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MODULES WITH SUPPLEMENTS

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Let M be an R -module and $N \subset M$. Any $H \subset M$ satisfying $H + N = M \dots \dots$ (i) and $H' \subset H, H' + N = M \Rightarrow H' = H \dots \dots$ (ii) will be referred to as a supplement of N in M . In general N need not have a supplement in M . A module M will be said to have property (P_1) if every $N \subset M$ has a supplement in M . If for every $A \subset M, N \subset M$ with $A + N = M$, there exists a supplement H of N in M satisfying $H \subset A$, we say that M has property (P_2) . Modules with property (P_2) play an important role in our study of dual Goldie dimension. In the present paper we determine the class of rings R with the property that every $M \in R\text{-mod}$ possesses property (P_2) . These ture out to be left perfect rings. Also the results obtained here throw more light on the differences between corank and P. Fleury's spanning dimension.

Introduction. While attempting to dualize the concept of Goldie dimension, Patrick Fleury [3] introduced the class of modules with finite spanning dimension. A module M is said to have finite spanning dimension if for every infinite, strictly decreasing chain $N_0 \supseteq N_1 \supseteq N_2 \supseteq \dots \dots$ of submodules of M , there exists an integer j (depending on the sequence) such that N_i is small in M for all $i \geq j$. A module H is said to be hollow if $H \neq 0$ and any $X \subsetneq H$ is small in H . One of the main results of [3] states that any module of finite spanning dimension can be expressed as an irredundant sum $M = \sum_{i=1}^n H_i$ of finitely many hollow submodules H_i of M and that their number n depends only on M . This number is referred to as the spanning dimension (abbreviated as $s \cdot d$) of M . In our earlier paper entitled Dual Goldie dimension [6] we indicated a completely different way of dualizing Goldie dimension and gave ample evidence to show that our approach has distinct advantages over Fleury's approach. Let k be an integer ≥ 1 . In [6] we defined M to have corank $\geq k$ if there was an epimorphism $f: M \rightarrow \prod_{i=1}^k N_i$ with each $N_i \neq 0$. For $M \neq 0$ we defined corank M to be k if corank $M \geq k$, but corank $M \not\geq (k + 1)$. If corank $M \geq n$ for all $n \geq 1$, we set corank $M = \infty$. When $M = 0$ we set corank $M = 0$. In [6] we showed that the invariant corank had many more desirable properties to be termed dual Goldie dimension than the invariant $s \cdot d$ introduced by P. Fleury. Every module with finite spanning dimension possesses property (P_2) . [3]. In our study of corank also [6], modules with property (P_2) played a special role,

though the theory of corank works very well for all modules. In the present paper we characterise completely the rings R with the property that every $M \in R\text{-mod}$ possesses property (P_2) .

1. Modules with property (P_2) . Throughout this paper all the rings considered are associative rings with an identity element. Unless otherwise mentioned all the modules considered are unitary left modules and all the concepts are left sided concepts. Let R be a ring and $M \in R\text{-mod}$. If M satisfies (P_2) it is clear that M satisfies (P_1) . In [6] we showed that if every submodule of M has (P_1) then M itself has (P_2) . In particular every $M \in R\text{-mod}$ has $(P_2) \Leftrightarrow$ every $M \in R\text{-mod}$ has property (P_1) .

An epimorphism $\varepsilon: M \rightarrow N$ is said to be minimal if $\text{Ker } \varepsilon$ is small in M .

PROPOSITION 1.1. *Let $N \subset M$ and $\varepsilon: P \rightarrow M/N$ a minimal epimorphism. Suppose there exists a lift $f: P \rightarrow M$ of ε . Then $H = f(P)$ is a supplement of N in M .*

Proof. Let $\eta: M/N$ denote the canonical quotient map and $K = \text{Ker } \varepsilon$. Then $\eta(H) = \eta \circ f(P) = \varepsilon(P) = M/N$. Hence $H + N = M$. Suppose $H' \subset H$ satisfies $H' + N = M$. Let $P' = f^{-1}(H')$. Then $f(P') = H'$ and $\varepsilon(P') = \eta \circ f(P') = \eta(H') = M/N$. This yields $P' + K = P$. Since K is small in P we get $P' = P$. Hence $H' = f(P) = H$.

Before proceeding further recall the definition of an M -projective module due to G. Azumaya [1].

DEFINITION 1.2. Let M be a fixed R -module. An R -module P is called M -projective, if for any exact sequence $M \xrightarrow{\varphi} M'' \rightarrow 0$, the sequence $\text{Hom}_R(P, M) \xrightarrow{\varphi^*} \text{Hom}_R(P, M'') \rightarrow 0$ is exact.

PROPOSITION 1.3. *Let $A \subset M, N \subset M$ satisfy $A + N = M$. Suppose there exists a minimal epimorphism $\varepsilon: P \rightarrow M/N$ with P an M -projective module. Then there exists a supplement H of N in M with $H \subset A$.*

Proof. Let $\alpha = \eta/A$ where $\eta: M \rightarrow M/N$ is the canonical quotient map. Then $A + N = M \Rightarrow \alpha(A) = M/N$. Hence $A \xrightarrow{\alpha} M/N \rightarrow 0$ is exact. Since $A \subset M$ and P is M -projective, it follows that P is A -projective [1]. Hence, there exists a map $f: P \rightarrow A$ with $\alpha \circ f = \varepsilon$. Lemma 1.1 shows that $f(P) = H$ is a supplement of N in M . Clearly $H \subset A$.

Since any projective R -module is M -projective for any $M \in R\text{-mod}$, we get the following as a corollary of Proposition 1.3.

COROLLARY 1.4. (a) *If R is left perfect then every $M \in R\text{-mod}$ possesses property (P_2) .*

(b) *If R is semi-perfect then every finitely generated $M \in R\text{-mod}$ possesses property (P_2) .*

Before proceeding further we state a result of F. Kasch and E. Mares [4] which we need.

PROPOSITION 1.5. (F. Kasch and E. Mares). *Let P be a projective module, $N \subset P$. Suppose H is a supplement of N in P and that H itself has a supplement L in P . Then $H \cap L = 0$, hence $P = H \oplus L$. In particular H is projective.*

THEOREM 1.6. *The following conditions on a ring R are equivalent.*

- (1) *Every free R -module F has property (P_1) .*
- (2) *Every $M \in R\text{-mod}$ has property (P_2) .*
- (3) *R is left-perfect.*

Proof. (1) \Rightarrow (3). Assuming (1) we will prove that any $M \in R\text{-mod}$ has a projective cover. Let $F \xrightarrow{f} M \rightarrow 0$ be exact with F free. Let $K = \text{Ker } f$. Let H be a supplement of K in F . Since H has a supplement in F , from Proposition 1.5 we see that H is projective. Also $H \cap K$ is small in H [4, or Lemma 2.7 of [6]]. It follows that $f|_H: H \rightarrow M$ is a projective cover of M .

- (3) \Rightarrow (2) Immediate from Corollary 1.4 (a).
- (2) \Rightarrow (1) Trivial.

The arguments used in the above proof also yield the following result:

THEOREM 1.7. *The following conditions on a ring R are equivalent.*

- (1) *$R \in R\text{-mod}$ has property (P_1) .*
- (2) *R is semi-perfect.*
- (3) *Every finitely generated $M \in R\text{-mod}$ has property (P_2) .*

2. **Surjective endomorphisms with small kernels.** Let P be a projective module, $N \subset P$. Suppose N has a supplement H in P . If H also has a supplement in P , then the result of Kasch and Mares asserts that H is a direct summand of P . In general, when

H need not possess a supplement in P , we do not know whether H will be projective. The results in the present section throw some light on this question.

It is well-known that any surjective endomorphism $f: M \rightarrow M$ of a noetherian module is an isomorphism. A result of Vasconcelos [7] asserts that if R is a commutative ring and M a finitely generated module over R , then any surjective endomorphism $f: M \rightarrow M$ is an isomorphism. In [5] M. Orzech obtained a generalization of the above results. Let R be a ring, $M \in R\text{-mod}$, $N \subset M$. Assume either M is noetherian or that R is commutative and M is finitely generated. Then any epimorphism $f: N \rightarrow M$ is an isomorphism. In this section we will first obtain a useful modification of the above result of Orzech. For any $M \in R\text{-mod}$, we denote the set of small submodules of M by $\Gamma(M)$ and partially order $\Gamma(M)$ under inclusion. The class of finitely generated R -modules will be denoted by $M(R)$.

LEMMA 2.1. *Let $M \in R\text{-mod}$. Then $\Gamma(M)$ satisfies $a \cdot c \cdot c \Leftrightarrow J(M)$ is noetherian.*

Proof. The implication \Leftarrow follows from the well-known fact that any $N \in \Gamma(M)$ satisfies $N \subset J(M)$. Conversely, suppose $J(M)$ is not noetherian. Let $X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \dots$ be an infinite ascending chain of submodules of $J(M)$. Let $x_1 \in X_1$ and $x_j \in X_j - X_{j-1}$ for each $j > 1$. For any $k \geq 1$ let $N_k = \sum_{j=1}^k R x_j$. Then $N_k \in M(R)$ and $N_k \subset J(M)$. Hence $N_k \in \Gamma(M)$. It is clear that $N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \dots$. Hence $\Gamma(M)$ fails to satisfy $a \cdot c \cdot c$. This proves the implication \Rightarrow .

LEMMA 2.2. *Let $M \xrightarrow{f} M''$ be an epimorphism with $\text{Ker } f$ small in M . Then $f(J(M)) = J(M'')$.*

Proof. Since $f(J(M)) \subset J(M'')$, to prove the equality $f(J(M)) = J(M'')$ we have only to show that $f^{-1}(J(M'')) \subset J(M)$. Let $y \in J(M'')$. Then Ry is small in M'' . Since $f: M \rightarrow M''$ is an epimorphism with $\text{Ker } f$ small in M , it follows that $f^{-1}(Ry)$ is small in M and hence $f^{-1}(Ry) \subset J(M)$. This proves $f^{-1}(J(M'')) \subset J(M)$.

PROPOSITION 2.3. *Let $M \in R\text{-mod}$, $N \subset M$ and $f: N \rightarrow M$ an epimorphism with $\text{Ker } f$ small in N . Assume that either $J(M)$ is noetherian or that R is commutative and $J(M)$ is finitely generated. Then f is an isomorphism.*

Proof. Let $K = \text{Ker } f$. Then $K \subset J(N) \subset J(M)$. By Lemma 2.2, $g = f|_{J(N)}: J(N) \rightarrow J(M)$ is an epimorphism. From Orzech's

result we see that g is an isomorphism. Hence $\text{Ker } g = \text{Ker } f = K = 0$. This proves that $f: N \rightarrow M$ is an isomorphism.

Let R be any ring and E any injective module over R , $N \subset E$ and L a relative complement of N in E (namely $N \cap L = 0$, $N \cap L' \neq 0$ for any $L \subsetneq L' \subset E$). Since E is injective we can assume that the injective hull $E(L)$ of L is a submodule of E . From $N \cap L = 0$ we get $N \cap E(L) = 0$. Hence $L = E(L)$. Thus any relative complement in E is injective. If P is a projective R -module, $K \subset P$ admitting a supplement H in P , in general we do not know whether H will be projective. The following is a result in this direction. For any $M \in R\text{-mod}$ let $S(M) = \{H \mid H \subset M, H \text{ a supplement of some } N \subset M\}$.

PROPOSITION 2.4. *Let P be a projective R module and $H \in S(P)$. If either $J(H)$ is noetherian or R is commutative and $J(H)$ is finitely generated over R then H is a direct summand of P and hence H is projective.*

Proof. Let H be a supplement of N in P and $\eta: P \rightarrow P/N$ denote the canonical quotient map. From Lemma 2.7 of [6] we see that $H \cap N$ is small in H . From $H + N = P$ we get $\eta(H) = P/N$. Let $\alpha = \eta|_H$. Then $H \xrightarrow{\alpha} P/N \rightarrow 0$ is exact. Since P is projective, there exists a $\varphi: P \rightarrow H$ with

$$\begin{array}{ccc}
 & P & \\
 \swarrow \varphi & \downarrow \eta & \\
 H & \xrightarrow{\alpha} & P/N \longrightarrow 0
 \end{array}$$

commutative. From $\alpha(\varphi(H)) = \eta(H) = P/N$ we get $\varphi(H) + \text{Ker } \alpha = H$. But $\text{Ker } \alpha = H \cap N$. Since $H \cap N$ is small in H we get $\varphi(H) = H$. Also $\text{Ker } \varphi \subset \text{Ker } \eta$. If $f = \varphi|_H: H \rightarrow H$, then $\text{Ker } f \subset N \cap H$ and f is an epimorphism. From Proposition 2.3 we see that f is an isomorphism. Let $\theta = j \circ f^{-1}: H \rightarrow P$ where j is the inclusion of H in P . Then $\varphi \circ \theta = \theta \circ j \circ f^{-1} = f \circ f^{-1} = \text{Id}_H$. Thus θ is a splitting of φ . If $h = f^{-1} \circ \varphi: P \rightarrow H$ we have $h \circ j = f^{-1} \circ \varphi \circ j = f^{-1} \circ f = \text{Id}_H$. This proves that H is a direct summand of P .

3. Corank versus spanning dimension.

REMARKS.

(1) In [5] we proved that if $s \cdot dM < \infty$, then $\text{corank } M = s \cdot dM$. Also in the same paper we proved that if R is any local ring, and

$M = R \times R$ in R -mod then $\text{corank } M = 2$. But if R is nonartinian, then $s \cdot dM = \infty$. Since any local ring is semi-perfect, it follows from Corollary 1.4(b) that M satisfies property (P_2) .

(2) For any M with property (P_2) and having $\text{corank } M = n < \infty$ we have proved the following two assertions in [6].

(a) $M = \sum_{i=1}^n H_i$ an irredundant sum with each H_i hollow.

(b) If $M = \sum_{j=1}^k H'_j$ is an irredundant sum with each H'_j hollow then $k = n$.

Thus P. Fleury's main result [Theorem 3.1 of [3]] was extended by us to the class of modules M with $\text{corank } M < \infty$ and satisfying (P_2) . The example $M = R \times R$ in R -mod, with R a nonartinian local ring shows that the above class is strictly larger than the class of modules with finite spanning dimension.

(3) Let $S(M) = \{H/H \subset M, H \text{ a supplement of some } N \subset M\}$. If M is a module with property (P_2) , we proved in [5] that $\text{corank } M = \text{Sup } \{k \mid \text{there exists a strict increasing chain } H_0 \subsetneq H_1 \subsetneq \dots \subsetneq H_k \text{ of length } k \text{ with each } H_i \in S(M)\}$. For the example $M = R \times R$ in R -mod with R a nonartinian local ring we have $\text{corank } M = 2$. Thus there exist strict increasing chains of supplement submodules in M of length 2 and not more. However $s \cdot dM = \infty$. Thus in this example $\text{Sup } \{k \mid \text{there exists a strict increasing chain } H_0 \subsetneq \dots \subsetneq H_k \text{ with each } H_i \in S(M) = 2 < s \cdot dM\}$. This is one instance where corank is better behaved than spanning dimension.

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Krishnaswami Alladi and Paul Erdős, <i>On the asymptotic behavior of large prime factors of integers</i>	295
Alfred David Andrew, <i>A remark on generalized Haar systems in L_p, $1 < p < \infty$</i>	317
John M. Baker, <i>A note on compact operators which attain their norm</i>	319
Jonathan Borwein, <i>Weak local supportability and applications to approximation</i>	323
Tae Ho Choe and Young Soo Park, <i>Wallman's type order compactification</i>	339
Susanne Dierolf and Ulrich Schwanengel, <i>Examples of locally compact noncompact minimal topological groups</i>	349
Michael Freedman, <i>A converse to (Milnor-Kervaire theorem) $\times R$ etc.</i>	357
George Golightly, <i>Graph-dense linear transformations</i>	371
H. Groemer, <i>Space coverings by translates of convex sets</i>	379
Rolf Wim Henrichs, <i>Weak Frobenius reciprocity and compactness conditions in topological groups</i>	387
Horst Herrlich and George Edison Strecker, <i>Semi-universal maps and universal initial completions</i>	407
Sigmund Nyrop Hudson, <i>On the topology and geometry of arcwise connected, finite-dimensional groups</i>	429
K. John and Václav E. Zizler, <i>On extension of rotund norms. II</i>	451
Russell Allan Johnson, <i>Existence of a strong lifting commuting with a compact group of transformations. II</i>	457
Bjarni Jónsson and Ivan Rival, <i>Lattice varieties covering the smallest nonmodular variety</i>	463
Grigori Abramovich Kolesnik, <i>On the order of Dirichlet L-functions</i>	479
Robert Allen Liebler and Jay Edward Yellen, <i>In search of nonsolvable groups of central type</i>	485
Wilfrido Martínez T. and Adalberto Garcia-Maynez Cervantes, <i>Unicoherent plane Peano sets are σ-unicoherent</i>	493
M. A. McKiernan, <i>General Pexider equations. I. Existence of injective solutions</i>	499
M. A. McKiernan, <i>General Pexider equations. II. An application of the theory of webs</i>	503
Jan K. Pachl, <i>Measures as functionals on uniformly continuous functions</i>	515
Lee Albert Rubel, <i>Convolution cut-down in some radical convolution algebras</i>	523
Peter John Slater and William Yslas Vélez, <i>Permutations of the positive integers with restrictions on the sequence of differences. II</i>	527
Raymond Earl Smithson, <i>A common fixed point theorem for nested spaces</i>	533
Indulata Sukla, <i>Generalization of a theorem of McFadden</i>	539
Jun-ichi Tanaka, <i>A certain class of total variation measures of analytic measures</i>	547
Kalathoor Varadarajan, <i>Modules with supplements</i>	559
Robert Francis Wheeler, <i>Topological measure theory for completely regular spaces and their projective covers</i>	565