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## SPLITTING AND MODULARLY PERFECT FIELDS

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Let  $K$  be a field of characteristic  $p \neq 0$ . A field extension  $L/K$  is said to split when there exist intermediate fields  $J$  and  $D$  of  $L/K$  where  $J$  is purely inseparable over  $K$ ,  $D$  is separable over  $K$  and  $L = J \otimes_K D$ .  $K$  is modularly perfect if  $[K:K^p] \leq p$ . Every finitely generated extension of a modularly perfect field splits. This paper develops criteria for an arbitrary extension  $L/K$  to split and presents an example of an extension of a modularly perfect field which does not split. Necessary and/or sufficient conditions are also developed for the following to hold for an extension  $L/K$ : (a)  $L'/K$  splits for every intermediate field  $L'$ ; (b)  $L'/K$  is modular for every intermediate field  $L'$ ; (c)  $L/L'$  splits for every intermediate field  $L'$ ; (d)  $L/L'$  is modular for every intermediate field  $L'$ .

**Introduction.** Let  $K$  be a field of characteristic  $p \neq 0$ . A field extension  $L/K$  is said to split when there exists intermediate fields  $J$  and  $D$  of  $L/K$  where  $J/K$  is purely inseparable,  $D/K$  is separable, and  $L = J \otimes_K D$ . It is a classic result that any normal algebraic field extension  $L/K$  must split. Recent papers have been concerned with nonalgebraic extensions  $L/K$ . Suppose there exists an intermediate field  $J$  of  $L/K$  such that  $L/J$  is separable and  $J/K$  is purely inseparable (hence  $J = L \cap K^{p^{-\infty}}$ ). Under certain conditions, namely, if  $L$  has a separating transcendence basis over  $J$  [4], or if  $J$  is of bounded exponent over  $K$  [5], then  $L/K$  must split. That some conditions must be put on  $L/J/K$  is illustrated by an example in [1].

A field extension  $L/K$  is called modular if  $L^{p^n}$  and  $K$  are linearly disjoint for all  $n$ . The importance of modular extensions was first observed by Sweedler [11] who used this property to characterize purely inseparable extensions of bounded exponent which were tensor products of simple extensions. In [4] it was shown that if  $L/K$  is an (arbitrary) modular extension then there must exist an intermediate field  $J$  such that  $L/J$  is separable and  $J/K$  is purely inseparable modular. It follows that any finitely generated modular extension must split. In [5], a field  $K$  such that  $[K:K^p] \leq p$  is called modularly perfect. Such fields are characterized by the fact that any extension  $L$  of such a field  $K$  must be modular over  $K$ . In view of the above results, a natural question is whether every extension of a modularly perfect field  $K$  must split. In part I we develop a number of criterion for a field extension to split. We construct an extension  $L$  of a

modularly perfect field  $K$  which does not split. The field  $L$  also does not have a distinguished separable subfield [3], where  $D$  is distinguished in  $L/K$  if and only if  $D/K$  is separable and  $L \subseteq D(K^{p^{-\infty}})$ . In the remainder of the paper we determine necessary and/or sufficient conditions for the following to hold for an extension  $L/K$ :

- (a)  $L'/K$  splits for every intermediate field  $L'$ ;
- (b)  $L'/K$  is modular for every intermediate field  $L'$ ;
- (c)  $L/L'$  splits for every intermediate field  $L'$ ;
- (d)  $L/L'$  is modular for every intermediate field  $L'$ .

1. Let  $L \supseteq K$  be fields of characteristic  $p \neq 0$ . An intermediate field  $D$  of  $L/K$  is called a distinguished separable intermediate field [3] if  $D$  is separable over  $K$  and  $L \subseteq D \otimes_K K^{p^{-\infty}}$ .

REMARK 1.1. The following conditions are equivalent on  $K$ .

- (1)  $L/K$  splits for every finitely generated field extension  $L/K$ .
- (2)  $[K:K^p] \leq p$ .
- (3)  $L/K$  splits for every field extension  $L/K$  which has a distinguished separable intermediate field.

*Proof.* 1  $\rightarrow$  2: Suppose  $[K:K^p] > p$ . Let  $x, y$  be  $p$ -independent in  $K$  and let  $z$  be transcendental over  $K$ . Then  $L = K(z, zx^{p^{-1}} + y^{p^{-1}})$  is a finitely generated extension of  $K$  which does not split.

2  $\rightarrow$  3: By [5, Theorem 6, p. 1180],  $L/K$  is modular and whence splits [8, Corollary, p. 607].

3  $\rightarrow$  1: If  $L/K$  is finitely generated, then  $L/K$  has a distinguished separable intermediate field.

In [5, Theorem 6, p. 1180] it was shown that  $[K:K^p] \leq p$  if and only if for every field extension  $L/K$  there exists a separable field extension  $S$  of  $K$  (not necessarily in  $L$ ) such that  $L \subseteq S \otimes_K (K^{p^{-\infty}} \cap L)$ . Obviously  $S$  can be chosen as an intermediate field of  $L/K$  if and only if  $L/K$  splits. We now develop criterion for an extension  $L/K$  to split and present an example of an extension of a modularly perfect field which does not split.

LEMMA 1.2. *Let  $D$  be an intermediate field of  $L/K$  such that  $L/D$  is purely inseparable and  $D/K$  is separable. Then  $D$  is maximal separable if and only if  $L^p \cap D \subseteq K(D^p)$ .*

*Proof.* Assume  $D$  is maximal and let  $b \in L \setminus D$ . If  $b^p \in D \setminus K(D^p)$ , then  $D(b)/K$  is separable as follows: Let  $G$  be a  $p$ -basis for  $K$ . Then  $G$  is  $p$ -independent in  $D$ . Since  $b^p \in D \setminus K(D^p)$ ,  $G \cup \{b^p\}$  is  $p$ -independent in  $D$ . Hence there does not exist  $c \in G$  such that  $c \in D^p(b^p, G \setminus \{c\})$ . Thus  $G$  is  $p$ -independent in  $D(b)$  and  $D(b)/K$  is separable. However

this contradicts the maximality of  $D$ . Thus  $b^p \in K(D^p)$  and  $L^p \cap D \subseteq K(D^p)$ .

Conversely, assume  $L^p \cap D \subseteq K(D^p)$ . If  $L = D$ , then  $D$  is maximal. Suppose  $L \supset D$  and let  $b \in L \setminus D$  be such that  $b^p \in D$ . Then  $b^p \in L^p \cap D \subseteq K(D^p)$  so  $b \in D \otimes_K K^{p^{-1}}$ . Thus  $D(b)/K$  is not separable and hence  $D$  is maximal.

**THEOREM 1.3.** *Suppose  $L \supseteq K^{p^{-\infty}}$  and  $L/L^{p^\infty}$  has a separating transcendence basis. Then  $L/K$  splits.*

*Proof.* Since  $L \supseteq K^{p^{-\infty}}$ ,  $L^{p^\infty} \supseteq K^{p^{-\infty}}$ . Let  $D$  be a maximal separable extension of  $K$  in  $L^{p^\infty}$ . We first show  $D^{p^{-1}} \cap L^{p^\infty} \subseteq D(K^{p^{-\infty}})$ . If  $b \in D^{p^{-1}} \cap L^{p^\infty}$ , then  $b^p \in K(D^p)$  by the previous lemma. Hence  $b \in D(K^{p^{-1}}) \subset D(K^{p^{-\infty}})$ , as desired. Now  $L^{p^\infty}/D$  is purely inseparable, and hence  $L^{p^\infty}/D(K^{p^{-\infty}})$  is also purely inseparable. We prove  $L^{p^\infty} = D(K^{p^{-\infty}})$  by showing that each element  $b \in L^{p^\infty}$  of exponent one over  $D(K^{p^{-\infty}})$  is actually in  $D(K^{p^{-\infty}})$ . For such  $b$ ,  $b^p = \sum d_i e_i$  where  $d_i \in D$  and  $e_i \in K^{p^{-\infty}}$ . Hence  $b = \sum d_i^{p^{-1}} e_i^{p^{-1}}$  where each  $d_i^{p^{-1}} \in L^{p^\infty}$  is of exponent one over  $D$ . As noted above, each  $d_i^{p^{-1}} \in D(K^{p^{-\infty}})$  and thus  $b \in D(K^{p^{-\infty}})$  and  $L^{p^\infty} = D \otimes_K K^{p^{-\infty}}$ . Now  $L/L^{p^\infty}$  has a separating transcendence basis and  $L^{p^\infty}/D$  is purely inseparable. Hence there exists an intermediate field  $D^*$  of  $L/D$  such that  $D^*/D$  is separable and  $L = D^* \otimes_D L^{p^\infty}$  [4, Proposition 1, p. 2]. Thus  $L = D^* \otimes_D (D \otimes_K K^{p^{-\infty}}) = D^* \otimes_K K^{p^{-\infty}}$ . Since  $D^*/D$  and  $D/K$  are separable,  $D^*/K$  is separable.

**COROLLARY 1.4.** (1) *If  $[K: K^p] \leq p$ , then  $L/K$  splits for every field extension  $L/K$  such that  $L/L^{p^\infty}$  has a separating transcendence basis.*

(2) *Conversely, suppose  $K/K^{p^\infty}$  has a separating transcendence basis. If  $L/K$  splits for every field extension  $L/K$  such that  $L/L^{p^\infty}$  has a separating transcendence basis, then  $[K: K^p] \leq p$ .*

*Proof.* (1) If  $L \supseteq K^{p^{-\infty}}$ , then  $L/K$  splits by 1.3. If  $L \not\supseteq K^{p^{-\infty}}$ , then  $(L \cap K^{p^{-\infty}})/K$  has bounded exponent. Since  $[K: K^p] \leq p$ ,  $L/K$  is modular [5, Theorem 1, p. 1177] and hence splits [5, Theorem 3, p. 1178].

(2) Suppose  $[K: K^p] > p$ . Let  $T$  be a separating transcendence basis for  $K/K^{p^\infty}$ . Then  $T$  is a  $p$ -basis for  $K$  and  $|T| > 1$ . Let  $\{x, y\} \subseteq T$  and set  $L = K(z, zx^{p^{-1}} + y^{p^{-1}})$  where  $z$  is transcendental over  $K$ . If we show  $L/L^{p^\infty}$  has a separating transcendence basis, we have a contradiction since  $L/K$  does not split. Now  $T \setminus \{y\} \cup \{z, zx^{p^{-1}} + y^{p^{-1}}\}$  is a  $p$ -basis for  $L$ .  $L^{p^\infty} \subseteq \bigcap_{i=1}^{\infty} K(L^{p^i}) = K$ , so  $L^{p^\infty} = K^{p^\infty}$ .  $K/L^{p^\infty}(T)$  is separable algebraic so  $L/L^{p^\infty}(T \setminus \{y\}, z, zx^{p^{-1}} + y^{p^{-1}})$  is separable algebraic

since  $y \in L^{p^\infty}(T \setminus \{y\}, z, zx^{p^{-1}} + y^{p^{-1}})$ . Thus  $L/L^{p^\infty}$  has a separating transcendence basis.

**PROPOSITION 1.5.** *If  $L/K$  splits for every field extension  $L/K$  such that  $L/L \cap K^{p^\infty}$  is separable, then  $[K:K^p] < \infty$ .*

*Proof.* Suppose  $[K:K^p] = \infty$  and let  $\{x_1, \dots, x_n, \dots\}$  be a  $p$ -independent subset of  $K$ . Let  $J = K(x_1^{p^{-1}}, x_2^{p^{-2}}, \dots)$  and  $L = J(z, z^{p^{-1}} + x_1^{p^{-2}}, \dots, z^{p^{-n}} + x_1^{p^{-n-1}} + \dots + x_n^{p^{-n-1}}, \dots)$  where  $z$  is transcendental over  $J$ . Since  $L$  is the union of a chain of simple transcendental extensions of  $J$ ,  $L/J$  is separable and  $L \cap K^{p^\infty} = J$ . The proof that  $L/K$  does not split is completely analogous to the proof in [1].

Suppose  $K$  is a modularly perfect field, i.e.,  $[K:K^p] \leq p$ . Then for any field  $L$  which contains  $K$ ,  $L$  is separable over  $L \cap K^{p^\infty}$ . If  $L \cap K^{p^\infty} \neq K^{p^\infty}$ , then  $L/K$  must split. We now present an example where  $L \supset K^{p^\infty}$  and yet  $L/K$  does not split. This example indicates that the result presented in [5, Theorem 6, p. 1180] is in some sense the best possible.

**EXAMPLE 1.6.** Let  $P$  be a perfect field and let  $x, y, w_0$  be algebraically independent indeterminates over  $P$ . Set  $K = P(y)$  and  $L = K^{p^\infty}(x, w_0, w_1^{p^{-1}}, \dots, w_n^{p^{-n}}, \dots)$  where  $w_1^{p^{-1}} = x^{p^{-1}} + y^{p^{-3}}w_0^{p^{-1}}$  and  $w_n^{p^{-n}} = x^{p^{-1}} + y^{p^{-(2n+1)}}w_{n-1}^{p^{-n}}$ . Then  $[K:K^p] = p$  and  $K^{p^\infty} \subset L$ . We show that  $L/K$  does not split. Assume  $L = S \otimes_K K^{p^\infty}$  where  $S$  is separable over  $K$ . Consider  $S' = S(x, w_0)$ . Since  $S'$  is finitely generated over  $S$ ,  $S'/K$  has bounded exponent and hence splits since  $K$  is modularly perfect [5, Theorem 6, p. 1180]. Let  $S' = S^* \otimes_K K(y^{p^{-t}})$ , and hence  $L = S' \otimes_{K(y^{p^{-t}})} K^{p^\infty}$ . Now by construction,  $K(y^{p^{-t}}, x, w_0) \equiv K_t \subseteq S'$  and  $K_t \otimes_{K(y^{p^{-t}})} K^{p^\infty} = K^{p^\infty}(x, w_0)$ . The fields which lie between  $K^{p^\infty}(x, w_0)$  and  $L$  are chained and each is a purely inseparable extension of exponent one of the previous one. Hence the same is true for the fields which lie between  $K_t$  and  $S'$ . Since  $K^{p^\infty}(x, w_0)/K_t$  is also modular, it follows that  $L/K_t$  is modular, and in fact  $[K_t^{p^{-n}} \cap L: K_t] = p^{2n}$  for all  $n$ . Since any finitely generated extension of  $K_t$  in  $L$  is contained in  $K_t^{p^{-n}} \cap L$  for some  $n$ , and  $K_t^{p^{-n}} \cap L$  is modular over  $K_t$  with two elements in any subbase, we conclude that any finitely generated extension of  $K_t$  in  $L$  must be modular over  $K_t$  [9, Proposition 2.5, p. 76].

We now show that there is a field  $M$  which lies between  $L$  and  $K_t$  which is not modular over  $K_t$ . Since  $w_n^{p^{-n}} = x^{p^{-1}} + y^{p^{-(2n+1)}}w_{n-1}^{p^{-n}}$ , for large  $n$ ,  $w_n^{p^{-n}}$  will not be of exponent  $n$  over  $K_t$ . Thus assume  $w_{n-1}^{p^{-(n-1)}}$  is of exponent  $n-1$  over  $K_t$  and  $w_n^{p^{-n}}$  is not of exponent  $n$ . It follows that  $(y^{p^{-(2n+1)}})^{p^n} = y^{p^{-(n+1)}}$  is not an element of  $K_t$ . Let  $M = K_t(y^{p^{-(2n+1)}}, w_n^{p^{-n}})$ . If we show every higher derivation on  $M$  over  $K_t$

maps  $y^{p^{-(n+1)}}$  to 0,  $M/K_t$  is not modular [11, Theorem 1, p. 403]. Recall that a higher derivation of  $M$  over  $K_t$  is a sequence of  $K_t$ -linear maps  $D^{(s)} = \{D_0 = I, D_1, \dots, D_s\}$  of  $M$  into itself such that  $D_m(bc) = \sum_{i=0}^m D_i(b)D_{m-i}(c)$  for all  $b, c \in M$ ,  $m = 0, \dots, s$ . We shall need the direct corollary to [13, p. 436] that  $D_m(b^{p^r}) = 0$  if  $p^r \nmid m$  and  $D_m(b^{p^r}) = (D_{m/p^r}(b))^{p^r}$  if  $p^r \mid m$ . We follow the method of Sweedler [11, Example 1.1, p. 405]. Assume there exists  $D_s$  such that  $D_s(y^{p^{-(n+1)}}) \neq 0$ . Then

$$\begin{aligned} w_{n-1}[D_{s/p^n}(y^{p^{-(2n+1)}})]^{p^n} &= D_s(w_{n-1}y^{p^{-(n+1)}}) \\ &= D_s(x^{p^{n-1}} + y^{p^{-(n+1)}}w_{n-1}) \\ &= [D_{s/p^n}(x^{p^{n-1}} + y^{p^{-(2n+1)}}w_{n-1}^{p^{-n}})]^{p^n}. \end{aligned}$$

Hence

$$w_{n-1} = \frac{[D_{s/p^n}(x^{p^{n-1}} + y^{p^{-(2n+1)}}w_{n-1}^{p^{-n}})]^{p^n}}{[D_{s/p^n}(y^{p^{-(2n+1)}})]^{p^n}}$$

which is an element of  $M^{p^n}$ .

Thus  $w_{n-1}^{p^{-n}} \in M$  and hence  $x^{p^{-1}} \in M$ , a contradiction since  $x^{p^{-1}}$  is not in  $L$ .

In the previous example,  $L$  is of transcendence degree two over  $K$ . The next result illustrates that this is the least degree possible for an extension which does not split.

**PROPOSITION 1.7.**  $[K:K^p] \leq p$  if and only if  $L/K$  splits for every field extension  $L/K$  of transcendence degree one.

*Proof.* Suppose  $[K:K^p] \leq p$ . As usual it suffices to consider the case where  $L \supseteq K^{p^\infty}$ . If  $L = L^p$ , then  $L = L^{p^\infty}$  and  $L/K$  splits by 1.4. If  $L \supset L^p$ , we show  $L$  has a separating transcendence basis over  $L^{p^\infty}$ . Since  $L \supset K^{p^\infty}$ ,  $L/L^{p^\infty}$  has transcendence degree 1. Let  $B$  be a  $p$ -basis for  $L$ . Since  $B$  is algebraically independent over  $L^{p^\infty}$ ,  $B$  consists of exactly one element and  $L/L^{p^\infty}(B)$  is algebraic. Since  $L/L^{p^\infty}(B)$  is separable,  $B$  is a separating transcendence basis for  $L/L^{p^\infty}$  and 1.3 applies.

Conversely, suppose  $[K:K^p] > p$ . Let  $x, y$  be  $p$ -independent in  $K$  and let  $z$  be transcendental over  $K$ . Then  $L/K$  does not split where  $L = K(z, zx^{p^{-1}} + y^{p^{-1}})$ .

As Example 1.6 illustrates, not every extension  $L/K$  of a modularly perfect field need split. The following result gives several criteria for such an extension to split.

**THEOREM 1.8.** Assume  $[K:K^p] = p$  and  $L \supseteq K^{p^\infty}$ . The following are equivalent.

(1)  $L/K$  splits.

(2) There exists a maximal separable extension  $D$  of  $K$  in  $L$  such that  $L$  is modular over  $D$ .

(3) There exists a maximal separable extension  $D$  of  $K$  in  $L$  such that some relative  $p$ -basis for  $D$  over  $K$  remains  $p$ -independent in  $L$ .

(4) There exists a proper intermediate field  $D$  of  $L/K$  such that  $L = D(K^{p^{-\infty}})$ .

*Proof.* Assume  $L = D \otimes_K K^{p^{-\infty}}$ . Then  $D$  satisfies properties (2), (3), and (4). Assume (2). Since  $D$  is a maximal separable extension of  $K$  in  $L$ ,  $L/D$  is purely inseparable. Since  $D((D(K^{p^{-\infty}}))^p) = D(K^{p^{-\infty}})$ ,  $D(K^{p^{-\infty}})$  is pure in  $L/D$  [12, Definition, p. 41]. We claim  $D^{p^{-1}} \cap L = D^{p^{-1}} \cap (D(K^{p^{-\infty}}))$ . Let  $b \in (D^{p^{-1}} \cap L) \setminus D$ . Then  $D(b)$  and  $K^{p^{-1}}$  are not linearly disjoint over  $K$  since  $D(b)$  is not separable over  $K$ . Since  $[K^{p^{-1}}:K] = p$ ,  $K^{p^{-1}} \subseteq D(b)$  and hence  $D(b) = D(K^{p^{-1}})$ . Thus  $b \in D(K^{p^{-\infty}})$  and the claim is established. By [12, Proposition 2.7, p. 44],  $D(K^{p^{-\infty}}) = D \otimes_K K^{p^{-\infty}} = L$  and (1) holds. Assume (3). A relative  $p$ -basis  $B$  for  $D$  over  $K$  is a  $p$ -basis for  $D(K^{p^{-\infty}})$ . Since this  $p$ -basis remains  $p$ - $p$ -independent in  $L$ ,  $L/D(K^{p^{-\infty}})$  is separable. Since  $L/D$ , whence  $L/D(K^{p^{-\infty}})$  is also purely inseparable,  $L = D \otimes_K K^{p^{-\infty}}$  and (1) holds. Assume (4). Since  $D$  is proper,  $D \not\subseteq K^{p^{-\infty}}$  and hence  $D/K$  splits, say  $D = D' \otimes_K K^{p^{-n}}$ . Then  $L = D' \otimes_K K^{p^{-\infty}}$ .

II. In this section we determine necessary and/or sufficient conditions for the following to hold for an arbitrary extension  $L/K$ ;

(a)  $L'/K$  splits for any intermediate field  $L'$ ;

(b)  $L'/K$  is modular for any intermediate field  $L'$ .

We will need the following result.

**LEMMA 2.1.** *Suppose  $L/K$  splits and the intermediate fields of  $(L \cap K^{p^{-\infty}})/K$  appear in a chain. If  $L'$  is an intermediate field of  $L/K$ , then  $L'/(L' \cap K^{p^{-\infty}})$  is separable.*

*Proof.* We first note that  $L \cap K^{p^{-\infty}}$  and  $L'$  are linearly disjoint over  $L' \cap K^{p^{-\infty}}$ . This follows since  $L \cap K^{p^{-\infty}}/L' \cap K^{p^{-\infty}}$  is purely inseparable and the intermediate fields are chained. Now since  $L/K$  splits,  $L/L \cap K^{p^{-\infty}}$  is separable and hence  $(L \cap K^{p^{-\infty}})(L')$  is separable over  $L \cap K^{p^{-\infty}}$ . By [6, Corollary 6, p. 266], we conclude  $L'/K^{p^{-\infty}} \cap L'$  is separable.

**THEOREM 2.2.** *Suppose  $L/K$  is inseparable but not purely inseparable. Then each condition in the following list implies the succeeding one.*

- (1)  $L/K$  splits and  $(L \cap K^{p^{-\infty}})/K$  is simple.
- (2)  $L'/K$  splits for every intermediate field  $L'$ .
- (3)  $L/K$  splits and the intermediate fields of  $(L \cap K^{p^{-\infty}})/K$  appear in a chain.

*Proof.* (1) implies (2): Let  $L'$  be an intermediate field. By 2.1,  $L'/(L' \cap K^{p^{-\infty}})$  is separable. Since  $(L \cap K^{p^{-\infty}})/K$  is simple,  $(L' \cap K^{p^{-\infty}})/K$  is of bounded exponent and so  $L'/K$  splits [5, Theorem 4, p. 1178].

(2) implies (3): Suppose the intermediate fields of  $(L \cap K^{p^{-\infty}})/K$  do not appear in a chain. Then there exist  $b, c$  in  $L \cap K^{p^{-\infty}}$  such that  $b \notin K(c)$ ,  $c \notin K(b)$ , and both  $b$  and  $c$  have some positive exponent  $i$  over  $K$ . Let  $z \in L \setminus K$  be such that  $K(z)/K$  is separable. Let  $L' = K(z, zb + c)$ . Now  $K(z)$  is a distinguish maximal separable intermediate field of  $L'/K$ . Since  $L'/K$  splits,  $L' = K(z) \otimes_K J$  by [8, Lemma, p. 607] where  $J = L' \cap K^{p^{-\infty}}$ . Let  $\mathcal{Z}$  be a linear basis of  $K(z)/K$  with  $1, z \in \mathcal{Z}$ . Now  $zb + c = \sum x_j d_j$  where  $x_j \in \mathcal{Z}$  and  $d_j \in J$ . Thus

$$(*) \quad z^{p^i} b^{p^i} + c^{p^i} = \sum x_j^{p^i} d_j^{p^i}, d_j^{p^i} \in J^{p^i} \cap K.$$

Since  $K(z)/K$  is separable,  $\mathcal{Z}^{p^i}$  is linearly independent over  $K$ . Hence by equating coefficients in (\*) we have  $b^{p^i}, c^{p^i} \in J^{p^i}$ . Thus  $b, c \in J$ . Hence  $[J:K] > p^i$  and  $[L':K(z)] > p^i$ , a contradiction. Hence the intermediate fields of  $(L \cap K^{p^{-\infty}})/K$  appear in a chain.

**COROLLARY 2.3.** *Suppose  $L/K$  has finite inseparability exponent. Then the following conditions are equivalent.*

- (1)  $L'/K$  is modular for every intermediate field  $L'$ .
- (2)  $L/K$  is modular and  $(L \cap K^{p^{-\infty}})/K$  is simple.
- (3)  $L'/K$  splits for every intermediate field  $L'$ .
- (4)  $L/K$  splits and  $(L \cap K^{p^{-\infty}})/K$  is simple.

*Proof.* A modular field extension with finite inseparability exponent splits. Also a field extension which splits and whose maximal purely inseparable subfield is simple is necessarily modular.

**COROLLARY 2.4.** *Suppose  $L/K$  is inseparable and  $L/(L \cap K^{p^{-\infty}})$  has a finite separating transcendence basis. Then the following conditions are equivalent.*

- (1)  $L'/K$  is modular for every intermediate field  $L'$ .
- (2)  $L'/K$  splits for every intermediate field  $L'$ .
- (3) The intermediate fields of  $(L \cap K^{p^{-\infty}})/K$  appear in a chain.

*Proof.* Assume (1). Let  $L'$  be an intermediate field of  $L/K$ . If  $L' \not\cong L \cap K^{p^{-\infty}}$ , then  $L'/L \cap K^{p^{-\infty}}$  has a finite separating transcendence



basis [10, Theorem 1, p. 418] so  $L'/K$  splits by [4, Proposition 1, p. 2]. If  $L' \not\cong L \cap K^{p^{-\infty}}$ , then  $L' \cap K^{p^{-\infty}}/K$  is a simple extension since the intermediate fields of  $L \cap K^{p^{-\infty}}/K$  must be chained. For if they are not chained, choose  $b, c, z$  as in the proof of 2.2 and  $K(z, zb + c)$  is not modular over  $K$ . Now we have  $(L' \cap K^{p^{-\infty}})/K$  simple of bounded exponent,  $L'/L' \cap K^{p^{-\infty}}$  separable since  $L'/K$  is assumed modular, and hence  $L'/K$  splits by [5, Theorem 4, p. 1178].

That (2) implies (3) is part of 2.2. Assume (3). Since  $L/L \cap K^{p^{-\infty}}$  has a finite separating transcendence basis,  $L/K$  splits. By 2.1, if  $L'$  is an intermediate field of  $L/K$ , then  $L'/L' \cap K^{p^{-\infty}}$  is separable. Since  $L' \cap K^{p^{-\infty}}/K$  is modular,  $L'/K$  is modular by [11, Lemma 5(3), p. 407].

**COROLLARY 2.5.** *Suppose  $L/K$  is inseparable but not purely inseparable. Then each statement in the following list implies the succeeding one.*

- (1)  $L/K$  is modular and  $(L \cap K^{p^{-\infty}})/K$  is simple.
- (2)  $L'/K$  is modular for every intermediate field  $L'$ .
- (3)  $L/K$  is modular and the intermediate fields of  $(L \cap K^{p^{-\infty}})/K$  appear in a chain.

*Proof.* Straightforward.

III. We now determine necessary and sufficient conditions for the following to hold;

- (a)  $L/L'$  splits for any intermediate field  $L'$ .
- (b)  $L/L'$  is modular for any intermediate field  $L'$ .

**THEOREM 3.1.** (1) *Suppose  $L \not\cong K^{p^{-\infty}}$ . Then  $L/L'$  splits for every intermediate field  $L'$  of  $L/K$  if and only if  $L/K$  is algebraic and  $L/K$  splits.*

(2) *Suppose  $L \cong K^{p^{-\infty}}$ . Then  $L/L'$  splits for every intermediate field  $L'$  of  $L/K$  if and only if  $L = L^p$ .*

*Proof.* (1) Suppose  $L/L'$  splits for every intermediate field  $L'$ . Let  $J = L \cap K^{p^{-\infty}}$  and let  $b \in J \setminus J^p$ . Then  $b^{p^{-1}} \notin L$ . Assume  $L/J$  is not algebraic and let  $z \in T$ , where  $T$  is a transcendence basis for  $L/J$ . Let  $\hat{J} = J(T \setminus \{z\})$  and  $w = -z^{2p}(z^p + b)^{-1}$ . Then  $z^{2p} + wz^p + wb = 0$ . The polynomial  $X^{2p} + wX^p + wb$  is irreducible over  $\hat{J}(w)$  by Eisenstein's criterion. However  $L/\hat{J}(w)$  does not split else  $w^{p^{-1}}, w^{p^{-1}}b^{p^{-1}} \in L$  by [9, Lemma 3.7, p. 102] and so  $b^{p^{-1}} \in L$ . Thus  $L/K$  is algebraic. Clearly  $L/K$  splits. Conversely, suppose  $L/K$  is algebraic and splits, say  $L = S \otimes_K J$  where  $S/K$  is separable algebraic. Then  $L =$

$SL' \otimes_{L'} JL'$  for every intermediate field  $L'$  of  $L/K$ .

(2) Suppose  $L/L'$  splits for every  $L'$ . If  $L/K$  is algebraic, then  $L$  is separable algebraic over the perfect field  $K^{p^{-\infty}}$ , and is thus perfect. Suppose  $L/K$  is not algebraic and  $L \neq L^p$ . Let  $b \in L \setminus L^p$ . Then  $L/\hat{J}(w)$  does not split, a contradiction, where  $\hat{J}(w)$  is the field defined in (1) above. Hence  $L = L^p$ . Conversely, suppose  $L = L^p$ . Let  $L'$  be an intermediate field of  $L/K$ . Since  $L = L^p, L'^{p^{-\infty}} \subseteq L$ . Hence  $L/L'$  splits by 1.3.

**THEOREM 3.2.** (1) *Suppose  $L \not\cong K^{p^{-\infty}}$ . Then  $L/L'$  is modular for every intermediate field  $L'$  of  $L/K$  if and only if  $L/K$  is algebraic splits and  $L/L'$  is modular for every intermediate field  $L'$  of  $L/S$  where  $S$  is the maximal separable intermediate field of  $L/K$ .*

(2) *Suppose  $L \cong K^{p^{-\infty}}$ . Then  $L/L'$  is modular for every intermediate field  $L'$  of  $L/K$  if and only if  $L = L^p$ .*

*Proof.* (1) Suppose  $L/L'$  is modular for every  $L'$ . If  $L/K$  is not algebraic, then we can construct the field  $\hat{J}(w)$  of 3.1. Since  $L/\hat{J}(w)$  is algebraic,  $L/\hat{J}(w)$  cannot be modular else it would split. Thus  $L/K$  is algebraic. Since  $L/K$  is also modular,  $L/K$  splits. Conversely, suppose  $L/K$  is algebraic and splits and  $L/L'$  is modular for every intermediate field  $L'$  of  $L/S$ . Then  $L/L'$  is modular for every intermediate field  $L'$  of  $L/K$  by [7, Lemma 4, p. 340] since  $L/L'$  necessarily splits.

(2) Suppose  $L/L'$  is modular for every  $L'$ . If  $L/K$  is algebraic, then  $L$  is separable algebraic over the perfect field  $K^{p^{-\infty}}$  and is thus perfect. Suppose  $L/K$  is not algebraic and  $L \neq L^p$ . Let  $b \in L \setminus L^p$ . Then  $L/\hat{J}(w)$  is not modular, a contradiction, where  $\hat{J}(w)$  is the field defined in 3.1. Hence  $L = L^p$ . Conversely, suppose  $L = L^p$ . Let  $L'$  be an intermediate of  $L/K$ . By 3.1(2),  $L/L'$  splits, say  $L = D \otimes_{L'} L'^{p^{-\infty}}$  where  $D$  is separable over  $L'$ . Since  $L'^{p^{-\infty}}/L'$  is modular,  $L/L'$  is modular.

**COROLLARY 3.3.** *Let  $L$  be a perfect field. Then  $L$  splits over every subfield.*

*Proof.*  $Z_p^{p^{-\infty}} = Z_p \subseteq L$  and  $L = L^p$ .

**COROLLARY 3.4.** *Consider the following statements:*

- (a)  $L/L'$  and  $L'/K$  split for every intermediate field  $L'$ .
- (b)  $L/L'$  and  $L'/K$  are modular for every intermediate field  $L'$ .
- (c)  $L/K$  is algebraic and splits and the intermediate fields of  $(L \cap K^{p^{-\infty}})/K$  appear in chain.
- (d)  $L = L^p$  and  $[K: K^p] \leq p$ .

Then; (1) Suppose  $L \not\cong K^{p^{-\infty}}$ . Then (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (c).  
 (2) Suppose  $L \cong K^{p^{-\infty}}$ . Then (a)  $\Rightarrow$  (b)  $\Leftrightarrow$  (d).

*Proof.* Straightforward.

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