A COMMON FIXED POINT THEOREM FOR NESTED SPACES

RAYMOND EARL SMITHSON
A COMMON FIXED POINT THEOREM FOR NESTED SPACES

R. E. SMITHSON

Let $X$ be an arcwise connected Hausdorff space in which the union of any nest of arcs is contained in an arc. Let $f, g : X \to X$ be commuting functions (not necessarily continuous), which satisfy (1) $f(A)$ and $g(A)$ are arcwise connected for each arc $A \subset X$, and (2) $f^{-1}(x)$ and $g^{-1}(x)$ are arcwise connected for each $x \in X$. The principal result of this paper is:

**Theorem.** The functions $f$ and $g$ have a common fixed point.

A space satisfying the conditions on $X$ is called a nested space. Functions which satisfy condition (1) are called arc preserving and those satisfying condition (2) are called strongly monotone. In [3] Harris showed that continuous functions are arc preserving. Thus we have:

**Corollary.** Two commuting, continuous strongly monotone selfmaps of a nested space have a common fixed point.

In this context an arc is a continuum with exactly two non-cutpoints. If $X$ is metrizable, then this coincides with the classical definition of an arc as the homeomorphic image of the closed unit interval.

Before proceeding to the proof of the main result we give an example which shows that an arc preserving, strongly monotone function is not necessarily continuous and then give a few historical remarks.

**Example.** Let $X_0 = \{(x, 0) : 0 \leq x \leq 2\}$ and $X_n = \{(1/n, y) : 0 \leq y \leq 1\}$ for $n \geq 1$. Next set $X_{-1} = \{(2, y) : 0 \leq y \leq 2\}$, $X_{-2} = \{(x, 2) : 0 \leq x \leq 2\}$ and $X_{-3} = \{(0, y) : 1 \leq y \leq 2\}$. Then set $X = \bigcup \{X_k : k \geq -3\}$. Define $f : X \to X$ by $f(z) = (2, 0)$ if $z \in \bigcup_{i=1}^{i=4} U_{-i}$ and $f(z) = z$ otherwise. We see that $f$ is arc preserving and strongly monotone but $f$ is not continuous.

In 1967 W. J. Gray [1] proved that an abelian semigroup of continuous, monotone functions on an hereditarily unicoherent, hereditarily decomposable continuum into itself had a common fixed point. Further, in 1975 Gray and Smith [2] proved an extension of this result for hereditarily unicoherent, arcwise connected con-
tinua. In this note we prove a common fixed point theorem for commuting functions on a nested space.

The notion of a nested space was used by G.S. Young [6] in 1946. In [6] Young showed that nested spaces have the fixed point property for continuous selfmaps. This theorem was subsequently extended to multifunctions by Smithson [5] and further extended by Muenzenberger and Smithson in [4].

REMARK. It is clear from the definition that a nested space is acyclic. Thus if \( x, y \in X \) where \( X \) is a nested space, we denote the unique arc in \( X \) with endpoints \( x, y \) by \([x, y]\). In the sequel \( X \) will denote a nested space, and the functions \( f, g \) satisfy conditions (1) and (2).

We define a partial order on \( X \) as follows: Let \( e \in X \). Then \( x \leq y \) if and only if \( x \in [e, y] \). The proof that \( \leq \) is a partial order is routine and is omitted. In the remainder of the paper we assume that \( X \) has this partial order.

**Lemma 1.** The partial order \( \leq \) satisfies the following:

(i) If \( x < y \), then there is a \( z \) such that \( x < z < y \).

(ii) If \( C \subset X \) is totally ordered and nonempty, then \( \sup C \) exists in \( X \).

(iii) For each \( x, y \in X \), \( \inf \{x, y\} = x \wedge y \) exists.

**Proof.** For (i) let \( z \in [x, y] \setminus \{x, y\} \). Then \( z \in [x, y] \subset [e, y] \) since \( x \in [e, y] \) and thus \( x < z < y \). For (ii) note that \( \{[e, c]: c \in C\} \) is a nested collection of arcs in \( X \) and so \( C \) is contained in an arc \([e, a]\). Let \( c_0 = \sup C \) in \([e, a]\). If \( C \subset [e, b] \), then \( C \subset [e, a] \setminus [e, b] \) which is an arc and so \( c_0 \leq b \). Thus \( c_0 = \sup C \) in \( X \). For (iii), let \( A = [e, x] \cap [e, y] \). Then \( A \) is an arc \([e, a]\) and \( a = x \wedge y \).

REMARK. We could also show that each nonempty subset of \( X \) has an infimum in \( X \) and that for each \( x \in X \), there is a maximal element \( m \in X \) with \( x \leq m \).

If \( x, y \) are not comparable, \( x \wedge y \) is a cutpoint of the arc \([x, y]\) and thus \([x, y] = [x \wedge y, x] \cup [x \wedge y, y]\).

Define the sets \( L(x) \) and \( M(x) \) by: \( L(x) = \{y \in X; y \leq x\} \) and \( M(x) = \{y: x \leq y\} \). Then, since \( L(x) = [e, x] \), \( L(x) \) is totally ordered. Also \( M(x) \) is arcwise connected. We have:

**Lemma 2.** If \( A \) is arcwise connected, if \( A \cap M(x) \neq \emptyset \neq A \cap (X - M(x)) \), then \( x \in A \).

**Proof.** Let \( y_1 \in A \cap M(x) \) and \( y_2 \in A \cap (X - M(x)) \). Then \( y_1 \wedge y_2 \in \)
LEMMA 3. If \( a < b \) and if \([a, b]\) contains a fixed point of \( f \), then \( x_0 = \inf \{x \in [a, b]: f(x) = x\} \) is a fixed point of \( f \). Hence, \([a, b]\) contains a smallest (in \([a, b]\)) fixed point of \( f \).

Proof. Let \( x_1 \in [a, b] \) be a fixed point of \( f \). Then if \( f(x_0) \neq x_0 \), \( x_0 < x_1 \) and we may assume that \( f(x_0) \leq x_1 \). Let \( z_0 = f(x_0)Ax_1 \). Since \( f \) is arc preserving, there is a \( z_1 \in [x_0, x_1] \) such that \( f(z_1) = z_0 \). Note \( z_1 \neq x_0 \). Now let \( x_2 \) be a fixed point of \( f \) such that \( x_0 < x_2 < z_1 \). Since \( x_2 < x_1 \), \( f(x_0)Ax_2 \leq f(x_0)Ax_1 \leq f(x_0) \). But \( x_2 \) and \( f(x_0) \) are elements of \( f[x_0, x_2] \) and thus so are \( f(x_0)Ax_2 \) and \( f(x_0) \). Hence, \( z_0 \in f[x_0, x_2] \). This implies that \( f^{-1}(z_0) \cap M(x_2) \neq \emptyset \) and \( f^{-1}(z_0) \cap (X - M(x_2)) \neq \emptyset \) and thus \( x_2 \in f^{-1}(z_0) \) since \( f \) is strongly monotone. This is a contradiction to \( f(x_2) = x_2 \) and \( x_2 < z_0 \).

Next we need another definition.

DEFINITION. Let \( a \in X \). The branch at a containing \( x_1 \in M(a) - a \) is the set \( B = \{x: a < x Ax_1\} \).

Thus if \( B_1, B_2 \) are two different branches at \( a \) and if \( x_i \in B_i, \ i = 1, 2 \), then \( a = x_1 Ax_2 \) and \( a \in [x_1, x_2] \).

Before proving the main result we need two more lemmas.

LEMMA 4. If \( A \subset X \) is a nonempty totally ordered set such that \( x \leq f(x) \) for \( x \in A \), then \( x_0 \leq f(x_0) \) where \( x_0 = \sup A \).

Proof. Suppose \( f(x_0) \not\in M(x_0) \). Let \( b = f(x_0)Ax_0 \) and let \( b \leq c \leq x_0 \). Since \( x_0 = \sup A \), there is an \( x_1, c < x_1 < x_0 \), such that \( x_1 \leq f(x_1) \). Then \( f[x_1, x_0] \) meets \( M(c) \) and \( X - M(c) \) and hence, contains \( c \). Let \( z_1 \in [x_1, x_0] \) with \( f(z_1) = c \). Next let \( x_2 \) be in \( A \) with \( z_1 < x_2 < x_0 \). Then \( f[x_2, x_0] \) meets both \( M(c) \) and \( X - M(c) \) and also contains \( c \). But then \( f^{-1}(c) \) meets \( M(x_2) \) and \( X - M(x_2) \) and therefore contains \( x_2 \). This contradicts the choice of \( x_2 \). Thus \( x_0 \leq f(x_0) \).

LEMMA 5. Let \( a < f(a) \). (i) If \( B \) is the branch at a containing \( f(a) \), then \( B \) contains a fixed point of \( f \). (ii) If \( f(f(a)) \not\in M(f(a)) \), then \( X - M(f(a)) \) contains a fixed point of \( f \).

Proof. For part (i) first let \( a < x < f(a) \). Then if \( f(x) \not\in M(x) \), \( f[a, x] \) contains \( x \). Hence, there is an \( x, a < x < f(a) \), with \( x \leq f(x) \). Now let \( C \) be a maximal totally ordered set containing \( x \) such that \( c \leq f(c) \) for all \( c \in C \). Let \( x_0 = \sup C \). Then \( f(x_0) \in M(x_0) \) follows from Lemma 4. Note that \( x_0 \) is in the branch at a containing \( f(a) \).
For $x_0 < x_1 < f(x_0)$, $f(x_1) \in M(x_1)$. Thus $x_0 \in f[x_0, x_1]$. Suppose $f(z_1) = x_1$ where $x_0 < z_1 < x_1$. But then $z_1 \in C$ which contradicts the definition of $x_0$ and $C$. Thus $f(x_0) = x_0$.

For statement (ii) set $X_0 = \{X - M(f(a))\} \cup \{f(a)\}$, and define $g: X_0 \to X_0$ by $g(x) = f(x)$ if $f(x) \not\in M(f(a))$ and $g(x) = f(a)$ if $f(x) \in M(f(a))$. Since $X - M(f(a))$ is arcwise connected, $X_0$ is a nested space and $g$ is arc preserving, and strongly monotone. Now let $C$ be a maximal totally ordered set in $X_0$ which contains $a$ such that $x \leq f(x)$ for all $x \in C$. Then let $x_0 = \sup C$. (We are using $\leq$ restricted to $X_0$.) From the hypothesis for (ii) $f(f(a)) \not\in M(f(a))$ and thus $x_0 \neq f(a)$. Then by the same argument used in part (i), $f(x_0) = x_0$ and part (ii) follows.

Now we give the proof of the main theorem.

Proof. Let $A = \{x \in X: x \leq f(x) \text{ and } x \leq g(x)\}$ and let $C$ be a maximal totally ordered subset of $A$. Set $a = \sup C$. Then by applying Lemma 4 to $f$ and $g$ we see that $a \in C$. The remainder of the proof is divided into a number of parts.

First suppose that $a < f(a) \land g(a)$; without loss of generality we may suppose that for $x < b = f(a) \land g(a)$ there is a $z$, $a < z < x$ with $f(z) \in M(z)$. Thus let $a < z_1 < b_1 < b$ with $f(z_1) \not\in M(z_1)$. Then $b_1 \in f[a, z_1]$. Say $a < y_1 < z_1$ and $f(y_1) = b_1$. But then there is a $z_2, a < z_2 < y_1$ with $f(z_2) \not\in M(z_2)$. Thus $f[a, z_2]$ also contains $b_1$ which contradicts the assumption that $f^{-1}(b_1)$ is arcwise connected.

Next we assume that $a \neq f(a) \land g(a)$, and that $a = f(a) \land g(a)$. For each $i = 1, 2$, let $B_i$ be the branch at a containing $x_i = f(a)$ and $x_2 = g(a)$ respectively. Since $f$ and $g$ commute $f(x_1) = g(x_1)$ and hence, either $g(x_1) \in B_2$ or $f(x_1) \not\in B_1$. Say $f(x_1) \not\in B_1$. Then $a \in f[a, x_1]$. By Lemmas 3 and 4 there is a fixed point $y_1$ of $f$ in $B_1$ such that the only $f$-fixed point in $[a, y_1]$ is $y_1$. Next note that $f(g(y_1)) = g(f(y_1)) = g(y_1)$ and so $g(y_1)$ is also a fixed point of $f$. Further, if $y_2 = g(y_1) \in B_1$, then $a \in [y_2, y_1]$ and thus $a \in f[y_2, y_1]$. This leaves two possibilities: either $y_z \in B_1$ or $y_z \in B_2$ for otherwise $a \in f^{-1}(a)$ which is a contradiction. So suppose $y_z \in B_1$ and let $b = y_1 \land y_2$. Then if $f(b) \in M(b)$, $b \in f[b, y_1] \cap f[b, y_2]$ and so $b \in f^{-1}(b)$ which contradicts the choice of $y_1$. Thus $f(b) \not\in M(b)$. Then $g(b) \in M(b) - b$ follows from the maximality of $a$. Next set $y_3 = g(y_z)$. If $y_z \in M(b)$ we have $b \in g[b, y_1] \cap g[b, y_2]$ which is a contradiction. Hence, $y_z \not\in M(b)$. Note $f(y_z) = y_3$. So we have $b \in f[y_3, b]$ and either $b \in f[b, y_1]$ or $b \in f[b, y_2]$ which is another contradiction.

We still have the subcase $y_z \in B_2$ to consider. By Lemma 3 we may assume that $f$ does not have another fixed point in $[a_z, y_z]$. Then set $y_z = g(y_z)$. As above either $y_z \in B_1$ or $y_z \in B_2$. In either of
these cases we can obtain the same contradiction as in the case where \( y_2 \in B_i \). Hence, the argument for \( f(a) \neq a \neq g(a) \) and \( f(a) \
eq g(a) = a \) is concluded.

Finally, assume \( f(a) = a \) and \( a < g(a) = y_1 \). Since \( y_1 \) is a fixed point of \( f, g(y_1) \in M(y_1) \). Thus there is a \( g \)-fixed point \( x_1 \in M(a) - M(y_1) \). Let \( b = x_1 A y_1 \). Then \( a \leq b < y_1 \) and so let \( c \) be such that \( b < c < y_1 \). Now if \( x_2 = f(x_1) \in M(a) \), then \( c \in g[x_2, a] \) and \( c \in g[a, x_2] \) which is a contradiction since \( a \in g^{-1}(c) \). Thus \( x_2 \in M(a) \) and \( x_2 \in M(x_2) \).

From Lemma 3 we may assume that \( x_1 \) is the only \( g \)-fixed point in \([a, x_1]\). Now let \( d = x_1 A x_2 \). As in the previous arguments, applied to \( g \) in this case, \( f(d) \in M(d) \). Finally we set \( x_3 = f(x_2) \) and the same arguments as above give the same contradiction. Thus we conclude that \( a \) is a fixed point of both \( f \) and \( g \).

**References**

5. R. E. Smithson, *Fixed point theorems for certain classes of multifunctions*, 31 (1972), 595-600.

Received July 14, 1978 and in revised form February 15, 1979. This paper was written while the author was a visiting professor of mathematics at California State University, Chico.

**University of Wyoming**

**Laramie, WY 82071**
Krishnaswami Alladi and Paul Erdős, On the asymptotic behavior of large prime factors of integers .......................................................... 295
Alfred David Andrew, A remark on generalized Haar systems in $L_p$, 1 < $p$ < $\infty$ .......................................................... 317
John M. Baker, A note on compact operators which attain their norm .................. 319
Jonathan Borwein, Weak local supportability and applications to approximation .......................................................... 323
Tae Ho Choe and Young Soo Park, Wallman’s type order compactification .......... 339
Susanne Dierolf and Ulrich Schwanengel, Examples of locally compact noncompact minimal topological groups .................................................. 349
Michael Freedman, A converse to (Milnor-Kervaire theorem) $\times R$ etc. ........ 357
George Golightly, Graph-dense linear transformations .................................. 371
H. Groemer, Space coverings by translates of convex sets ................................. 379
Rolf Wim Henrichs, Weak Frobenius reciprocity and compactness conditions in topological groups .......................................................... 387
Horst Herrlich and George Edison Strecker, Semi-universal maps and universal initial completions .......................................................... 407
Sigmund Nyrop Hudson, On the topology and geometry of arcwise connected, finite-dimensional groups .......................................................... 429
K. John and Václav E. Zizler, On extension of rotund norms. II .......................... 451
Russell Allan Johnson, Existence of a strong lifting commuting with a compact group of transformations. II .......................................................... 457
Bjarni Jónsson and Ivan Rival, Lattice varieties covering the smallest nonmodular variety ............................................................................. 463
Grigori Abramovich Kolesnik, On the order of Dirichlet L-functions .................. 479
Robert Allen Liebler and Jay Edward Yellen, In search of nonsolvable groups of central type .......................................................... 485
Wilfrido Martín T. and Adalberto Garcia-Maynez Cervantes, Unicoherent plane Peano sets are $\sigma$-unicoherent .......................................................... 493
M. A. McKiernan, General Pexider equations. I. Existence of injective solutions .......................................................... 499
M. A. McKiernan, General Pexider equations. II. An application of the theory of webs ............................................................................. 503
Jan K. Pachl, Measures as functionals on uniformly continuous functions ........ 515
Lee Albert Rubel, Convolution cut-down in some radical convolution algebras .... 523
Peter John Slater and William Yslas Vélez, Permutations of the positive integers with restrictions on the sequence of differences. II .......................................................... 527
Raymond Earl Smithson, A common fixed point theorem for nested spaces ....... 533
Indulata Sukla, Generalization of a theorem of McFadden .................................. 539
Jun-ichi Tanaka, A certain class of total variation measures of analytic measures ............................................................................. 547
Kalathoor Varadarajan, Modules with supplements ............................................. 559
Robert Francis Wheeler, Topological measure theory for completely regular spaces and their projective covers .......................................................... 565