

Pacific Journal of Mathematics

GENERAL PEXIDER EQUATIONS. I. EXISTENCE OF INJECTIVE SOLUTIONS

M. A. MCKIERNAN

GENERAL PEXIDER EQUATIONS (PART I): EXISTENCE OF INJECTIVE SOLUTIONS

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Given open connected Ω , $\tilde{\Omega} \subseteq \mathbf{R}^n$ and given $T: \Omega \rightarrow \mathbf{R}$ continuous, $F: \tilde{\Omega} \rightarrow \mathbf{R}$ strictly monotonic, in each variable separately. The equation is $h \circ T = F \circ \pi$ for the unknowns $h: T(\Omega) \rightarrow \mathbf{R}$, $\pi: \Omega \rightarrow \tilde{\Omega}$ with $\pi = (f_1, \dots, f_n)$ a product mapping - e.g., $h\{T(x, y)\} = F\{f(x), g(y)\}$. If T is one-one in each variable, then any continuous solution π must be injective or constant on Ω ; conversely, if an injective solution π exists then T must be one-one in each variable separately.

1. Introduction. Given a subset $\Omega \subseteq \mathbf{R}^n$ for $n \geq 2$, let Ω_i denote its projection on the i th coordinate axis. By a *product mapping* $\pi: \Omega \rightarrow \tilde{\Omega} \subset \mathbf{R}^n$ is understood the restriction to Ω of a map $(f_1, \dots, f_n): X_1^n \Omega_i \rightarrow \mathbf{R}^n$ defined by n functions $f_i: \Omega_i \rightarrow \tilde{\Omega}_i \subseteq \mathbf{R}$. For given $T: \Omega \rightarrow \mathbf{R}$ and $F: \tilde{\Omega} \rightarrow \mathbf{R}$, equations of the form

$$(1) \quad h\{T(x_1, \dots, x_n)\} = F\{f_1(x_1), \dots, f_n(x_n)\}$$

for the unknowns $h: T(\Omega) \rightarrow \mathbf{R}$ and $\pi: \Omega \rightarrow \tilde{\Omega}$ are generalizations of Pexider equations¹. For the most part the literature concerns the case in which T and F are specified, usually the sum and/or product of the arguments. In [3] C. T. Ng recently gave a uniqueness theorem for continuous solutions π , assuming T continuous but with $F(u_1, \dots, u_n) = u_1 + \dots + u_n$; a generalization to certain topological spaces appears in Ng [4] and [2]. A simple case of (1) was used by J. Lester and the author [5] to characterize Lorentz transformations in \mathbf{R}^n .

2. Formulation of results. Given $\Omega, \tilde{\Omega} \subseteq \mathbf{R}^n$ for $n \geq 2$ and given $T: \Omega \rightarrow \mathbf{R}$, $F: \tilde{\Omega} \rightarrow \mathbf{R}$. Henceforth assume:

- (A-1) T continuous in each variable separately,
- (A-2) F one-to-one in each variable separately,
- (A-3) Ω open and connected.

THEOREM 1. *With (A-1, 2, 3) assume $T \circ h = F \circ \pi$ satisfied on Ω , where $h: T(\Omega) \rightarrow \mathbf{R}$ and where $\pi: \Omega \rightarrow \tilde{\Omega}$ is an injective product mapping. Then T must be strictly monotonic in each variable separately on Ω .*

The existence of an injective solution π then places a severe

¹ For literature see [1]; J. V. Pexider studied $h(x+y) = f(x) + g(y)$.

condition on T ; the following theorems indicate that if continuous solutions π are to exist, injectivity or at least some local one-to-one property of π is to be expected. A function will be called *locally nonconstant* if it is not constant on any open set.

THEOREM 2. *If in addition to (A-1, 2, 3), T is locally nonconstant in each variable separately then for any continuous product map $\pi: \Omega \rightarrow \tilde{\Omega}$ and corresponding $h: T(\Omega) \rightarrow \mathbf{R}$ satisfying $h \circ T = F \circ \pi$ on Ω , either π is also locally nonconstant in each variable separately or π is constant on Ω .*

The following theorem is a partial converse to Theorem 1.

THEOREM 3. *If in addition to (A-1, 2, 3), both T and F are strictly monotonic in each variable separately, then for any continuous product map, $\pi: \Omega \rightarrow \tilde{\Omega}$ and corresponding $h: T(\Omega) \rightarrow \mathbf{R}$ satisfying $h \circ T = F \circ \pi$ on Ω , either π is injective or π is constant on Ω .*

3. **Proof of Theorem 1.** By symmetry it suffices to consider T in its first variable for all choices of the remaining variables, denoted by $X = (x_2, \dots, x_n)$. If (a, X) and (b, X) are elements of Ω with $a \neq b$, then by (A-2), $T(a, X) = T(b, X)$ implies $\pi(a, X) = \pi(b, X)$ for product functions π ; π would not be injective. Hence each X determines a line λ parallel to the x_1 axis and $T(\cdot; X)$ is one-to-one on $\lambda \cap \Omega$. Hence T is one-to-one and continuous in each variable separately. Since Ω was not assumed convex, the domain of $T(\cdot; X)$ for given X need not be connected (in \mathbf{R}) and it remains to prove that T is in fact strictly monotonic in each variable for all choices of the remaining variables (either increasing for all, or decreasing for all). For each point $(x_1, \dots, x_n) \in \Omega$, some open ball around this point is contained in Ω and define $V: \Omega \rightarrow \mathbf{R}^n$ by $V(x_1, \dots, x_n) = (\pm 1, \dots, \pm 1)$ according as T is strictly increasing (+1) or decreasing (-1) in each variable within that open ball. Since V is constant on some neighborhood of each point in Ω , V is continuous on Ω and all of the 2^n sets $V^{-1}(\pm 1, \dots, \pm 1)$ are closed and disjoint. Since Ω is connected, all but one of these sets must be empty.

4. **Proof of Theorem 2.** Consider the two dimensional case $h\{T(x, y)\} = F\{f(x), g(y)\}$, valid on some open connected $\Omega \subset \mathbf{R}^2$; Ω_x, Ω_y denote the projections of Ω on the x and y axes, f and g are continuous on Ω_x and Ω_y respectively. Let $N_\varepsilon(x) :=]x - \varepsilon, x + \varepsilon[$, the open interval.

LEMMA 2. *For (x_0, y_0) in Ω , if f is constant on some $N_\varepsilon(x_0)$*

then g is also constant on some $N_\delta(y_0)$ and conversely.

Proof. Choose $\varepsilon > 0$ sufficiently small so that $N_\varepsilon(x_0) \times N_\varepsilon(y_0) \subset \Omega$ with $f(x) = k$ constant on $N_\varepsilon(x_0)$. Since $T(\cdot, y_0)$ is locally non-constant and continuous, $T(N_\varepsilon(x_0), y_0)$ contains an open interval I ; since $h\{T(x, y_0)\} = F\{k, g(y_0)\}$ is also constant, h must be constant on I . With x_1 chosen in $N_\varepsilon(x_0)$ such that $T(x_1, y_0)$ is in I , so also is $T(x_1, y)$ in I for all y in some $N_\delta(y_0)$; hence $h\{T(x_1, y)\} = F\{k, g(y)\}$ is constant, that is, $g(y)$ is constant by (A-2) for y in $N_\delta(y_0)$. Similarly for the converse.

LEMMA 3. *If f is constant on some closed interval $[a, b] \subset \Omega_x$, $a < b$, then for some $\delta > 0$, f is also constant on $]a - \delta, b + \delta[\subset \Omega_x$. Similarly for g relative to intervals in Ω_y .*

Proof. With $b \in \Omega_x$ so also $(b, y_0) \in \Omega$ for some y_0 and since Ω is open, $[b - \varepsilon, b + \varepsilon] \times [y_0 - \varepsilon, y_0 + \varepsilon] \subset \Omega$ for some $\varepsilon > 0$. Choose $x_0 \in]b - \varepsilon, b[= N(x_0)$, a neighbourhood of x_0 on which f is constant; by Lemma 2, g is constant on some $N(y_0)$. But again $(b, y_0) \in \Omega$ with g constant on $N(y_0)$ implies f constant on some $N(b)$, thus extending $[a, b]$ to $[a, b + \delta[$. Similarly for the end point a and for g relative to Ω_y .

If f is constant on some open interval, so also on the closure in Ω_x of the maximal extension of the interval on which f is constant; this maximal extension must also be open in Ω_x by Lemma 3. Since Ω_x is connected, f must be constant on Ω_x itself. In view of Lemma 2, g will be constant on Ω_y . A similar argument applied to any two of the arguments of π in \mathbf{R}^n proves the theorem.

5. **Proof of Theorem 3².** With T strictly monotonic in each variable separately, T is locally nonconstant in each variable also; the results of § 4 are therefore applicable and it remains only to prove that if π is not injective on Ω , then some f_i is constant on some open set in Ω_i . Consider again the \mathbf{R}^2 case using f, g as before. If $f(a) = f(b)$ for some $a < b$, then for some $a < c < b$, $f(c)$ extremizes f (choosing max. or min. as required) on $[a, b]$ and in every $N_\varepsilon(c)$ two points x_0, x_2 can be found satisfying $f(x_0) = f(x_2)$. With $c \in \Omega_x$ so also $(c, y_0) \in \Omega$ for some $y_0 \in \Omega_y$, and for sufficiently small $\varepsilon > 0$ so also $N_\varepsilon(c) \times \{y_0\} \subset \Omega$. Hence $f(x_0) = f(x_2)$ with $[x_0, x_2] \times \{y_0\} \subset \Omega$; for this x_0, x_2 choose x_1 in the open interval $]x_0, x_2[$ such that $f(x_1)$ extremizes f on $[x_0, x_2]$. Assume $f(x_1) \geq f(x)$ for all

² A similar argument may be found in [3].

$x_0 \leq x \leq x_2$ and note that $x_0 < x_1 < x_2$. Since T is strictly monotonic in each variable assume $T(x_0, y_0) < T(x_1, y_0) < T(x_2, y_0)$ and define $\Gamma_1, \Gamma_2, \Gamma_3 \subset \Omega_y$ as follows: $\Gamma_1 = \{y \mid T(x_0, y) < T(x_1, y) < T(x_2, y)\}$, $\Gamma_2 = \{y \mid T(x_0, y) < T(x_1, y)\}$, and $\Gamma_3 = \{y \mid T(x_1, y) < T(x_2, y)\}$. By continuity each Γ_i is open and $y_0 \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ thus defining a neighborhood $N(y_0)$ of y_0 . For every $y \in N(y_0)$ follows $T(x_0, y) < T(x_1, y) < T(x_2, y)$ and $T(x_0, y) < T(x_1, y) < T(x_2, y)$; therefore there exist points $\alpha, \beta \in]x_0, x_1[$ satisfying $T(\alpha, y_0) = T(x_1, y)$ and $T(\beta, y) = T(x_1, y_0)$. The equation $h \circ T = F \circ \pi$ then implies $F\{f(\alpha), g(y_0)\} = F\{f(x_1), g(y)\}$ and $F\{f(\beta), g(y)\} = F\{f(x_1), g(y_0)\}$. But $f(x_1) \geq f(\alpha)$ and $\geq f(\beta)$ and since F is now strictly monotonic in each variable, $g(y) = g(y_0)$ follows. Hence g is constant on $N(y_0)$, and by § 4, g is constant on Ω_y and f is constant on Ω_x . When applied to any two arguments of the original equation in \mathbf{R}^n , $n \geq 2$, the theorem follows.

REFERENCES

1. J. Aczel, *Lectures on Functional Equations and their Applications*, Academic Press, New York, 1966.
2. J. L. Denny, *Cauchy's equation and sufficient statistics on arcwise connected spaces*, Ann. Math. Statistics, **41** (1970), 401-411.
3. C. T. Ng, *Local boundedness and continuity for a functional equation on topological spaces*, Proc. Amer. Math. Soc., **39-3** (1973), 525-529.
4. ———, *On the functional equation $f(x) + \sum g_i(y_i) = h\{T(x, y_1, \dots, y_n)\}$* , Annales Pol. Math., **27** (1973), 329-336.
5. J. A. Lester, and M. A. McKiernan, *On null cone preserving mappings*, Math. Proc. Camb. Phil. Soc., **81** (1977), 455-462.

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Krishnaswami Alladi and Paul Erdős, <i>On the asymptotic behavior of large prime factors of integers</i>	295
Alfred David Andrew, <i>A remark on generalized Haar systems in L_p, $1 < p < \infty$</i>	317
John M. Baker, <i>A note on compact operators which attain their norm</i>	319
Jonathan Borwein, <i>Weak local supportability and applications to approximation</i>	323
Tae Ho Choe and Young Soo Park, <i>Wallman's type order compactification</i>	339
Susanne Dierolf and Ulrich Schwanengel, <i>Examples of locally compact noncompact minimal topological groups</i>	349
Michael Freedman, <i>A converse to (Milnor-Kervaire theorem) $\times R$ etc.</i>	357
George Golightly, <i>Graph-dense linear transformations</i>	371
H. Groemer, <i>Space coverings by translates of convex sets</i>	379
Rolf Wim Henrichs, <i>Weak Frobenius reciprocity and compactness conditions in topological groups</i>	387
Horst Herrlich and George Edison Strecker, <i>Semi-universal maps and universal initial completions</i>	407
Sigmund Nyrop Hudson, <i>On the topology and geometry of arcwise connected, finite-dimensional groups</i>	429
K. John and Václav E. Zizler, <i>On extension of rotund norms. II</i>	451
Russell Allan Johnson, <i>Existence of a strong lifting commuting with a compact group of transformations. II</i>	457
Bjarni Jónsson and Ivan Rival, <i>Lattice varieties covering the smallest nonmodular variety</i>	463
Grigori Abramovich Kolesnik, <i>On the order of Dirichlet L-functions</i>	479
Robert Allen Liebler and Jay Edward Yellen, <i>In search of nonsolvable groups of central type</i>	485
Wilfrido Martínez T. and Adalberto Garcia-Maynez Cervantes, <i>Unicoherent plane Peano sets are σ-unicoherent</i>	493
M. A. McKiernan, <i>General Pexider equations. I. Existence of injective solutions</i>	499
M. A. McKiernan, <i>General Pexider equations. II. An application of the theory of webs</i>	503
Jan K. Pachl, <i>Measures as functionals on uniformly continuous functions</i>	515
Lee Albert Rubel, <i>Convolution cut-down in some radical convolution algebras</i>	523
Peter John Slater and William Yslas Vélez, <i>Permutations of the positive integers with restrictions on the sequence of differences. II</i>	527
Raymond Earl Smithson, <i>A common fixed point theorem for nested spaces</i>	533
Indulata Sukla, <i>Generalization of a theorem of McFadden</i>	539
Jun-ichi Tanaka, <i>A certain class of total variation measures of analytic measures</i>	547
Kalathoor Varadarajan, <i>Modules with supplements</i>	559
Robert Francis Wheeler, <i>Topological measure theory for completely regular spaces and their projective covers</i>	565