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CONVOLUTION CUT-DOWN IN SOME RADICAL CONVOLUTION ALGEBRAS

LEE ALBERT RUBEL

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Let $\mathcal{A} = L^1_{\text{loc}}(\mathbb{R}^+)$ be the algebra of locally integrable functions on the positive real axis, with convolution as multiplication, given by

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt .$$

Sometimes it is convenient to think of our functions as being defined on all of \mathbb{R} , but vanishing for negative x . We are interested in subalgebras of \mathcal{A} that are Banach algebras in some norm, and that are radical in the sense that there exist no (nontrivial) complex homomorphisms. We call these algebras *radical convolution algebras*. Such algebras A present a challenge because there is no Fourier transform for them.

We are concerned with the problem of "convolution cut-down"; namely whether given a radical convolution algebra A and an $f \in A$, there must exist an $h \in A$ ($h \neq 0$) such that $f * h \in L^1(\mathbb{R}^+)$. We show, at least, that one cannot always choose $h \in L^1$. As a corollary, we show that *simultaneous convolution cut-down* is not always possible.

A class of examples is formed by certain Beurling algebras $A_w = L^1_w(\mathbb{R}^+)$, where w is a positive weight function that satisfies

(i)
$$w(x + y) \leq w(x)w(y)$$

and where

(ii)
$$\|f\| = \int_0^\infty |f(t)|w(t)dt .$$

It is not hard to show that for A_w to be a radical algebra, it is necessary and sufficient that

(iii)
$$[w(x)]^{1/x} \longrightarrow 0 \text{ as } x \rightarrow +\infty .$$

For example, we could choose $w(x) = \exp(-x^\alpha)$ for any $\alpha > 1$. (Note that the restriction, assumed by many authors, that $w(x) \geq 1$, certainly does not apply here.)

Problem. Given a radical convolution algebra A and an f in A , does there exist an h in A ($h \neq 0$) such that $f * h \in L^1$?

This problem arose in discussion with Jamil A. Siddiqi, whom the author thanks for his help. An affirmative answer to the pro-

blem would be useful in solving certain natural problems about the ideal structure of A . To tell the truth, an affirmative answer is hardly to be expected, since convolution is known as a smoothing operator, and not as a reducing operator. We cannot at present prove a general negative answer. At least for some nonradical algebras, a negative answer is furnished by studying the distribution of the zeros of the Fourier transforms. We provide a partial negative solution in the radical case.

From now on, we assume that A is a radical convolution algebra, and we discard the function identically 0.

THEOREM. *It is not the case that for every $f \in A$ there is a $g \in L^1$ such that $f * g \in L^1$.*

COROLLARY. *There exists a pair $f, g \in A$ such that for no $h \in L^1$ are both $f * h$ and $g * h$ in L^1 .*

We summarize this by saying that simultaneous convolution cut-down is impossible.

Proof of corollary. We proceed by contradiction. Given $f \in A$, choose $g \in A$ so that $f * g \in L^1$. Now choose $h \in A$ so that $g * h \in L^1$ and $(f * g) * h \in L^1$. Then with $k = g * h$, we would have $f * k \in L^1$ where $k \in L^1$, which contradicts the theorem.

Proof of theorem. We observe that for a function $h \in L^1(\mathbf{R}^+)$, its Fourier transform

$$\hat{h}(x) = \int_0^{\infty} e^{-itx} f(t) dt$$

has a natural extension as a function holomorphic in the lower half-plane $L = \{z: \text{Im } z < 0\}$. If now $f * g = h$ with $g, h \in L^1(\mathbf{R}^+)$, we define for every $z \in L$:

$$V_z f = \text{ord}_z \hat{h} - \text{ord}_z \hat{g}$$

where $\text{ord}_z H$ is the order of the zero of H at the point z . It is easy to see that V_z is well defined, for if $f * g = h$ and $f * G = H$ then $f * (g * H - G * h) = 0$ and by the famous Titchmarsh theorem that there are no divisors of zero in \mathcal{A} , we have $g * H - G * h = 0$, etc. Roughly, V_z plays the role of "the order of vanishing at z of the Fourier transform of f ". Now it is easy to see that V_z extends to a valuation on the field A/A of formal quotients of functions in A . (That A/A makes sense as a field follows from the Titchmarsh theorem.)

By "valuation" we mean that

(i)
$$V_z f g = V_z f + V_z g$$

(ii)
$$V_z(f + g) \geq \min(V_z f, V_z g) .$$

We appeal now to a theorem of Royden ([2], Proposition 5, p. 274) that any valuation V on A/A must be nonnegative on A . When applied to V_z , this implies that h^\wedge/g^\wedge has no pole at z . If we now let

$$\phi_z f = \frac{h^\wedge(z)}{g^\wedge(z)}$$

then it follows that ϕ_z is a complex homomorphism of A . Since A is assumed to be radical, $\phi_z f = 0$, and thus $h^\wedge(z) = 0$ for all $z \in L$.

Therefore, by a simple argument, $h^\wedge(x)$ vanishes for all real x , and consequently $h = 0$, which is not allowed. This contradiction proves the theorem.

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