\sigma)} C^\omega \log^3 \tau .$$

We need the following result of E. Bombieri:

LEMMA. *Let N and m be nonnegative integers. Let α_j, β_j be numbers such that $|\alpha_j - \beta_j| \leq (2\pi m N^j)^{-1}$ for $1 \leq j \leq m$, and let $f(x) = \alpha_1 x + \dots + \alpha_m x^m$, $g(x) = \beta_1 x + \dots + \beta_m x^m$. Let c_1, c_2, \dots be complex, and let*

$$S(\bar{\alpha}, N) = \max_{1 \leq n_1 < N} \left| \sum_{1 \leq n \leq N_1} c_n e(f(n)) \right| ,$$

where $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$. Then $S(\bar{\beta}, N) \leq 6S(\bar{\alpha}, N)$.

Proof. For every $N_1 \in [1, N]$ we have:

$$\begin{aligned} \sum_{1 \leq n \leq N_1} c_n e(g(n)) &= \sum_{1 \leq n \leq N_1} c_n e(f(n)) \prod_{j=1}^m e((\beta_j - \alpha_j)n^j) \\ &= \sum_{k_1, \dots, k_m=0}^{\infty} \left(\prod_{j=1}^m \frac{\{2\pi i(\beta_j - \alpha_j)\}^{k_j}}{k_j!} \right) \sum_{1 \leq n \leq N_1} c_n n^{m k_m + \dots + k_1} e(f(n)) . \end{aligned}$$

Using Abel's summation formula, we obtain:

$$\begin{aligned} S(\bar{\beta}, N) &\leq \sum_{k_1, \dots, k_m=0}^{\infty} \prod_{j=1}^m \frac{|2\pi(\beta_j - \alpha_j)|^{k_j}}{k_j!} \cdot N^{m k_m + \dots + k_1} \cdot 2S(\bar{\alpha}, N) \\ &\leq 2S(\bar{\alpha}, N) \cdot \sum_{k_1, \dots, k_m=0}^{\infty} \prod_{j=1}^m \frac{|(2\pi(\beta_j - \alpha_j)N^j)|^{k_j}}{k_j!} \\ &\leq 2S(\bar{\alpha}, N) \left(\sum_{k=0}^{\infty} m^{-k}/k! \right)^m \leq 6S(\bar{\alpha}, N) . \end{aligned}$$

LEMMA 2. *Let $q \geq 2$ and let M, N be integers. Let χ be a primitive character (mod q). Then*

$$\left| \sum_{1 \leq n \leq N} \chi(n + M) \right| \leq \sqrt{N} q^{3/16} C^\omega .$$

This lemma can be proven similarly to Theorem 2, [1].

LEMMA 3. *Let q and N be integers such that $q \geq 2$ and $N \leq \tau q$. Let χ be a primitive character (mod q). Then*

$$|S = \max_{N \leq N_1 \leq 2N} \left| \sum_{N+1 \leq n \leq N_1} \chi(n) n^{-it} \right| \ll \sqrt{N\tau} \log \tau \cdot q^{3/16} C^\omega .$$

Proof. We can obviously suppose that $\tau \log \tau q \leq N$ since otherwise the estimate is trivial. Taking $H = [N(\tau \log \tau q)^{-1}]$ and $m = [\log \tau q]$, and dividing the sum in S into $\leq 2NH^{-1}$ subsums, we obtain:

$$|S| \leq 2NH^{-1} \max_{N \leq M \leq 2N} \max_{1 \leq H_1 \leq H} \left| \sum_{M+1 \leq n \leq M+H_1} \chi(n)n^{-it} \right|.$$

For every M and H_1 in the above range, we get

$$\begin{aligned} (6) \quad \sum_{M+1 \leq n \leq M+H_1} \chi(n)n^{-it} &= \left| \sum_{1 \leq n \leq H_1} X(n+M) \left(\frac{n+M}{M}\right)^{-it} \right| \\ &\leq \left| \sum_{1 \leq n \leq H_1} \chi(n+M)e \left(-\frac{t}{2\pi} \left\{ \frac{n}{M} - \frac{n^2}{2M^2} + \dots + \frac{(-1)^m \cdot n^m}{mM^m} \right\} \right) \right| \\ &\quad + \frac{|t|H^{m+2}}{M^{m+1}}. \end{aligned}$$

Let $\beta_j = 0$ and $\alpha_j = (-1)^j t / 2\pi j M^j$. Then for $1 \leq j \leq m$ $|\alpha_j - \beta_j| = |t| \cdot (2\pi j M^j)^{-1} \leq (2\pi m H^j)^{-1}$. Applying Lemmas 1 and 2, we obtain:

$$\begin{aligned} |S| &\leq |2NH^{-1} \max_{N \leq M \leq 2N} \max_{1 \leq H_1 \leq H} \left| \sum_{1 \leq n \leq H} \chi(n+M) \right| + 2 \frac{\tau H^{m+1}}{N^m} \\ &\ll NH^{-1} \sqrt{H} q^{3/16} C^\omega + N\tau(\tau \log \tau q)^{-\log \tau q} \\ &\ll \sqrt{N \cdot \tau \log \tau q} q^{3/16} C^\omega. \end{aligned}$$

From this, the result is easily obtained.

Now we can prove Theorem 1. First, we suppose that χ is primitive. Let $N = [\tau q]$, $M = [\tau q^{3/8}]$, $L = \log(N/M) / \log 2$, $N_l = M2^l$ ($l = 0, \dots, L$). Using Abel's formula, the Polya-Vinogradov estimate for character sums and Lemma 3, we get:

$$\begin{aligned} |L(s, \chi)| &\leq \sum_{n < M} n^{-\sigma} + \left| \sum_{M \leq n \leq N} \chi(n)n^{-\sigma-it} \right| + \left| \sum_{n > N} \chi(n)n^{-s} \right| \\ &\ll M^{1-\sigma} \log M + \sum_{l=0}^L \max_{N_l \leq N_l^1 \leq 2N_l, N_l \leq n \leq N_l^1} \left| \sum \chi(n)n^{-\sigma-it} \right| \\ &\quad + \sum_{n > N} \tau n^{-\sigma-1} \left| \sum_{N \leq n \leq n} \chi(n) \right| \\ &\ll M^{1-\sigma} \log M + \sum_{l=0}^L N_l^{-\sigma} \max_{N_l \leq N_l^1 \leq 2N_l, N_l \leq n \leq N_l^1} \left| \sum \chi(n)^{-it} \right| \\ &\quad + \tau \sqrt{q} N^{-\sigma} \log q \\ &\ll M^{1-\sigma} \log M + \sum_{l=0}^L N_l^{1/2-\sigma} \sqrt{\tau} q^{3/16} C^\omega \sqrt{\log \tau} + \tau \sqrt{q} N^{-\sigma} \log q \\ &\ll M^{1-\sigma} \log M + LM^{1/2-\sigma} \sqrt{\tau} q^{3/16} C^\omega \sqrt{\log \tau} + \tau \sqrt{q} N^{-\sigma} \log q \\ &\ll \tau^{1-\sigma} q^{3/8(1-\sigma)} C^\omega \log \tau. \end{aligned}$$

If X is not primitive, then there is a $q_1|q$ and a primitive

character $\chi_1 \pmod{q_1}$, associated with χ , such that we can write (see, for example, [5, (6.12)]):

$$|L(s, \chi)| = |L(s, \chi_1)| \prod_{p|q} \left| 1 - \frac{\chi_1(p)}{p^s} \right| \leq |L(s, \chi_1)| \cdot \prod_{p|q} 2 \leq |L(s, \chi_1)| \cdot 2^\omega,$$

and the theorem follows.

4. The proof of Theorem 2. To prove Theorem 2, we need two lemmas.

LEMMA 4. Let $t \geq 0$, $0 \leq a \leq 1$, and let X and X_1 be integers such that $0 < X \leq X_1 \leq 2X \leq \tau^{143/108}$. Then

$$S_1 \equiv \sum_{X \leq x \leq X_1} e(t \log(x+a)) \ll \sqrt{X} \tau^{35/216} \log^2 \tau.$$

Proof. If $X \leq \sqrt{\tau}$, then the result can be proven similarly to Corollary 2, [4]. The same method yields

$$(9) \quad \sum_{X \leq x \leq X_1} e(t \log x - ax) \ll \sqrt{X} \tau^{35/216} \log^2 \tau,$$

for $X \leq \sqrt{\tau}$. If $\sqrt{\tau} \leq X \leq \tau^{143/108}$, then, by Lemma 3 of [4]

$$|S_1| \leq \sum_{t/(X_1+a) \leq n \leq t/(X+a)} \frac{\sqrt{t}}{n} |e(t \log n - an)| + O(X\tau^{-1/2}).$$

Here $t/(X+a) \leq \sqrt{\tau}$. With the use of Abel's inequality, (9) yields the result for $\sqrt{\tau} \leq X \leq \tau^{143/108}$.

LEMMA 5. Let $1/2 \leq \sigma \leq 1$, $t \geq 1$ and $0 \leq a \leq 1$. Then

$$\zeta(s, a) \equiv \sum_{n=0}^{\infty} (n+a)^{-s} \ll a^{-\sigma} + \tau^{35(1-\sigma)/108} \log^3 \tau.$$

Proof. Let $N = \tau^{143/108}$. Using the Euler-Maclaurin formula [see, for example, [5], (1.7), p. 372)], we obtain similarly to [5], (5.8), p. 114:

$$\begin{aligned} \zeta(s, a) - \sum_{n=0}^{N-1} (n+a)^{-s} &= \frac{(N+a)^{1-s}}{1-s} - s \int_N^{\infty} \frac{x - [x]}{(x+a)^{s+1}} dx \\ &= \frac{(N+a)^{1-s}}{1-s} - \frac{1}{2} s \frac{(x - [x])^2}{(x+a)^{s+1}} \Big|_N^{\infty} + \frac{1}{2} s(s+1) \int_N^{\infty} \frac{(x - [x])^2}{(x+a)^{s+2}} dx \\ &\ll 1 + \tau^2 \int_N^{\infty} u^{-\sigma-2} du \leq 1 + \tau^2 \cdot N^{-\sigma-1} \ll \tau^{35(1-\sigma)/108}. \end{aligned}$$

If we denote $M = [\tau^{35/108}]$, $L = [\log(N/M)/\log 2]$, $N_l = M \cdot 2^l$ for $l = 0, \dots, L$ and $N_{L+1} = N$, then we have

$$S \equiv \sum_{n=0}^{N-1} (n+a)^{-s} \ll \sum_{0 < n < M} (n+a)^{-\sigma} + \sum_{0 \leq t \leq L} \left| \sum_{N_l \leq n < N_{l+1}} (n+a)^{-s} \right|.$$

Using Abel's formula and Lemma 4, we obtain:

$$\begin{aligned} S &\ll a^{-\sigma} + M^{1-\sigma} \log M + \sum_{0 \leq t \leq L} N_l^{-\sigma} \max_{N_l \leq N'_l \leq N_{l+1}} \left| \sum_{N_l \leq n \leq N'_l} (n+a)^{-it} \right| \\ &\ll a^{-\sigma} + M^{1-\sigma} \log M + \sum_{0 \leq t \leq L} N_l^{1/2-\sigma} \cdot \tau^{35/216} \log^2 \tau \ll a^{-\sigma} \\ &\quad + \tau^{35(1-\sigma)/108} \log^3 \tau. \end{aligned}$$

This proves the lemma.

To prove Theorem 2, we can obviously suppose $t \geq 1$, otherwise the result follows from (1). Using Lemma 5, we obtain:

$$\begin{aligned} |L(s, \chi)| &= |q^{-s} \sum_{m=1}^q \chi(m) \zeta(s, m/q)| \\ &< q^{-\sigma} \sum_{m=1}^q ((q/m)^\sigma + \tau^{35(1-\sigma)/108} \log^3 \tau) \ll \tau^{35(1-\sigma)/108} q^{1-\sigma} \log^3 \tau q. \end{aligned}$$

Note Added in Proof. We would like to draw attention to a recent paper by D. R. Heath-Brown, "Hybrid bounds for Dirichlet L -function," *Inventiones Mathematicae*, 44 (1978), 149-170, which contains a better result than our Theorem 7.

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