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# GENERALIZATION OF A THEOREM OF MCFADDEN

INDULATA SUKLA

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# GENERALIZATION OF A THEOREM OF McFADDEN

INDULATA SUKLA

**McFadden's relation $|N, 1/(n+1)| \subset |C, k|(k>0)$ is strengthened to $|N, p_n| \subset |R, \lambda(w), k|(k>0)$ for suitable $\{p_n\}$ and $\lambda(w)$.**

1. Let $\{p_n\}$ be a sequence of complex numbers such that for $n > 0$,

$$P_n = p_0 + p_1 + \cdots + p_n \neq 0 .$$

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with $\{s_n\}$ as its partial sums. We define the *$(N, p_n)$-transform* $\{t_n(s_n)\}$ of $\{s_n\}$ *generated by the sequence* $\{p_n\}$ *by the formula*

$$t_n(s_n) = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v . \tag{1.1}$$

Similarly, $\{t_n(na_n)\}$ denotes the $(N, p_n)$-transform of the sequence $\{na_n\}$ generated by the sequence $\{p_n\}$. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|N, p_n|$ if $\{t_n(s_n)\} \in BV$, i.e., $\sum_{n=1}^{\infty} |t_n - t_{n-1}|$ is convergent. (See [7], [5].) In the special case when $p_n = \binom{n+k-1}{k-1}$, $(k > -1)$, summability $|N, p_n|$ is summability $|C, k|$.

Let $\lambda = \lambda(w)$ be a differentiable, monotonically increasing function of $w$ in $(A, \infty)$, where $A$ is a finite positive number; and let $\mu(w)$ be its differential and let $\lambda(w)$ tend to infinity with $w$. For $k \geqq 0$, we write

$$R_\lambda^k(w) = \sum_{n \leqq w} \{\lambda(w) - \lambda(n)\}^k a_n .$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|R, \lambda, k|$ if

$$\int_A^{\infty} |d[R_\lambda^k(w)/\lambda^k(w)]| < \infty , \tag{1.2}$$

see [8], [9]. For $k > 0$, $N < w < N+1$ $(N = 1, 2, \cdots)$

$$\frac{d}{dw}[R_\lambda^k(w)/\lambda^k(w)] = \frac{k\mu(w)}{\lambda^{k+1}(w)} \sum_{n \leqq w} \{\lambda(w) - \lambda(n)\}^{k-1} \lambda(n) a_n .$$

Hence, summability $|R, \lambda, k|$ is equivalent to the convergence of the integral

$$\int_A^{\infty} \frac{\mu(w)}{\lambda^{k+1}(w)} \left| \sum_{n \leqq w} \{\lambda(w) - \lambda(n)\}^{k-1} \lambda(n) a_n \right| dw .$$

If every series summable by the method $P$ is summable by the

method $Q$, we write $P \subseteqq Q$. If $P \subseteqq Q$ and $Q \subseteqq P$, we write $P \sim Q$.

We now define a sequence of constants $\{c_n\}$ by the identity

$$\left[\sum_{n=0}^{\infty} p_n x^n\right]^{-1} = \sum_{n=0}^{\infty} c_n x^n, \quad c_{-1} = 0 .$$

If, for $n = 0, 1, 2, \cdots$

$$p_n > 0, \quad \frac{p_{n+1}}{p_n} \leqq \frac{p_{n+2}}{p_{n+1}} \leqq 1 ,$$

we shall write $\{p_n\} \in \mathscr{M}$. We write

$$d_n = c_0 + c_1 + \cdots + c_n ;$$

$$e_n = d_0 + d_1 + \cdots + d_n .$$

We write $P(v)$, $d(v)$, $e(v)$ in place of $P_v$, $d_v$, $e_v$ respectively when $v$ is replaced by a more complicated expression. We let $\Delta f_n = f_n - f_{n+1}$, for any sequence $\{f_n\}$.

The following inclusion theorems are known:

$$|C, 0| \subset \left|N, \frac{1}{n+1}\right| \subset |C, k| \sim |R, n, k|, \ (k > 0) .$$

The first one is due to Mears [6], the second one is due to McFadden [4] and the equivalence is due to Hyslop [3].

Our object is to prove that under certain conditions on $\{p_n\}$ and $\lambda(w)$,

$$|N, p_n| \subset |R, \lambda(\omega), k| \ (k > 0) .$$

2. We establish the following.

THEOREM. *Let*

$$\{p_n\} \in \mathscr{M} , \tag{2.1}$$

$$P(v^2) = O(P_v) , \tag{2.2}$$

$\lambda(w)$ be an indefinite integral of some function $\mu(w)$, (2.3)

$$(n+1)\left\{\frac{\lambda(n+1) - \lambda(n)}{\lambda(n+1)}\right\} = O(1) . \tag{2.4}$$

*Then* $|N, p_n|$ *implies* $|R, \lambda(w), k|$ *for all* $k > 0$.

For the proof we require the following lammas.

LEMMA 1 [1]. *Let* $\{p_n\} \in \mathscr{M}$. *A necessary and sufficient condi-*

*tion for* $\Sigma a_n \in |N, p_n|$ *is*

$$\sum_{n=1}^{\infty} \frac{|t_n(na_n)|}{n} < \infty \,. \tag{2.4'}$$

LEMMA 2 [1]. *Let* $\{p_n\} \in \mathscr{M}$. *Then*

(i) $c_0 > 0$, $c_n \leqq 0$ $(n = 1, 2, 3, \cdots)$

(ii) $\sum_{n=v+1}^{\infty} |c_n| \leqq d_v$

(iii) $d_n \geqq 0$ *and monotonic nonincreasing*

(iv) $P_n d_n \leqq 1$

(v) $P_n e_n \leqq (2n + 1)$.

*For* (i), (ii) *see Hardy* [2] *Theorem* 22, *p.* 68.

LEMMA 3. *Let* $\{p_n\} \in \mathscr{M}$. *Then for any fixed* $k$ *with* $0 < k < 1$, (2.2) *is equivalent to*

$$P_v = O(P(u)), \text{ where } u = [v^k]. \tag{2.5}$$

*Proof.* If (2.2) holds, by successive application of (2.2) we see that for any fixed integer $r$,

$$P(v^{2^r}) = O(P_v) \,. \tag{2.6}$$

Choose $r$ so that $2^r > 1/k$. Then if $u = [v^k]$, $v \leqq (u)^{1/k} < (u)^{2^r}$. So, since $P_v$ is increasing, (2.5) follows from (2.6).

Conversely, suppose that (2.5) holds. Given any positive integer $v$, define $v_r$ inductively (on $r$) by taking $v_0 = v$ and defining $v_r (r > 1)$ as the least integer greater than or equal to $v_{r-1}^{1/k}$. Since $\{p_n\} \in \mu$ implies that

$$\frac{p_r}{p_{r-1}} \longrightarrow 1 \,, \tag{2.7}$$

as $r \to \infty$, we see that (2.5) is equivalent to

$$P(v_1) = O(P_v) \,. \tag{2.8}$$

By successive application of (2.8) we deduce that, for any fixed $r$,

$$P(v_r) = O(P_v) \,. \tag{2.9}$$

Choose $r$ so that $(1/k)^r > 2$. Then $v_r > v^2$ so that (again since $P_v$ is increasing) (2.9) implies (2.2).

For the proof of the theorem we require (2.5). The condition (2.2) is preferable to (2.5) because the former is simpler and independent of $k$.

LEMMA 4. *If* (2.4) *is satisfied then*

$$\frac{\lambda(n+1)}{\lambda(n)} = O(1), \text{ as } n \longrightarrow \infty .$$

This is obvious.

3. **Proof of the theorem.** It is enough to consider the case $0 < k < 1$. This implies the result for $k \geqq 1$. We can assume without loss of generality that $a_0 = 0$. Then by Lemma 1 and (1.2) it is enough to show that (2.4′) implies

$$\int_1^\infty \left| \frac{d}{dw}(R_\lambda^k(w)/\lambda^k(w)) \right| dw < \infty .$$

Now,

$$na_n = \sum_{v=1}^{n} c_{n-v} P_v t_v(va_v) .$$

Then

$$\frac{d}{dw}\frac{R_\lambda^k(w)}{\lambda^k(w)} = \frac{k\mu(w)}{\lambda^{k+1}(w)}\left[ \sum_{n=1}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} \sum_{v=1}^{n} c_{n-v} P_v t_v(va_v) \right]$$

$$= \frac{k\mu(w)}{\lambda^{k+1}(w)}\left[ \sum_{v=1}^{[w]} P_v t_v(va_v) \sum_{n=v}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} c_{n-v} \right].$$

Then

$$\int_1^\infty \left| \frac{d}{dw}(R_\lambda^k(w)/\lambda^k(w)) \right| dw$$

$$= O(1)\left[ \int_1^\infty \frac{\mu(w)}{\lambda^{k+1}(w)} \sum_{v=1}^{[w]} vP_v \frac{|t_v(va_v)|}{v} \left| \sum_{n=v}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} c_{n-v} \right| dw \right]$$

$$= O(1)\left[ \sum_{v=1}^{\infty} vP_v \frac{|t_v(va_v)|}{v} \int_v^\infty \frac{\mu(w)}{\lambda^{k+1}(w)} \left| \sum_{n=v}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} c_{n-v} \right| dw \right].$$

Thus it is enough to prove that uniformly in $v \geqq 1$,

$$J(v) = \int_v^\infty \frac{\mu(w)}{\lambda^{k+1}(w)} \left| \sum_{n=v}^{[w]} \{\lambda(w) - \lambda(n)\}^{k-1} \frac{\lambda(n)}{n} c_{n-v} \right| dw \tag{3.1}$$

$$= O\left(\frac{1}{vP_v}\right).$$

Write $m = \min([w], v+u)$. Let $a = v+u-1$, $b = v+u+1$.

Applying partial summation to the sum over the range $v \leqq n \leqq m$, we see that the expression inside the modulus in (3.1) is equal to

$$(3.2)\quad \sum_{n=v}^{m-1} \Delta_n\left[(\lambda(w)-\lambda(n))^{k-1}\frac{\lambda(n)}{n}\right]d_{n-v} + (\lambda(w)-\lambda(m))^{k-1}\frac{\lambda(m)}{m}d_{m-v}$$
$$+ \sum_{n=m+1}^{[w]} (\lambda(w)-\lambda(n))^{k-1}\frac{\lambda(n)}{n}c_{n-v}$$
$$= \sum_{n=v}^{m-1} (\lambda(w)-\lambda(n))^{k-1}\Delta\left(\frac{\lambda(n)}{n}\right)d_{n-v}$$
$$+ \sum_{n=v}^{m-1} \Delta_n(\lambda(w)-\lambda(n))^{k-1}\frac{\lambda(n+1)}{n+1}d_{n-v}$$
$$+ (\lambda(w)-\lambda(m))^{k-1}\frac{\lambda(m)}{m}d_{m-v} + \sum_{n=m+1}^{[w]} (\lambda(w)-\lambda(n))^{k-1}\frac{\lambda(n)}{n}c_{n-v}\,.$$

Here the last term is to be omitted when $m=[w]$, i.e., when $w<b$. Hence

$$J(v) \leqq J_1(v) + J_2(v) + J_3(v) + J_4(v)\,,$$

where $J_1(v)$, $J_2(v)$, $J_3(v)$, $J_4(v)$ denote the expressions obtained by replacing the expression inside the modulus in (3.1) by each of the four terms on the right of (3.2). First,

$$J_1(v) \leqq \sum_{n=v}^{a} d_{n-v}\left|\Delta\left(\frac{\lambda(n)}{n}\right)\right| \int_{n+1}^{\infty} \frac{(\lambda(w)-\lambda(n))^{k-1}\mu(w)}{\lambda^{k+1}(w)}dw$$
$$= O(1)\sum_{n=v}^{a} d_{n-v}\frac{|\Delta\lambda(n)|}{(n+1)\lambda(n)} + O(1)\sum_{n=v}^{a} d_{n-v}\frac{\lambda(n)}{n(n+1)\lambda(n)}\,.$$

Using (2.4) and Lemma 4,

$$J_1(v) = O(1)\sum_{n=v}^{a}\frac{d_{n-v}}{n^2}$$
$$= O(1)\frac{e(u-1)}{v^2} = O\left(\frac{1}{vP_v}\right),$$

by Lemma 2(v) and Lemma 3. Next,

$$J_2(v) \leqq \sum_{n=v}^{a} d_{n-v}\frac{\lambda(n+1)}{n+1}$$
$$\times \int_{n+1}^{\infty} \frac{[(\lambda(w)-\lambda(n+1))^{k-1} - (\lambda(w)-\lambda(n))^{k-1}]\mu(w)}{\lambda^{k+1}(w)}dw\,.$$

The inner integral can be evaluated and is equal to

$$\frac{1}{k}\left[\frac{1}{\lambda(n+1)} - \frac{1}{\lambda(n)}\left\{1-\left(\frac{\lambda(n+1)-\lambda(n)}{\lambda(n+1)}\right)^k\right\}\right]$$
$$= \frac{1}{k\lambda(n)}\left[\left(\frac{\lambda(n+1)-\lambda(n)}{\lambda(n+1)}\right)^k - \left(\frac{\lambda(n+1)-\lambda(n)}{\lambda(n+1)}\right)\right]$$
$$= O\left(\frac{1}{n^k\lambda(n)}\right),$$

by (2.4). Hence, by Lemma 4,

$$J_2(v) = O(1)\sum_{n=v}^{a}\frac{d_{n-v}}{n^{k+1}} = O(1)\frac{e(u-1)}{v^{k+1}}$$
$$= O\Big(\frac{1}{vP_v}\Big),$$

by Lemma 2(v) and Lemma 3.

Suppose $N \leqq w < N+1$. Then,

$$J_3(v) \leqq \int_v^{v+u}(\lambda(w)-\lambda(N))^{k-1}\frac{\lambda(N)}{N}d_{N-v}\frac{\mu(w)}{\lambda^{k+1}(w)}dw$$
$$+\frac{\lambda(v+u)}{v+u}d(u)\int_{v+u}^{\infty}(\lambda(w)-\lambda(v+u))^{k-1}\frac{\mu(w)}{\lambda^{k+1}(w)}dw$$
$$= J_{31}(v)+J_{32}(v)\ .$$

Since

$$\int_{v+u}^{\infty}(\lambda(w)-\lambda(v+u))^{k-1}\frac{\mu(w)}{\lambda^{k+1}(w)}dw = \frac{1}{k\lambda(v+u)}\ ,$$

clearly

$$J_{32}(v) = O\Big(\frac{d(u)}{v+u}\Big) = O\Big(\frac{1}{vP_v}\Big),$$

by Lemma 2(iv) and Lemma 3.

Now,

$$J_{31}(v) \leqq \sum_{\sigma=0}^{u-1}\int_{v+\sigma}^{v+\sigma+1}(\lambda(w)-\lambda(N))^{k-1}\frac{\lambda(N)}{N}d_{N-v}\frac{\mu(w)}{\lambda^{k+1}(w)}dw$$
$$\leqq \sum_{\sigma=0}^{u-1}\frac{d_\sigma}{(v+\sigma)\lambda^k(v+\sigma)}\int_{v+\sigma}^{v+\sigma+1}(\lambda(w)-\lambda(N))^{k-1}\mu(w)dw$$
$$\leqq \sum_{\sigma=0}^{u-1}\frac{d_\sigma}{(v+\sigma)\lambda^k(v+\sigma)}\int_{v+\sigma}^{v+\sigma+1}(\lambda(w)-\lambda(N))^{k-1}\mu(w)dw$$
$$= O(1)\sum_{\sigma=0}^{u-1}\frac{d_\sigma}{(v+\sigma)\lambda^k(v+\sigma)}\frac{\lambda^k(v+\sigma+1)}{(v+\sigma+1)^k}$$
$$= O\Big(\frac{1}{vP_v}\Big),$$

by Lemma 4, Lemma 2(v), and Lemma 3. Hence

$$J_3(v) = O(1/vP_v)\ .$$

Lastly,

$$J_4(v) \leqq \int_b^\infty \frac{\mu(w)}{\lambda^{k+1}(w)} \left[ \sum_{n=m+1}^{[w]} (\lambda(w) - \lambda(n))^{k-1} \frac{\lambda(n)}{n} |c_{n-v}| \right] dw$$

$$= \sum_{n=b}^{\infty} |c_{n-v}| \frac{\lambda(n)}{n} \int_n^\infty (\lambda(w) - \lambda(n))^{k-1} \frac{\mu(w)}{\lambda^{k+1}(w)} dw$$

$$= \frac{1}{k} \sum_{n=b}^{\infty} \frac{|c_{n-v}|}{n} \leqq \frac{1}{k} \frac{d(u)}{b}$$

$$= O(1/vP_v) ,$$

by Lemma 2 (ii), (iv), and Lemma 3. Hence (3.1) is proved.

This completes the proof of the theorem.

By putting $p_n = 1/(n+1)$, $\lambda(w) = w$ (integer) we get the inclusion $|N, 1/(n+1)| \subset |R, n, k|$, $k > 0$ due to McFadden [4].

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SAMBALPUR UNIVERSITY
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