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Finitely additive measures whose Lebesgue decomposition projections have refinement integral representations are characterized in terms of certain atomic properties.

In [2] it was shown that under certain fairly weak conditions the dual of $ba(S, F)$ does not have a refinement integral representation since these conditions lead to a $\mu \in ba(S, F)$ for which T_μ, the linear functional associated with the Lebesgue decomposition projection P_μ, is not representable. In [1] it was shown that for any $\mu > 0$ there is a maximal "nonrepresentable" part. Here we combine these results to give a necessary and sufficient condition under which T_μ is representable.

A nonnegative μ in $ba(S, F)$ is *atomic* if $I \in F$ such that $\mu(I) > 0$ implies I contains a *μ-atom* J (i.e., $J \in F$, $\mu(J) > 0$ and for $K \in F$ and $K \subseteqq J$, $\mu(K) \in \{0, \mu(J)\}$. In this paper we are interested in a stronger notion of atomic.

DEFINITION. If $\mu \in ba(S, F)$ and $\mu \geqq 0$, then μ is *totally atomic* if each $\lambda \in ba(S, F)$ such that $0 \leqq \lambda \leqq \mu$ is atomic.

A totally atomic μ is a sum of two-valued measures, but the converse statement is false. It was noted in [3] that for the σ-field $P(N)$ one may select a $\mu' \geqq 0$ which is a sum of two-valued measures and for which there are no μ'-atoms in $P(N)$. Letting $\lambda \geqq 0$ be a sum of λ_n, where λ_n is two-valued and $\{n\}$ is a λ_n-atom, for every n, we then have that $\mu' + \lambda$ is a sum of two-valued measures which is also atomic but still not totally atomic.

It may be noted that $T_\mu(\lambda) = P_\mu(\lambda)(S)$, for $\lambda \in ba(S, F)$, defines a member of the dual of $ba(S, F)$. If $\eta \in ba(S, F)$, and $f\colon F \to R$, then $T_\mu(\lambda) = \int_S f\lambda$, for $\lambda \in ba(S, F)$("T_μ is representable") iff $P_\mu(\lambda) = \int f\lambda$, for $\lambda \in ba(S, F)$("μ is representable" in [1]). The equivalence is an immediate consequence of the definition of T_μ, and is useful below.

THEOREM. *Let $\mu \in ba(S, F)$ and $\mu \geqq 0$. The following are equivalent.*

(i) *T_μ is representable.*

(ii) *μ is totally atomic.*

Proof. In Theorem 2 of [1] it was shown that $\mu=\lambda+\eta$, where λ is representable and if $I\in F$, then η^I is representable iff $\eta^I = 0$ ($\eta^I(V) = \eta(I\cap V)$ for $I, V\in F$). If I is an η-atom, then clearly η^I is representable. Thus η has no atoms. If μ is totally atomic, then $\eta = 0$. Therefore $\mu = \lambda$ and we have that μ is representable.

If μ is not totally atomic, then there is $\lambda\in ba(S, F)$ such that $0\leqq\lambda\leqq\mu$ and $I\in F$ such that $\lambda(I) > 0$ and I contains no λ-atom. Thus if $J\in F$ and $\lambda^I(J) > 0$, then there is $K\in F$ for which $K\subseteqq J$ and $\lambda^I(K)\notin\{0, \lambda^I(J)\}$. By Theorem 2 of [2], λ^I is not representable, and by 3.c.1. of [1] it follows that μ is not representable.

REFERENCES

1. Wayne C. Bell, *A decomposition of additive set functions*, Pacific J. Math., **72** (1977), 305-311.

2. Michael Keisler, *Integral representations for elements of the dual of* $ba(S, \Sigma)$, Pacific J. Math., **83** (1979), 177-183.

3. Dorothy Maharam, *Finitely additive measures on the integers*, Sankhyā, series A, **38** (1976), 44-59.

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