ON THE SOBRIFICATION REMAINDER $sX - X$

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The topics of this paper are (1) a study of the sobri- fication remainder $*X - X$ (hence our title), (2) a new, simple proof of the characterization of $T_\delta$-spaces $Y$ as those spaces $Y$ such that $Y$ is the smallest subspace $X$ of $*Y$ for which the embedding $X \hookrightarrow *Y$ is the universal sobrification, (3) an elegant characterization of Noetherian sober spaces. These themes are linked by the common tool by aid of which they are investigated, the so-called $b$-topology L. Skula [28].

Recall that a space $Y$ is called irreducible iff $O_1 \cap O_2 \neq \emptyset$ for every pair of nonempty open subsets $O_i$ of $Y$ ($i = 1, 2$) — sometimes, in addition, $Y \neq \emptyset$ is assumed. A space $X$ is called "sober" ([3] IV 4.2.1) iff every irreducible, nonempty, closed subset $M$ of $X$ has a unique "generic" point $m$, i.e., $M = \text{cl}\{m\}$ (hence $T_\delta$ = "sober" = $T_\sigma$). To every space $X$ one associates a sober space $*X$ whose elements are all irreducible, closed, nonempty subsets of $X$. The open sets of $*X$ are all sets of the form $^*O := \{M \in *X | M \cap O \neq \emptyset\}$ for some open set $O$ of $X$. The map $\chi: x \mapsto \text{cl}\{x\}$ is the reflection morphism for the category $\text{Top}$ of topological spaces and continuous maps into its full subcategory $\text{cob}$ of sober spaces. If $X$ is a $T_\delta$-space, then $\chi_X$ is an embedding; we shall sometimes identify $X$ with the subspace $\chi_X[X]$ of $*X$, in particular we shall write $*X - X$ for a $T_\delta$-space $X$ instead of $*X - \chi_X[X]$. For further information on sober spaces see [19], [20] (3.1), [21] and some recent work of S. S. Hong [22], J. R. Isbell [23], L. D. Nel [26], L. D. Nel and R. G. Wilson [27] (to the historical survey of [21] p. 365/366 a reference to [8] II, (1) on p. 17 has to be added).

An essential tool for the investigation of sober spaces is the $b$-topology introduced by L. Skula ([28]; cf. also [11] p. 288). The $b$-topology associated with a space $X$ is the topology which has $\{O \cap A | O \text{ open in } X, A \text{ closed in } X\}$ as an open basis. The members of this basis are called locally closed sets (N. Bourbaki [6] Chap. I, §3.3). The terms "$b$-dense", "$b$-isolated" etc. will refer to the $b$-topology, i.e., the topological space $bX$ associated with a given space $X$; in particular, a $b$-dense subspace $Y$ of $X$ is a subspace of $X$ which is a dense subset of $bX$. A subspace $Y$ of $X$ is $b$-dense, iff whenever $O_1, O_2$ are open subsets of $X$, $O_1 \neq O_2$, then $O_1 \cap Y \neq O_2 \cap Y$. In [7] G.C.L. Brümmer looks at the uniformity (canonically) associated with the Pervin quasi-uniformity of a topological space $X$; this uniformity induces a topology which is easily seen to be the $b$-topology associated
to the space $X$: thus $bX$ is uniformizable by a distinguished uniformity ([7] p. 408). We note further that $bX$ is $O$-dimensional, i.e., it has an open basis of sets which are both closed and open.

Recall that a space $X$ is $T_b$ iff for every $x \in X$ there is an open neighborhood $U$ of $x$ with $U \cap \text{cl} \{x\} = \{x\}$, i.e., every point of $X$ is locally closed. The $T_b$-axiom was introduced by G. Bruns [8] II p. 7 ("$T_{1/2}$") and C. E. Aull and W. J. Thron [4] p. 29. For characterizations of $T_b$ see [21] 2.1 and, in addition, [30] 2.1 (g). As a recent application of the $T_b$-axiom, we note that C. C. Moore and J. Rosenberg have shown that the space of primitive ideals of the group $C^*$-algebra of a connected and locally compact group $G$ is $T_b$ ([25] Thm. 1). Furthermore cf. [14] (§§3.2, 3.3).

To a preordered set $(X, \leq)$ one may associate a topological space with the same carrier set and open basis $\{U_a \mid a \in X\}$ with $U_a := \{y \in X \mid a \leq y\}$. Such a space is called $A$-discrete (or Alexandrov-discrete) [1]. A topological space is $A$-discrete iff every union of closed sets is closed. Nowadays, $A$-discrete spaces are also known as finitely generated spaces, since they form the co-reflective hull of the class of finite spaces ([16] 22.2(4)). An $A$-discrete $T_0$-space is $T_b$ ([8] II, p. 18, [4] p. 35). For some further information see [2].

I am indebted to B. Banaschewsky (Hamilton) and J. R. Isbell (Buffalo) for discussions (during the Oberwolfach meeting on category theory, August 1977) on some themes of this paper.

**Lemma 1.1.** Suppose $\beta$ is a basis of the open sets of a space $X$, then

$$\{U \cap \text{cl} \{x\} \mid x \in U \in \beta\}$$

is a basis of the $b$-topology associated with $X$.

From this easily proved lemma we immediately obtain

**Lemma 1.2.** For topological spaces $X$ and $Y$ holds $bX \times bY = b(X \times Y)$.

**Proof.** Let $\tau_X$ and $\tau_Y$ denote the topologies of $X$ and $Y$ respectively, then $\{U \times V \mid U \in \tau_X, V \in \tau_Y\}$ is a basis for $X \times Y$, hence

$$\{(U \times V) \cap (\text{cl}_X \{x\} \times \text{cl}_Y \{y\})$$

$$= (U \cap \text{cl}_X \{x\}) \times (V \cap \text{cl}_Y \{y\}) \mid U \in \tau_X, V \in \tau_Y, x \in X, y \in Y\}$$

is a basis for $b(X \times Y)$ and, obviously, also for $bX \times bY$.

**Proposition 1.3.** Let $\{X_i \}_{i \in I}$ be a family of nonempty topological spaces. $b(\prod_I X_i) = \prod_I (bX_i)$ iff $K := \{i \in I \mid X_i$ is not indiscrete}$ is finite.
Proof. For every \( i \in K \), there is some \( x_i \in X_i \) with \( \text{cl}\{x_i\} \neq X_i \). If \( K \) is infinite, then \( \prod_{i} \text{cl}\{x_i\} \times \prod_{i \in K \setminus \{i\}} X_i \) is open in \( b(\prod_{i} X_i) \), but not open in a product topology arising from any modifications of the topologies of \( X_i \). If \( K \) is finite, then

\[
b(\prod_{i} X_i \times \prod_{i \in K \setminus \{i\}} X_i) = b(\prod_{i} X_i) \times \prod_{i \in K \setminus \{i\}} X_i = \prod_{i} (bX_i) \times \prod_{i \in K \setminus \{i\}} X_i = \prod_{i} bX_i
\]

(via some obvious identifications).

It is shown in [20] 3.1.2 that a sober space is the universal sobrification of every \( b \)-dense subspace via its embedding.

**Theorem 1.4.** For a family \( \{X_i\}_i \) of topological spaces holds \( ^s \prod_i X_i = \prod_i ^s X_i \). In other words, the reflection functor \( ^s(-) : \mathcal{Z}_{op} \to \mathcal{Sob} \) preserves products.

**Proof.** (i) We observe first the \( \mathcal{Z}_o \)-reflector \( \mathcal{Z}_{op} \to \mathcal{Z}_o \) preserves products. Recall that the canonical \( T_o \)-identification space \( X_o \) of a space \( X \) is defined by the equivalence relation \( x \approx y \iff \text{cl}\{x\} = \text{cl}\{y\} \).

(ii) Because of (i) we may assume now that every \( X_i \) is \( T_o \). Since \( \mathcal{Sob} \) is reflective in \( \mathcal{Z}_{op} \), \( \Pi_i ^s X_i \) is sober. Thus it suffices to show that \( \Pi_i X_i \) is \( \Pi_i \mathcal{X}_{x_i} \) — a \( b \)-dense subspace of \( \Pi_i ^s X_i \). Suppose \( (C_i)_i \in \Pi_i \mathcal{X}_{x_i} \); then let \( \Pi_i U_i \) be an open neighborhood of \( (C_i)_i \) with \( U_i \) open in \( X_i \); hence \( U_i = X_i \) for all but finitely many indices \( i \). Since \( U_i \cap C_i \neq \emptyset \) for every \( i \in I \), we choose some \( x_i \in U_i \cap C_i \), then \( \mathcal{X}_{x_i}(x_i) \in ^s U_i \cap cl_{X_i} \{C_i\} \). In consequence, \( \Pi_i X_i \) is \( \Pi_i \mathcal{X}_{x_i} \) — a \( b \)-dense subspace of \( \Pi_i ^s X_i \).

**Remark 1.5.** Let \( X \) be an infinite space with co-finite topology. \( ^sX - X \) consists of the unique element \( X \). Let \( \pi : X \to X \) be a permutation of \( X \) without fixed point. The equalizer of \( \text{id}_X \) and \( \pi \) is the inclusion of the empty space \( \emptyset \) into \( X \), whereas the equalizer of \( \text{id}_{^sX} \) and \( \pi : ^sX \to ^sX \) is the inclusion of the one-element set \( \{X\} \). Thus \( ^s(-) : \mathcal{Z}_{op} \to \mathcal{Sob} \) does not preserve equalizers, hence is not right adjoint.

Similarly, by two different constant selfmaps of a two point indiscrete space it is shown that the \( \mathcal{Z}_o \)-reflection functor does not preserve equalizers.

Let \( N = \{0, 1, 2, \cdots\} \) denote the space of natural numbers with its \( A \)-discrete topology, i.e., \( \emptyset \) and \( \{n, n + 1, \cdots\}(n \in N) \) are open in \( N \). Let \( ^sN \) denote the sobrification space; if we designate the unique element \( N \) of \( ^sN - N \) by \( \infty \), then \( \emptyset \) and \( \{\infty\} \cup \{n, n + 1, \cdots\} \) are the open sets of \( ^sN \)(cf. [18] Theorem 2). For an arbitrary \( T_o \)-
space $X$ let $N_x: = (sN \times sX) - ((00} \times X)$ with the topology induced from $sN \times sX$ ($X$ is to be considered as a subspace of $sX$).

**Theorem 1.6.** For every $T_0$-space $X$ holds $X \cong sN_x - N_x$, i.e., every $T_0$-space is a sobrification remainder.

**Proof.** It is sufficient to show that $sN \times sX$ is the sobrification of $N_x$ via its embedding. Thus — by the result of [20] 3.1.2 quoted above — it suffices to show that $N_x$ is $b$-dense in $sN \times sX$. This is clear from $N \times X \subseteq N_x \subseteq sN \times sX = s(N \times X)$, since $N \times X$ is $b$-dense in $s(N \times X)$ by the other implication of [20] (3.1.2).

The statement of (1.6) is analogous to the fact that every completely regular $T_2$-space is a Stone—Čech—remainder — cf. [13] (9K6, p. 138). The proof of (1.6) above is, in some sense, even more simple, since there is no straightforward analogue of (1.4) in the case of compact $T_1$-spaces. Maybe it is also worth noting that in (1.6) a single space $sN$ of ordinals suffices — other than in [13] (8K5, p. 138).

Since every $T_0$-space is a sobrification remainder of some $T_0$-space (1.6), it may be of interest to look at the sobrification remainders of certain distinguished subclasses of the class of all $T_0$-spaces, e.g., $T_D$-spaces. When is $N_x$ (1.6) a $T_D$-space?

**Lemma 1.7.** (a) If $Y$ is a $T_D$-space, then $sY - Y$ is sober.
(b) $N_x$ is $T_D$ iff $X$ is both sober and $T_D$.

**Proof.** (a) By (2.1) every element of $Y$ is $b$-isolated in $sY$, hence $Y$ is $b$-open in $sY$. Thus $sY - Y$ is $b$-closed in $sY$, hence sober.
(b) Suppose $N_x$ is $T_D$, then $N \times X = N_x$, since $N \times X$ is $b$-dense in $sN \times sX$, hence in $N_x$ (a discrete space has no proper dense subspace). In consequence, ($X = sX$ and) $X$ is $T_D$. If $X$ is sober and $T_D$, then $N_x = N \times X$ is $T_D$.

**Remark 1.8.** The sobrification process also gives rise to a (new?) cardinal invariant of a $T_0$-space $X$. Let

\[ r_nX: = X, \quad u_0X: = sX - X, \quad u_nX: = \delta(r_nX), \quad r_nX, \quad r_{n+1}X: = \delta(u_nX) - u_nX. \]

Here $\delta(-)$ denotes the $b$-closure of ($-$) in $sX$. By [20] 3.1.2

\[ u_nX \cong s(r_nX) - r_nX \]

and
We observe that
\[ r_{n+1}X \subseteq r_nX \text{ and } u_{n+1}X \subseteq u_nX. \]

For \( n \rightarrow 0 \) and, similarly, for every limit number \( \lambda \) we may define
\[ r_\lambda X := \bigcap_{\gamma < \lambda} r_\gamma X \]
and
\[ u_\lambda X := \delta(r_\lambda X) - r_\lambda X. \]

There is a smallest cardinal \( \alpha \leq \text{card } X \) such that \( r_{\alpha+1}X = r_\alpha X \). \( Y := r_\alpha X \) has the property \( r_1Y = Y \). Such \( T_\emptyset \)-spaces \( Y \) may be called periodic. \( Y = r_\alpha X \) is the largest \( b \)-closed periodic subspace of \( X \). \( \alpha \) may be called the periodicity index of \( X \). (It is not difficult to describe a categorical setting in which such an index arises.)

**Example 1.9.** Let \( R \) denote the set of real numbers. The "left topology" on \( R \cup \{\infty\} \) has \( \emptyset, R \cup \{\infty\} \) and \( \{\infty\} \cup \{x \in R \mid r < x \}(r \in R) \) as its open sets. This space \( R^\ast \) is sober. Its \( b \)-dense subset \( Q \) of rational numbers is a periodic space in the induced topology. \( R^\ast \) is easily identified with the sobrification remainder of \( (R, \subseteq) \) in its \( A \)-discrete topology: If \( X \) is \( T_D \), then \( sX - X \) need not be also \( T_D \).

2. In [9] J. R. Büchi discusses the problem of "minimal" representation of a lattice by a "set lattice" ([9] def. 37, Cor. 40); the case of a minimal representation of a lattice of open sets of a topological space has been investigated by G. Bruns [8] §§7, 8 who has obtained a characterization of those lattices, which admit such a minimal representation. Our result (2.1) below in part overlaps with the results of G. Bruns (cf. [8] §8, Satz 5, p. 13). The theme has been independently dealt with by D. Drake and W. J. Thron ([12], in particular Thm. 5.4). In the following we briefly rephrase part of Bruns' representation theory (and we add some information obtained in the meantime).

Let \( (L, \leq) \) denote a complete lattice. A reduced, isomorphic, topological representation \( (\varphi; X, \Gamma) \), for short: an \( r.i.t. \)-representation of \( (L, \leq) \) consists of a \( T_\emptyset \)-space \( (X, \Gamma) \) — whose lattice of closed subspaces is designated by \( (\Gamma, \subseteq) \) — and a lattice-isomorphism \( \varphi: (L, \leq) \rightarrow (\Gamma, \subseteq) \). The class of \( r.i.t. \)-representations receives the following pre-order: \( (\varphi; X, \Gamma) \leq (\psi; Y, \Delta) \) iff there is an embedding \( e \) of \( (X, \Gamma) \) into \( (Y, \Delta) \) such that
\[ e^{-1}[\psi(a)] = \varphi(a) \]
for every \( a \in L \). This class contains — if it is nonempty\(^1\) — a greatest element \( (\mathcal{L}; L, \mathcal{S}, \mathcal{S}) \) with \( L = \{ a | a \text{ "(join-)prime" in } L, \text{i.e., } \neq 0 \) and whenever \( a \leq \text{sup} \{ a_1, a_2 \} \) for \( a_1, a_2 \in L \), then \( a \leq a_1 \) or \( a \leq a_2 \) and \( \mathcal{S} = \{ \mathcal{S} \} | a \in L \} \) with \( \mathcal{S} : = \{ a \in L | a \leq c \} \), and \( \mathcal{L} : = \mathcal{S} \) for every \( c \in L \). Every \( r.-i.-t.-\) representation \( (\varphi; X, \mathcal{S}) \) of \( (L, \leq) \) is equivalent to (i.e., both smaller and greater than) an \( r.-i.-t.-\) representation \( (\psi; Y, \mathcal{S}) \) arising from (and uniquely determined by) a subspace \( (Y, \mathcal{S}) \) of \( (\mathcal{S}, \mathcal{S}) \): \[
Y = \{ a \in L \} \varphi(a) \text{ is a point closure } cl_X \{ x \} \text{ in } X
\]
such that the canonical inclusion \( e: (Y, \mathcal{S}) \rightarrow (\mathcal{S}, \mathcal{S}) \) gives \( \psi(a) : = e^{-1}[\mathcal{L}(a)] \). The subspaces \( (Y, \mathcal{S}) \) of \( (\mathcal{S}, \mathcal{S}) \) thus obtained are easily seen to be precisely the \( b \)-dense subspaces of \( (\mathcal{S}, \mathcal{S}) \). Thus an \( r.-i.-t.-\) representation of \( (L, \leq) \) is an embedding of a \( b \)-dense subspace into \( (\mathcal{S}, \mathcal{S}) \); the pre-order for \( r.-i.-t.-\) representations becomes the (partial) order between these inclusions\(^2\).

Recall that a point \( c \) of a space \( X \) is "isolated" iff \( \{ c \} \) is open in \( X \). A space \( X \) is \( T_0 \) iff every point of \( X \) is \( b \)-isolated, i.e., iff \( bX \) is discrete ([7] 4.1, cf. also [27], [18] Bemerkung).

**Theorem 2.1.** Let \( X \) be a \( T_0 \)-space, then the following conditions are equivalent:

(i) \( X \) has a smallest \( b \)-dense subspace \( Y_1 \).
(ii) \( X \) has a minimal \( b \)-dense subspace \( Y_2 \).
(iii) \( X \) has a \( b \)-dense subspace \( Y_3 \) which satisfies \( T_0 \).
(iv) \( X \) has a \( b \)-dense subspace \( Y_4 \) consisting of points which are \( b \)-isolated in \( X \).
(v) The set \( Y_5 \) of all \( b \)-isolated points of \( X \) is \( b \)-dense in \( X \).

*If one (hence all) of these conditions is satisfied, then \( Y_1 = Y_2 = Y_3 = Y_4 = Y_5 \).*

**Proof.** Note that the \( b \)-topology of a subspace is the induced \( b \)-topology. \( X \) is \( T_0 \), iff its \( b \)-topology is \( T_0 \) (hence \( T_0 \), etc.). Thus the questions reduce to minimality of discrete dense subspaces, and discreteness of minimal dense subspaces.

(i) \( \Rightarrow \) (ii): Trivial.

(ii) \( \Leftrightarrow \) (iii): A dense subset is minimal-dense, iff it is discrete as a subspace.

(ii) \( \Rightarrow \) (v): Suppose \( Z \) is a \( T_1 \)-space, \( P, Q \subseteq Z \) dense, \( P \) is the

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1 It is nonempty iff every element of \( L \) is a join of "(join-)prime" elements [9] p. 157 (Th. 15), cf. [8] pp. 198-199.

2 Note that the inclusions and not the \( b \)-dense subspaces themselves are to be considered as 'representative' representations, since it may happen that two different \( b \)-dense subspaces are homeomorphic, e.g., \( Q \) and \( j + Q \) in \( R^* \) for an irrational number \( j \).
set of all isolated points of $Z$, $p \in P - Q$. Since $P$ is discrete, there is an open set $O$ of $Z$ with $O \cap P = \{p\}$. Since $Q$ is dense, there is some $q \in Q \cap O$. Since $Z$ is $T_1$, there is an open set $V \subseteq O$ with $q \in V$, $p \notin V$, hence $V \cap P = \emptyset$ — contradiction. Thus $P \subseteq Q$.

(v) $\Rightarrow$ (iv): Trivial.

(iv) $\Rightarrow$ (i): A dense subspace necessarily contains all isolated points, hence $Y_i = Y_1$.

Let $\mathcal{D}(X)$ denote the lattice of open sets of the space $X$. From (2.1) one easily deduces

**Corollary 2.2.** ([8] II p. 18, [30] p. 673). Suppose $X$ and $Y$ are $T_D$-spaces and let $\varphi: \mathcal{D}(X) \to \mathcal{D}(Y)$ be a lattice-isomorphism, then there is a homeomorphism $f: Y \to X$ with $f^{-1}[?] = \varphi(?) : \mathcal{D}(X) \to \mathcal{D}(Y)$. In particular, a sober space is the sobrification space of at most one $T_D$-subspace.

**Definition 2.3.** A topological space $X$ is called a $\mathcal{B}$-space iff $X$ is $T_0$ and $^\delta X \cong ^\delta Y$ for some $T_D$-space $Y$.

The above Theorem 2.1 describes the class of $\mathcal{B}$-spaces $X$ as those $T_0$-spaces $X$ whose set of $b$-isolated points is $b$-dense in $X$.

Note that the property of a space to be a $\mathcal{B}$-space is lattice-invariant relative to $T_0$. Recall that a class $\mathcal{R}$ (resp. a "property" $\mathcal{R}$) of topological spaces is called lattice-invariant ("verwandtschaftstreu" [24] p. 298) relative to a class $\mathcal{L}$ of spaces with $\mathcal{R} \subseteq \mathcal{L}$ iff property $\mathcal{R}$ is expressible (relative to $\mathcal{L}$) in terms of the lattice $\mathcal{D}(X)$ of open sets of the space $X$ with the inclusion order, i.e., iff whenever $X \in \mathcal{R}$, $Y \in \mathcal{L}$, $\mathcal{D}(X) \cong \mathcal{D}(Y)$, then $Y \in \mathcal{R}$. (Remember that $\mathcal{D}(X) \cong \mathcal{D}(Y)$ iff $^\delta X \cong ^\delta Y$; clearly, a property expressible in terms of $\mathcal{D}(X)$ is also expressible in terms of the opposite lattice $\mathcal{U}(X)$ of closed subsets of $X$ ordered by inclusion).

We give the following explicit description of this fact. Recall that an element $a$ of a complete lattice $L$ is strongly (join-)irreducible iff $a = \sup_{i \in I} a_i$ implies $a = a_i$ for some $i \in I$.

**Theorem 2.4.** A $T_0$-space $X$ is a $\mathcal{B}$-space iff its lattice $\mathcal{U}(X)$ of closed subsets enjoys the following property: Every element of $\mathcal{U}(X)$ is the supremum (\(\equiv\) join) of strongly irreducible elements.

**Proof.** (1) We note that $x \in X$ is $b$-isolated iff $cl\{x\}$ is strongly (join-)irreducible in $\mathcal{U}(X)$. (Cf. [30] 2.1(g).)

(2) Suppose that there is an open neighborhood $V$ of some $x \in X$ such that $V \cap cl\{x\}$ does not contain a $b$-isolated point, then the
supremum of all strongly irreducible elements of $\mathfrak{A}(X)$ which are smaller than $cl\{x\}$ is smaller than $cl\{x\} - V \in A(X)$. 

In order to avoid any confusion with Büchi's theorem quoted by G. Bruns [8] I, p. 198 we note that the concept of $\mathfrak{M}$-$\delta$-subirreducible element in a lattice $L$ is usually different from the above concept.

**Example 2.5.** (a) An infinite power $\prod_i S$ of the Sierpinski space $S (\{0, 1\}$ with open sets $\emptyset, \{1\}, \{0, 1\})$ is not $T_D$ (cf. [7] p. 408, [18] Thm. 1), but it is a $\mathfrak{B}$-space, since its subspace of $b$-isolated points \($(x_i)_{i \in \{0, 1\}, \{i \in I \mid x_i \neq 0\} \text{ is finite}} \text{ is } b\text{-dense in } \prod_i S$. We note in passing that this subspace is even $A$-discrete. A general criterion, when a space contains a $b$-dense $A$-discrete subspace, will be given elsewhere ("Topological spaces admitting a dual", in: Categorical Topology Springer Lecture Notes in Math., 719 (1978), 157–166).

(b) $R^*$ (1.9), does not contain any $b$-isolated point, hence $R^*$ is not the sobrification of any $T_D$-space. Of course, the same holds for every $T_\delta$-space containing a $b$-dense periodic subspace. (cf. 1.8).

One readily observes that a point $(x_i)_i$ of a product space $\prod_i X_i$ is $b$-isolated iff it satisfies (1) and (2):

1. The set $K: = \{i \in I \mid \{x_i\} \text{ is not closed in } X_i\}$ is finite.
2. For every $i \in I$, $x_i$ is $b$-isolated in $X_i$.

For the formulation of (2.6) below we need the following property:

(※) For every point $x$ of a space $X$ there is a closed point \(y \in X(i.e., cl\{y\} = y)\) with \(y \in cl\{x\}\).

**Theorem 2.6.** $\prod_i X_i$ with topological spaces $X_i \neq \emptyset (i \in I)$ is a $\mathfrak{B}$-space, iff conditions (i) and (ii) are satisfied:

(i) Every $X_i$ is a $\mathfrak{B}$-space

(ii) $K: = \{i \in I \mid X_i \text{ does not satisfy property (※)}\}$ is finite.

**Proof.** Since a finite product of $T_D$-spaces is $T_D$, a finite product of $\mathfrak{B}$-spaces is a $\mathfrak{B}$-space by (1.2). Suppose $\prod_i X_i$ is a product of $\mathfrak{B}$-spaces $X_i$ satisfying (※), let $(x_i)_i \in \prod_i X_i$ and let $\prod_i U_i$ be a neighborhood of $(x_i)_i$ in $\prod_i X_i$ with $U_i$ open in $X_i$; hence $L: = \{i \in I \mid U_i \neq X_i\}$ is finite. For every $i \in L$ let $y_i$ denote a $b$-isolated point of $X_i$ contained in $U_i \cap cl\{x_i\}$; for $i \in L$ let $y_i$ denote a closed point contained in $cl\{x_i\}$. By the remark preceding the theorem, $(y_i)_i$ is a $b$-isolated point of $\prod_i X_i$ contained in $(\prod_i U_i) \cap cl\{x_i\}$. — Conditions (i) and (ii) are easily seen (by similar considerations) to be necessary.

**Remark 2.7.** A space $X$ may be called a $\mathfrak{B}^*$-space iff it is a $\mathfrak{B}$-space satisfying condition (※). Since (※) is productive, so is the class
of \( \mathcal{B}^* \)-spaces by (2.6), hence it is the greatest productive class of \( \mathcal{B} \)-spaces. Of course, every \( T_1 \)-spaces is a \( \mathcal{B}^* \)-space. However, a \( \mathcal{B}^* \)-space satisfying \( T_D \) need not be \( T_1 \).

**Lemma 2.8.** Every finite \( T_0 \)-space is a \( \mathcal{B}^* \)-space. An \( A \)-discrete \( T_0 \)-space is a \( \mathcal{B}^* \)-space iff every element — in terms of the associated pre-order — has a lower bound which is a minimal element.

*Proof.* A finite \( T_0 \)-space, and moreover ([8, 4]) an \( A \)-discrete \( T_0 \)-space is \( T_D \), hence a \( \mathcal{B} \)-space.

**Lemma 2.9.** The class of \( \mathcal{B}^* \)-spaces is lattice-invariant relative to \( T_0 \).

*Proof.* Property (*) may be rephrased in \( \mathcal{U}(X) \): Every (nonempty) irreducible element is minorized by an atom.

**Remark 2.10.** We note that the class of sober \( \mathcal{B}^* \)-spaces is productive, but not reflective in \( \mathcal{T}_0 \mathcal{P} \), since there are sober spaces which are not \( \mathcal{B} \)-spaces — cf. (2.5b) and [19] 1.3.

**Remark 2.11.** A \( T_0 \)-space \( X \) is called a Jacobson space\(^3\) ([10] 0.2.8.1) iff its subset of closed points is \( b \)-dense in \( X \) — cf. also [24] 5.7 (p. 311). Every Jacobson space is a \( \mathcal{B}^* \)-space; \( S \) is a \( \mathcal{B}^* \)-space, but not a Jacobson space. The proof of 2.6 shows that a product of nonempty topological spaces is a Jacobson space iff so is every coordinate space. Also the characterization Theorem 2.1 has an analogue; the following conditions (a), (b), (c), (d) are pairwise equivalent for a \( T_0 \)-space \( X \):

(a) \( X \) is a Jacobson space;
(b) \( X \) has a \( b \)-dense subspace which satisfies \( T_1 \);
(c) \( X \) has a \( b \)-dense subspace consisting of closed points of \( X \);
(d) there is \( T_1 \)-space \( Y \) with \( ^sX \cong ^sY \).

A Jacobson space is a \( \mathcal{B} \)-space all of whose \( b \)-isolated points are closed points, i.e., a \( \mathcal{B} \)-space satisfying the property \( \mathcal{L}^* \) of [30] p. 675: Every strongly irreducible element of \( \mathcal{U}(X) \) is an atom\(^4\). Thus 2.4 with “strongly irreducible” replaced by “atom” characterizes Jacobson spaces.

3. Since for a space \( X, bX \) is uniformizable, i.e., completely

\(^3\) We observe that in [10] (0.2.8.1) the requirement of the \( T_0 \)-property is omitted.

\(^4\) Recall from [21] p. 374 that \( T_0 + \mathcal{L}^* \) ([30] p. 675) = sober + \( T_1 \). Furthermore, we observe that sober + \( T_D = T_0 + \) “every irreducible element of \( A(X) \) is strongly irreducible“.
regular, it is natural to ask: When is \(bX\) a compact \(T_2\)-space? The answer is essentially based upon a result of M. Hochster [17] (Thm. 1, p. 45).

Recall that a space \(X\) is said to be Noetherian (N. Bourbaki, [5] II, 4.2, p. 123) iff every ascending chain of open subsets is eventually stationary, i.e., iff every open subspace is quasi-compact (for a detailed study see [29]). — A Noetherian sober space is sometimes called a Zariski space ([15] 3.17, p. 93).

**Theorem 3.1.** A topological space \(X\) is both Noetherian and sober iff \(bX\) is a compact \(T_2\)-space.

**Proof.** (i) Suppose that \(bX\) is compact and Hausdorff, and let \(V\) be open in \(X\). Then \(bV\) is a closed subspace of \(bX\), hence \(bV\) is quasi-compact. Since \(V\) is coarser than \(bV\), \(V\) is also quasi-compact. — Now let \(C\) be an irreducible, closed, nonempty subspace of \(X\). \(\mathcal{O} = \{V \cap C \mid V \text{ open in } X, V \cap C \neq \emptyset\}\) is a family of \(b\)-closed subsets of \(X\) with the property that every finite subfamily has a nonempty intersection. Since \(bC\) is closed in \(bX\), hence compact, there is an element \(x \in \bigcap \mathcal{O}\), hence \(C = \overline{\{x\}}\). Since \(bX\) is \(T_2\), \(X\) is \(T_0\).

(ii) Suppose that \(X\) is a Zariski space, then, of course, \(X\) is a "spectral space" in the sense of M. Hochster, and the \(b\)-topology coincides with M. Hochster's "patch topology" ([17] p. 45, p. 52), thus [17] (Theorem 1, p. 45) applies.

A space is called quasi-sober [22] (2.1) iff every irreducible, closed, nonempty subset has at least one generic point (cf. also [20] 2.6).

**Corollary 3.2.** \(bX\) is quasi-compact, iff \(X\) is a quasi-sober Noetherian space.

**Proof.** Suppose \(bX\) is quasi-compact. Then the \(T_\gamma\)-identification space \((bX)_0 = b(X_0)\) is compact and \(T_\gamma\), hence \(X_0\) is a Zariski space (3.1), i.e., \(\mathcal{O}(X) \cong \mathcal{O}(X_0)\) is "Noetherian" and \(X\) is quasi-sober ([22] 2.2). — The other implication is established by reversing these conclusions.

Note that the \(A\)-discrete space \(N\) above is both Noetherian and \(T_\gamma\), but not sober, hence \(bN\) is not quasi-compact.

**Note Added in Proof.** The space \(sN\) appearing in 1.6 above was characterized in [18] Theorem 2. By the aid of this result (and 2.1 above!), we obtain an interesting characterization of the space
ON THE SOBRIFICATION REMAINDER \( \mathring{s}X - X \)

\( N \) of natural numbers in in \( A \)-discrete topology: Up to a homeomorphism \( N \) is the only \( T_0 \)-space \( M \) which enjoys the following properties:

(i) \( M \) (is a \( T_D \)-space which) is not sober.

(ii) Whenever \( X \) is a \( T_0 \)-space which fails to be \( T_D \), then there exists a continuous surjective map \( f: X \to \mathring{s}M \).

**Proof.** By 2.1 above, \( \mathring{s}M \) cannot be a \( T_D \)-space, since \( M \neq \mathring{s}M \). Thus, by [18] Theorem 2, \( \mathring{s}M \) is homeomorphic to \( \mathring{s}N \). Now—by 2.1 above—\( M \) is either homeomorphic to \( N \) or to \( \mathring{s}N = \{N \cup \{\infty\}\} \). By (i), \( N \) is homeomorphic to \( M \).

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