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RUDOLF-EBERHARD HOFFMANN

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# ON THE SOBRIFICATION REMAINDER ${}^sX - X$

RUDOLF-E. HOFFMANN

**The topics of this paper are (1) a study of the sobrification remainder ${}^sX - X$ (hence our title), (2) a new, simple proof of the characterization of $T_D$-spaces $Y$ as those spaces $Y$ such that $Y$ is the smallest subspace $X$ of ${}^sY$ for which the embedding $X \hookrightarrow {}^sY$ is the universal sobrification, (3) an elegant characterization of Noetherian sober spaces. These themes are linked by the common tool by aid of which they are investigated, the so-called $b$-topology L. Skula [28].**

Recall that a space $Y$ is called *irreducible* iff $O_1 \cap O_2 \neq \emptyset$ for every pair of nonempty open subsets $O_i$ of $Y (i = 1, 2)$ — sometimes, in addition, $Y \neq \emptyset$ is assumed. A space $X$ is called "*sober*" ([3] IV 4.2.1) iff every irreducible, nonempty, closed subset $M$ of $X$ has a unique "generic" point $m$, i.e., $M = cl\{m\}$(hence $T_2 \Rightarrow$ "sober" $\Rightarrow T_0$). To every space $X$ one associates a sober space ${}^sX$ whose elements are all irreducible, closed, nonempty subsets of $X$. The open sets of ${}^sX$ are all sets of the form ${}^sO := \{M \in {}^sX \mid M \cap O \neq \emptyset\}$ for some open set $O$ of $X$. The map $\chi: x \mapsto cl\{x\}$ is the *reflection morphism* for the category $\mathfrak{Top}$ of topological spaces and continuous maps into its full subcategory $\mathfrak{Sob}$ of sober spaces. *If $X$ is a $T_0$-space, then $\chi_X$ is an embedding*; we shall sometimes identify $X$ with the subspace $\chi_X[X]$ of ${}^sX$, in particular we shall write ${}^sX - X$ for a $T_0$-space $X$ instead of ${}^sX - \chi_X[X]$. For further information on sober spaces see [19], [20] (3.1), [21] and some recent work of S. S. Hong [22], J. R. Isbell [23], L. D. Nel [26], L. D. Nel and R. G. Wilson [27] (to the historical survey of [21] p. 365/366 a reference to [8] II, (1) on p. 17 has to be added).

An essential tool for the investigation of sober spaces is the *b-topology* introduced by L. Skula ([28]; cf. also [11] p. 288). The $b$-topology associated with a space $X$ is the topology which has $\{O \cap A \mid O \text{ open in } X, A \text{ closed in } X\}$ as an open basis. The members of this basis are called *locally closed sets* (N. Bourbaki [6] Chap. I, §3.3). The terms "*b-dense*", "*b-isolated*" etc. will refer to the $b$-topology, i.e., the topological space $bX$ associated with a given space $X$; in particular, a $b$-dense subspace $Y$ of $X$ is a subspace of $X$ which is a dense subset of $bX$. A subspace $Y$ of $X$ is $b$-dense, iff whenever $O_1$, $O_2$ are open subsets of $X$, $O_1 \neq O_2$, then $O_1 \cap Y \neq O_2 \cap Y$. In [7] G.C.L. Brümmer looks at the uniformity (canonically) associated with the *Pervin quasi-uniformity* of a topological space $X$; *this uniformity induces a topology which is easily seen to be the b-topology associated* *to the space* $X$: thus $bX$ is uniformizable by a distinguished uniformity ([7] p. 408). We note further that $bX$ is $O$-dimensional, i.e., it has an open basis of sets which are both closed and open.

Recall that a space $X$ is $T_D$ iff for every $x \in X$ there is an open neighborhood $U$ of $x$ with $U \cap cl\{x\} = \{x\}$, i.e., every point of $X$ is locally closed. The $T_D$-axiom was introduced by G. Bruns [8] II p. 7 ("$T_{1/2}$") and C. E. Aull and W. J. Thron [4] p. 29. For characterizations of $T_D$ see [21] 2.1 and, in addition, [30] 2.1 (g). As a recent application of the $T_D$-axiom, we note that C. C. Moore and J. Rosenberg have shown that the space of primitive ideals of the group $\mathfrak{C}^*$-algebra of a connected and locally compact group $G$ is $T_D$ ([25] Thm. 1). Furthermore cf. [14] (§§3.2, 3.3).

To a preordered set $(X, \leqq)$ one may associate a topological space with the same carrier set and open basis $\{U_a | a \in X\}$ with $U_a := \{y \in X | a \leqq y\}$. Such a space is called *A-discrete* (or *Alexandrov-discrete*) [1]. A topological space is $A$-discrete iff every union of closed sets is closed. Nowadays, $A$-discrete spaces are also known as *finitely generated spaces,* since they form the co-reflective hull of the class of finite spaces ([16] 22.2(4)). An $A$-discrete $T_0$-space is $T_D$ ([8] II, p. 18, [4] p. 35). For some further information see [2].

I am indebted to B. Banaschewsky (Hamilton) and J. R. Isbell (Buffalo) for discussions (during the Oberwolfach meeting on category theory, August 1977) on some themes of this paper.

LEMMA 1.1. *Suppose $\beta$ is a basis of the open sets of a space $X$, then*

$$\{U \cap cl\{x\} | x \in U \in \beta\}$$

*is a basis of the b-topology associated with $X$.*

From this easily proved lemma we immediately obtain

LEMMA 1.2. *For topological spaces $X$ and $Y$ holds $bX \times bY = b(X \times Y)$.*

*Proof.* Let $\tau_X$ and $\tau_Y$ denote the topologies of $X$ and $Y$ respectively, then $\{U \times V | U \in \tau_X, V \in \tau_Y\}$ is a basis for $X \times Y$, hence

$$\{(U \times V) \cap (cl_X\{x\} \times cl_Y\{y\}) = (U \cap cl_X\{x\}) \times (V \cap cl_Y\{y\}) | U \in \tau_X, V \in \tau_Y, x \in X, y \in Y\}$$

is a basis for $b(X \times Y)$ and, obviously, also for $bX \times bY$.

PROPOSITION 1.3. *Let $\{X_i\}_{i \in I}$ be a family of nonempty topological spaces. $b(\prod_I X_i) = \prod_I (bX_i)$ iff $K := \{i \in I | X_i$ is not indiscrete$\}$ is finite.*

*Proof.* For every $i \in K$, there is some $x_i \in X_i$ with $cl\{x_i\} \neq X_i$. If $K$ is infinite, then $\prod_K cl\{x_i\} \times \prod_{I-K} X_i$ is open in $b(\prod_I X_i)$, but not open in a product topology arising from any modifications of the topologies of $X_i$. If $K$ is finite, then

$$b\Big(\prod_K X_i \times \prod_{I-K} X_i\Big) = b\Big(\prod_K X_i\Big) \times \prod_{I-K} X_i = \prod_K (bX_i) \times \prod_{I-K} X_i = \prod_I bX_i$$

(via some obvious identifications).

It is shown in [20] 3.1.2 that *a sober space is the universal sobrification of every b-dense subspace via its embedding.*

THEOREM 1.4. *For a family $\{X_i\}_I$ of topological spaces holds ${}^s\prod_I X_i = \prod_I {}^sX_i$. In other words, the reflection functor ${}^S(-)\colon \mathfrak{Top} \to \mathfrak{Sob}$ preserves products.*

*Proof.* (i) We observe first the $\mathfrak{T}_0$-reflector $\mathfrak{Top} \to \mathfrak{T}_0$ preserves products. Recall that the canonical $T_0$-identification space $X_0$ of a space $X$ is defined by the equivalence relation $x \approx y \Leftrightarrow cl\{x\} = cl\{y\}$.

(ii) Because of (i) we may assume now that every $X_i$ is $T_0$. Since $\mathfrak{Sob}$ is reflective in $\mathfrak{Top}$, $\prod_I {}^SX_i$ is sober. Thus it suffices to show that $\prod_I X_i$ is $-$via $\prod_I \chi_{X_i}$ $-$ a $b$-dense subspace of $\prod_I {}^SX_i$. Suppose $(C_i)_{i \in I} \in \prod_I {}^SX_i$, then let $\prod_I {}^SU_i$ be an open neighborhood of $(C_i)_I$ with $U_i$ open in $X_i$; hence $U_i = X_i$ for all but finitely many indices $i$. Since $U_i \cap C_i \neq \varnothing$ for every $i \in I$, we choose some $x_i \in U_i \cap C_i$, then $\chi_{X_i}(x_i) \in {}^SU_i \cap cl_{{}^SX_i}\{C_i\}$. In consequence, $\prod_I X_i$ is $-$ via $\prod_I \chi_{X_i}$ $-$ a $b$-dense subspace of $\prod_I {}^SX_i$.

REMARK 1.5. Let $X$ be an infinite space with co-finite topology. ${}^SX - X$ consists of the unique element $X$. Let $\pi\colon X \to X$ be a permutation of $X$ without fixed point. The equalizer of $id_X$ and $\pi$ is the inclusion of the empty space $\varnothing$ into $X$, whereas the equalizer of $id_{{}^SX}$ and ${}^s\pi\colon {}^SX \to {}^SX$ is the inclusion of the one-element set $\{X\} \hookrightarrow {}^SX$. *Thus ${}^S(-)\colon \mathfrak{Top} \to \mathfrak{Sob}$ does not preserve equalizers, hence is not right adjoint.*

Similarly, by two different constant selfmaps of a two point indiscrete space it is shown that *the $\mathfrak{T}_0$-reflection functor does not preserve equalizers.*

Let $N = \{0, 1, 2, \cdots\}$ denote the space of natural numbers with its $A$-discrete topology, i.e., $\varnothing$ and $\{n, n+1, \cdots\} (n \in N)$ are open in $N$. Let ${}^SN$ denote the sobrification space; if we designate the unique element $N$ of ${}^SN - N$ by $\infty$, then $\varnothing$ and $\{\infty\} \cup \{n, n+1, \cdots\}$ are the open sets of ${}^SN$(cf. [18] Theorem 2). For an arbitrary $T_0$-

space $X$ let $N_X := ({}^SN \times {}^SX) - (\{\infty\} \times X)$ with the topology induced from ${}^SN \times {}^SX$ ($X$ is to be considered as a subspace of ${}^SX$).

THEOREM 1.6. *For every $T_0$-space $X$ holds $X \cong {}^SN_X - N_X$, i.e., every $T_0$-space is a sobrification remainder.*

*Proof.* It is sufficient to show that ${}^SN \times {}^SX$ is tne sobrification of $N_X$ via its embedding. Thus — by the result of [20] 3.1.2 quoted above — it suffices to show that $N_X$ is $b$-dense in ${}^SN \times {}^SX$. This is clear from $N \times X \subseteqq N_X \subseteqq {}^SN \times {}^SX = {}^S(N \times X)$, since $N \times X$ is $b$-dense in ${}^s(N \times X)$ by the other implication of [20] (3.1.2).

The statement of (1.6) is analogous to the fact that every completely regular $T_2$-space is a Stone — Čech — remainder — cf. [13] (9K6, p. 138). The proof of (1.6) above is, in some sense, even more simple, since there is no straightforward analogue of (1.4) in the case of compact. $T_2$-spaces. Maybe it is also worth noting that in (1.6) a *single* space ${}^SN$ of ordinals suffices — other than in [13] (8K5, p. 138).

Since every $T_0$-space is a sobrification remainder of some $T_0$-space (1.6), it may be of interest to look at the sobrification remainders of certain distinguished subclasses of the class of all $T_0$-spaces, e.g., $T_D$-spaces. When is $N_X$ (1.6) a $T_D$-space?

LEMMA 1.7. (a) *If $Y$ is a $T_D$-space, then ${}^SY - Y$ is sober.*
(b) *$N_X$ is $T_D$ iff $X$ is both sober and $T_D$.*

*Proof.* (a) By (2.1) every element of $Y$ is $b$-isolated in ${}^SY$, hence $Y$ is $b$-open in ${}^SY$. Thus ${}^SY - Y$ is $b$-closed in ${}^SY$, hence sober.

(b) Suppose $N_X$ is $T_D$, then $N \times X = N_X$, since $N \times X$ is $b$-dense in ${}^SN \times {}^SX$, hence in $N_X$ (a discrete space has no proper dense subspace). In consequence, ($X = {}^SX$ and) $X$ is $T_D$. If $X$ is sober and $T_D$, then $N_X = N \times X$ is $T_D$.

REMARK 1.8. The sobrification process also gives rise to a (new?) cardinal invariant of a $T_0$-space $X$. Let

$$r_0X := X\,, \qquad u_0X := {}^SX - X\,,$$
$$u_nX := \delta(r_nX) - r_nX\,,$$
$$r_{n+1}X := \delta(u_nX) - u_nX\,.$$

Here $\delta(-)$ denotes the $b$-closure of $(-)$ in ${}^SX$. By [20] 3.1.2

$$u_nX \cong {}^S(r_nX) - r_nX$$

and

$$r_{n+1} X \cong {}^S(u_n X) - u_n X .$$

We observe that

$$r_{n+1} X \subseteqq r_n X \quad \text{and} \quad u_{n+1} X \subseteqq u_n X .$$

For $\aleph_0$ and, similarly, for every limit number $\lambda$ we may define

$$r_\lambda X := \bigcap_{\gamma\alpha<\lambda} r_\gamma X$$

and

$$u_\lambda X := \delta(r_\lambda X) - r_\lambda X .$$

There is a smallest cardinal $\alpha \leqq \operatorname{card} X$ such that $r_{\alpha+1} X = r_\alpha X$. $Y := r_\alpha X$ has the property $r_1 Y = Y$. Such $T_0$-spaces $Y$ may be called *periodic*. $Y = r_\alpha X$ is the largest $b$-closed periodic subspace of $X$. $\alpha$ may be called the *periodicity index* of $X$. (It is not difficult to describe a categorical setting in which such an index arises.)

EXAMPLE 1.9. Let $\boldsymbol{R}$ denote the set of real numbers. The "left topology" on $\boldsymbol{R} \cup \{\infty\}$ has $\emptyset$, $\boldsymbol{R} \cup \{\infty\}$ and $\{\infty\} \cup \{x \in \boldsymbol{R} \mid r < x\} (r \in \boldsymbol{R})$ as its open sets. This space $\boldsymbol{R}^*$ is sober. Its $b$-dense subset $\boldsymbol{Q}$ of rational numbers is a periodic space in the induced topology. $\boldsymbol{R}^*$ is easily identified with the sobrification remainder of $(\boldsymbol{R}, \leqq)$ in its $A$-discrete topology: If $X$ is $T_D$, then ${}^SX - X$ need not be also $T_D$.

2. In [9] J. R. Büchi discusses the problem of *"minimal" representation* of a lattice by a *"set lattice"* ([9] def. 37, Cor. 40); the case of a minimal representation of a lattice of open sets of a topological space has been investigated by G. Bruns [8] §§7, 8 who has obtained a characterization of those lattices, which admit such a minimal representation. Our result (2.1) below in part overlaps with the results of G. Bruns (cf. [8] §8, Satz 5, p. 13). The theme has been independently dealt with by D. Drake and W. J. Thron ([12], in particular Thm. 5.4). In the following we briefly rephrase part of Bruns' representation theory (and we add some information obtained in the meantime).

Let $(L, \leqq)$ denote a complete lattice. A *reduced, isomorphic, topological representation* $(\varphi; X, \Gamma)$, for short: an *r.-i.-t.-representation* of $(L, \leqq)$ consists of a $T_0$-space $(X, \Gamma)$ — whose lattice of closed subspaces is designated by $(\Gamma, \subseteqq)$ — and a lattice-isomorphism $\varphi: (L, \leqq) \to (\Gamma, \subseteqq)$. The class of *r.-i.-t*-representations receives the following pre-order: $(\varphi; X, \Gamma) \leqq (\psi; Y, \Delta)$ iff there is an embedding $e$ of $(X, \Gamma)$ into $(Y, \Delta)$ such that

$$e^{-1}[\psi(a)] = \varphi(a)$$

for every $a \in L$. This class contains — if it is nonempty[1] — a greatest element $(\chi_L; {}^sL, {}^s\Gamma)$ with ${}^sL = \{a \mid a$ "(join-)prime" in $L$, i.e., $\neq 0$ and whenever $a \leqq \sup\{a_1, a_2\}$ for $a_1, a_2 \in L$, then $a \leqq a_1$ or $a \leqq a_2\}$ and ${}^s\Gamma = \{{}^sc \mid c \in L\}$ with ${}^sc := \{a \in {}^sL \mid a \leqq c\}$, and $\chi_L(c) := {}^sc$ for every $c \in L$. Every *r.-i.-t.*-representation $(\varphi; X, \Gamma)$ of $(L, \leqq)$ is equivalent to (i.e., both smaller and greater than) an *r.-i.-t.*-representation $(\psi; Y, \Delta)$ arising from (and uniquely determined by) a subspace $(Y, \Delta)$ of $({}^sL, {}^s\Gamma)$:

$$Y = \{a \in {}^sL \mid \varphi(a) \text{ is a point closure } cl_X\{x\} \text{ in } X\}$$

such that the canonical inclusion $e: (Y, \Delta) \hookrightarrow ({}^sL, {}^s\Gamma)$ gives $\psi(a) := e^{-1}[\chi_L(a)]$. *The subspaces $(Y, \Delta)$ of $({}^sL, {}^s\Gamma)$ thus obtained are easily seen to be precisely the b-dense subspaces of $({}^sL, {}^s\Gamma)$.* Thus an *r.-i.-t.*-representation of $(L, \leqq)$ is an embedding of a $b$-dense subspace into $({}^sL, {}^s\Gamma)$; the pre-order for *r.-i.-t.*-representations becomes the (partial) order between these inclusions[2].

Recall that a point $c$ of a space $X$ is "isolated" iff $\{c\}$ is open in $X$. *A space $X$ is $T_D$ iff every point of $X$ is b-isolated, i.e., iff $bX$ is discrete* ([7] 4.1, cf. also [27], [18] Bemerkung).

THEOREM 2.1. *Let $X$ be a $T_0$-space, then the following conditions are equivalent:*

(i) *$X$ has a smallest b-dense subspace $Y_1$.*

(ii) *$X$ has a minimal b-dense subspace $Y_2$.*

(iii) *$X$ has a b-dense subspace $Y_3$ which satisfies $T_D$.*

(iv) *$X$ has a b-dense subspace $Y_4$ consisting of points which are b-isolated in $X$.*

(v) *The set $Y_5$ of all b-isolated points of $X$ is b-dense in $X$. If one (hence all) of these conditions is satisfied, then $Y_1 = Y_2 = Y_3 = Y_4 = Y_5$.*

*Proof.* Note that the $b$-topology of a subspace is the induced $b$-topology. $X$ is $T_0$, iff its $b$-topology is $T_1$ (hence $T_2$, etc.). Thus the questions reduce to minimality of discrete dense subspaces, and discreteness of minimal dense subspaces.

(i) $\Rightarrow$ (ii): Trivial.

(ii) $\Leftrightarrow$ (iii): A dense subset is minimal-dense, iff it is discrete as a subspace.

(ii) $\Rightarrow$ (v): Suppose $Z$ is a $T_1$-space, $P, Q \subseteqq Z$ dense, $P$ is the

[1] It is nonempty iff every element of $L$ is a join of "(join-)prime" elements [9] p. 157 (Th. 15), cf. [8] pp. 198–199.

[2] Note that the inclusions and not the $b$-dense subspaces themselves are to be considered as 'representative' representations, since it may happen that two different $b$-dense subspaces are homeomorphic, e.g., $\boldsymbol{Q}$ and $j + \boldsymbol{Q}$ in $\boldsymbol{R}^*$ for an irrational number $j$.

set of all isolated points of $Z$, $p \in P - Q$. Since $P$ is discrete, there is an open set $O$ of $Z$ with $O \cap P = \{p\}$. Since $Q$ is dense, there is some $q \in Q \cap O$. Since $Z$ is $T_1$, there is an open set $V \subseteq O$ with $q \in V$, $p \notin V$, hence $V \cap P = \varnothing$ — contradiction. Thus $P \subseteq Q$.

(v) $\Rightarrow$ (iv): Trivial.

(iv) $\Rightarrow$ (i): A dense subspace necessarily contains all isolated points, hence $Y_4 = Y_1$.

Let $\mathfrak{O}(X)$ denote the lattice of open sets of the space $X$. From (2.1) one easily deduces

COROLLARY 2.2. ([8] II p. 18, [30] p. 673). *Suppose $X$ and $Y$ are $T_D$-spaces and let $\varphi: \mathfrak{O}(X) \to \mathfrak{O}(Y)$ be a lattice-isomorphism, then there is a homeomorphismf: $Y \to X$ with $f^{-1}[?] = \varphi(?): \mathfrak{O}(X) \to \mathfrak{O}(Y)$. In particular, a sober space is the sobrification space of at most one $T_D$-subspace.*

DEFINITION 2.3. A topological space $X$ is called a $\mathfrak{B}$-*space* iff $X$ is $T_0$ and ${}^{S}X \cong {}^{S}Y$ for some $T_D$-space $Y$.

The above Theorem 2.1 describes the class of $\mathfrak{B}$-spaces $X$ as those $T_0$-spaces $X$ whose set of $b$-isolated points is $b$-dense in $X$.

Note that the property of a space to be a $\mathfrak{B}$-space is lattice-invariant relative to $T_0$. Recall that a class $\mathfrak{K}$ (resp. a "property" $\mathfrak{K}$) of topological spaces is called *lattice-invariant* ("*verwandtschaftstreu*" [24] p. 298) relative to a class $\mathfrak{L}$ of spaces with $\mathfrak{K} \subseteq \mathfrak{L}$ iff property $\mathfrak{K}$ is expressible (relative to $\mathfrak{L}$) in terms of the lattice $\mathfrak{O}(X)$ of open sets of the space $X$ with the inclusion order, i.e., iff whenever $X \in \mathfrak{K}$, $Y \in \mathfrak{L}$, $\mathfrak{O}(X) \cong \mathfrak{O}(Y)$, then $Y \in \mathfrak{K}$. (Remember that $\mathfrak{O}(X) \cong \mathfrak{O}(Y)$ iff ${}^{S}X \cong {}^{S}Y$; clearly, a property expressible in terms of $\mathfrak{O}(X)$ is also expressible in terms of the opposite lattice $\mathfrak{A}(X)$ of closed subsets of $X$ ordered by inclusion).

We give the following explicit description of this fact. Recall that an element $a$ of a complete lattice $L$ is *strongly (join-)irreducible* iff $a = \sup_{i \in I} a_i$ implies $a = a_i$ for some $i \in I$.

THEOREM 2.4. *A $T_0$-space $X$ is a $\mathfrak{B}$-space iff its lattice $\mathfrak{A}(X)$ of closed subsets enjoys the following property: Every element of $\mathfrak{A}(X)$ is the supremum ($\equiv$join) of strongly irreducible elements.*

*Proof.* (1) We note that $x \in X$ is $b$-isolated iff $cl\{x\}$ is strongly (join-)irreducible in $\mathfrak{A}(X)$. (Cf. [30] 2.1(g).)

(2) Suppose that there is an open neighborhood $V$ of some $x \in X$ such that $V \cap cl\{x\}$ does not contain a $b$-isolated point, then the

supremum of all strongly irreducible elements of $\mathfrak{A}(X)$ which are smaller than $cl\{x\}$ is smaller than $cl\{x\} - V \in \mathfrak{A}(X)$.

In order to avoid any confusion with Büchi's theorem quoted by G. Bruns [8] I, p. 198 we note that the concept of $\mathfrak{M}$-$\delta$-subirreducible element in a lattice $L$ is usually different from the above concept.

EXAMPLE 2.5. (a) An infinite power $\prod_I S$ of the Sierpinki space $S$ ($\{0, 1\}$ with open sets $\varnothing$, $\{1\}$, $\{0, 1\}$) is not $T_D$ (cf. [7] p. 408, [18] Thm. 1), but it is a $\mathfrak{B}$-space, since its subspace of $b$-isolated points $\{(x_i)_I \mid x_i \in \{0, 1\}, \{i \in I \mid x_i \neq 0\}$ is finite$\}$ is $b$-dense in $\prod_I S$. We note in passing that this subspace is even $A$-discrete. A general criterion, when a space contains a $b$-dense $A$-discrete subspace, will be given elsewhere ("*Topological spaces admitting a dual*", in: Categorical Topology Springer Lecture Notes in Math., **719** (1978), 157–166).

(b) $\boldsymbol{R}^*$ (1.9), does not contain any $b$-isolated point, hence $\boldsymbol{R}^*$ is not the sobrification of any $T_D$-space. Of course, the same holds for every $T_0$-space containing a $b$-dense periodic subspace. (cf. 1.8).

One readily observes that a point $(x_i)_I$ of a product space $\prod_I X_i$ is $b$-isolated iff it satisfies (1) and (2):

(1) The set $K := \{i \in I \mid \{x_i\}$ is not closed in $X_i\}$ is finite.

(2) For every $i \in I$, $x_i$ is $b$-isolated in $X_i$.

For the formulation of (2.6) below we need the following property:

(*) For every point $x$ of a space $X$ there is a closed point $y \in X$(i.e., $cl\{y\} = y$) with $y \in cl\{x\}$.

THEOREM 2.6. *$\prod_I X_i$ with topological spaces $X_i \neq \varnothing (i \in I)$ is a $\mathfrak{B}$-space, iff conditions* (i) *and* (ii) *are satisfied:*

(i) *Every $X_i$ is a $\mathfrak{B}$-space*

(ii) *$K := \{i \in I \mid X_i$ does not satisfy property (*)$\}$ is finite.*

*Proof.* Since a finite product of $T_D$-spaces is $T_D$, a finite product of $\mathfrak{B}$-spaces is a $\mathfrak{B}$-space by (1.2). Suppose $\prod_I X_i$ is a product of $\mathfrak{B}$-spaces $X_i$ satisfying (*), let $(x_i) \in \prod_I X_i$ and let $\prod_I U_i$ be a neighborhood of $(x_i)$ in $\prod_I X_i$ with $U_i$ open in $X_i$; hence $L := \{i \in I \mid U_i \neq X_i\}$ is finite. For every $i \in L$ let $y_i$ denote a $b$-isolated point of $X_i$ contained in $U_i \cap cl\{x_i\}$; for $i \in I - L$ let $y_i$ denote a closed point contained in $cl_{X_i}\{x_i\}$. By the remark preceding the theorem, $(y_i)_I$ is a $b$-isolated point of $\prod_I X_i$ contained in $(\prod_I U_i) \cap cl_{\prod_I X_i}\{(x_i)_I\}$. — Conditions (i) and (ii) are easily seen (by similar considerations) to be necessary.

REMARK 2.7. A space $X$ may be called a $\mathfrak{B}^*$-space iff it is a $\mathfrak{B}$-space satisfying condition (*). Since (*) is productive, so is the class of $\mathfrak{B}^*$-spaces by (2.6), hence it is the greatest productive class of $\mathfrak{B}$-spaces. Of course, every $T_1$-spaces is a $\mathfrak{B}^*$-space. However, a $\mathfrak{B}^*$-space satisfying $T_D$ need not be $T_1$.

LEMMA 2.8. *Every finite $T_0$-space is a $\mathfrak{B}^*$-space. An A-discrete $T_0$-space is a $\mathfrak{B}^*$-space iff every element — in terms of the associated pre-order — has a lower bound which is a minimal element.*

*Proof.* A finite $T_0$-space, and moreover ([8, 4]) an $A$-discrete $T_0$-space is $T_D$, hence a $\mathfrak{B}$-space.

LEMMA 2.9. *The class of $\mathfrak{B}^*$-spaces is lattice-invariant relative to $T_0$.*

*Proof.* Property (*) may be rephrased in $\mathfrak{A}(X)$: Every (nonempty) irreducible element is minorized by an atom.

REMARK 2.10. We note that the class of sober $\mathfrak{B}^*$-spaces is productive, but not reflective in $\mathfrak{Top}$, since there are sober spaces which are not $\mathfrak{B}$-spaces — cf. (2.5b) and [19] 1.3.

REMARK 2.11. A $T_0$-space $X$ is called a *Jacobson space*[3] ([10] 0.2.8.1) iff its subset of closed points is $b$-dense in $X$ — cf. also [24] 5.7 (p. 311). Every Jacobson space is a $\mathfrak{B}^*$-space; $S$ is a $\mathfrak{B}^*$-space, but not a Jacobson space. The proof of 2.6 shows that a product of nonempty topological spaces is a Jacobson space iff so is every coordinate space. Also the characterization Theorem 2.1 has an analogue; the following conditions (a), (b), (c), (d) are pairwise equivalent for a $T_0$-space $X$:

(a) $X$ is a Jacobson space;
(b) $X$ has a $b$-dense subspace which satisfies $T_1$;
(c) $X$ has a $b$-dense subspace consisting of closed points of $X$;
(d) there is $T_1$-space $Y$ with ${}^SX \cong {}^SY$.

A Jacobson space is a $\mathfrak{B}$-space all of whose $b$-isolated points are closed points, i.e., a $\mathfrak{B}$-space satisfying the property $\mathfrak{L}^*$ of [30] p. 675: *Every strongly irreducible element of $\mathfrak{A}(X)$ is an atom*[4]. Thus 2.4 with "strongly irreducible" replaced by "atom" characterizes Jacobson spaces.

3. Since for a space $X, bX$ is uniformizable, i.e., completely

[3] We observe that in [10] (0.2.8.1) the requirement of the $T_0$-property is omitted.

[4] Recall from [21] p. 374 that $T_0 + \mathfrak{L}^{**}$ ([30] p. 675) = sober + $T_1$. Furthermore, we observe that sober + $T_D = T_0$ + "every irreducible element of $A(X)$ is strongly irreducible".

regular, it is natural to ask: When is $bX$ a compact $T_2$-space? The answer is essentially based upon a result of M. Hochster [17] (Thm. 1, p. 45).

Recall that a space $X$ is said to be *Noetherian* (N. Bourbaki, [5] II, 4.2, p. 123) iff every ascending chain of open subsets is eventually stationary, i.e., iff every open subspace is quasi-compact (for a detailed study see [29]). — A Noetherian sober space is sometimes called a *Zariski space* ([15] 3.17, p. 93).

THEOREM 3.1. *A topological space $X$ is both Noetherian and sober iff $bX$ is a compact $T_2$-space.*

*Proof.* (i) Suppose that $bX$ is compact and Hausdorff, and let $V$ be open in $X$. Then $bV$ is a closed subspace of $bX$, hence $bV$ is quasi-compact. Since $V$ is coarser than $bV$, $V$ is also quasi-compact. — Now let $C$ be an irreducible, closed, nonempty subspace of $X$. $\mathfrak{D}: = \{V \cap C \mid V \text{ open in } X, V \cap C \neq \varnothing\}$ is a family of $b$-closed subsets of $X$ with the property that every finite subfamily has a nonempty intersection. Since $bC$ is closed in $bX$, hence compact, there is an element $x \in \bigcap \mathfrak{D}$, hence $C = cl\{x\}$. Since $bX$ is $T_2$, $X$ is $T_0$.

(ii) Suppose that $X$ is a Zariski space, then, of course, $X$ is a "spectral space" in the sense of M. Hochster, and the $b$-topology coincides with M. Hochster's *"patch topology"* ([17] p. 45, p. 52), thus [17] (Theorem 1, p. 45) applies.

A space is called *quasi-sober* [22] (2.1) iff every irreducible, closed, nonempty subset has *at least one* generic point (cf. also [20] 2.6).

COROLLARY 3.2. *$bX$ is quasi-compact, iff $X$ is a quasi-sober Noetherian space.*

*Proof.* Suppose $bX$ is quasi-compact. Then the $T_0$-identification space $(bX)_0 = b(X_0)$ is compact and $T_2$, hence $X_0$ is a Zariski space (3.1), i.e., $\mathfrak{D}(X) \cong \mathfrak{D}(X_0)$ is "Noetherian" and $X$ is quasi-sober ([22] 2.2). — The other implication is established by reversing these conclusions.

Note that the $A$-discrete space $N$ above is both Noetherian and $T_0$, but not sober, hence $bN$ is not quasi-compact.

NOTE ADDED IN PROOF. The space ${}^{S}N$ appearing in 1.6 above was characterized in [18] Theorem 2. By the aid of this result (and 2.1 above!), we obtain an interesting characterization of the space

$N$ of natural numbers in in $A$-discrete topology: *Up to a homeomorphism $N$ is the only $T_0$-space $M$ which enjoys the following properties*:

(i) *$M$ (is a $T_D$-space which) is not sober.*

(ii) *Whenever $X$ is a $T_0$-space which fails to be $T_D$, then there exists a continuous surjective map $f: X \to {}^{s}M$.*

*Proof.* By 2.1 above, ${}^{s}M$ cannot be a $T_D$-space, since $M \neq {}^{s}M$. Thus, by [18] Theorem 2, ${}^{s}M$ is homeomorphic to ${}^{s}N$. Now—by 2.1 above—$M$ is either homeomorphic to $N$ or to ${}^{s}N$ $(= N \cup \{\infty\})$. By (i), $N$ is homeomorphic to $M$.

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UNIVERSITÄT BREMEN
FACHBEREICH MATHEMATIK
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