

# Pacific Journal of Mathematics

## **A CHARACTERIZATION OF THE REPRESENTABLE LEBESGUE DECOMPOSITION PROJECTIONS**

WAYNE C. BELL AND MICHAEL KEISLER

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**Finitely additive measures whose Lebesgue decomposition projections have refinement integral representations are characterized in terms of certain atomic properties.**

In [2] it was shown that under certain fairly weak conditions the dual of  $ba(S, F)$  does not have a refinement integral representation since these conditions lead to a  $\mu \in ba(S, F)$  for which  $T_\mu$ , the linear functional associated with the Lebesgue decomposition projection  $P_\mu$ , is not representable. In [1] it was shown that for any  $\mu > 0$  there is a maximal "nonrepresentable" part. Here we combine these results to give a necessary and sufficient condition under which  $T_\mu$  is representable.

A nonnegative  $\mu$  in  $ba(S, F)$  is *atomic* if  $I \in F$  such that  $\mu(I) > 0$  implies  $I$  contains a  $\mu$ -atom  $J$  (i.e.,  $J \in F$ ,  $\mu(J) > 0$  and for  $K \in F$  and  $K \subseteq J$ ,  $\mu(K) \in \{0, \mu(J)\}$ ). In this paper we are interested in a stronger notion of atomic.

**DEFINITION.** If  $\mu \in ba(S, F)$  and  $\mu \geq 0$ , then  $\mu$  is *totally atomic* if each  $\lambda \in ba(S, F)$  such that  $0 \leq \lambda \leq \mu$  is atomic.

A totally atomic  $\mu$  is a sum of two-valued measures, but the converse statement is false. It was noted in [3] that for the  $\sigma$ -field  $P(N)$  one may select a  $\mu' \geq 0$  which is a sum of two-valued measures and for which there are no  $\mu'$ -atoms in  $P(N)$ . Letting  $\lambda \geq 0$  be a sum of  $\lambda_n$ , where  $\lambda_n$  is two-valued and  $\{n\}$  is a  $\lambda_n$ -atom, for every  $n$ , we then have that  $\mu' + \lambda$  is a sum of two-valued measures which is also atomic but still not totally atomic.

It may be noted that  $T_\mu(\lambda) = P_\mu(\lambda)(S)$ , for  $\lambda \in ba(S, F)$ , defines a member of the dual of  $ba(S, F)$ . If  $\eta \in ba(S, F)$ , and  $f: F \rightarrow R$ , then  $T_\mu(\lambda) = \int f \lambda$ , for  $\lambda \in ba(S, F)$  (" $T_\mu$  is representable") iff  $P_\mu(\lambda) = \int f \lambda$ , for  $\lambda \in ba(S, F)$  (" $\mu$  is representable" in [1]). The equivalence is an immediate consequence of the definition of  $T_\mu$ , and is useful below.

**THEOREM.** Let  $\mu \in ba(S, F)$  and  $\mu \geq 0$ . The following are equivalent.

- (i)  $T_\mu$  is representable.
- (ii)  $\mu$  is totally atomic.

*Proof.* In Theorem 2 of [1] it was shown that  $\mu = \lambda + \eta$ , where  $\lambda$  is representable and if  $I \in F$ , then  $\eta^I$  is representable iff  $\eta^I = 0$  ( $\eta^I(V) = \eta(I \cap V)$  for  $I, V \in F$ ). If  $I$  is an  $\eta$ -atom, then clearly  $\eta^I$  is representable. Thus  $\eta$  has no atoms. If  $\mu$  is totally atomic, then  $\eta = 0$ . Therefore  $\mu = \lambda$  and we have that  $\mu$  is representable.

If  $\mu$  is not totally atomic, then there is  $\lambda \in ba(S, F')$  such that  $0 \leq \lambda \leq \mu$  and  $I \in F$  such that  $\lambda(I) > 0$  and  $I$  contains no  $\lambda$ -atom. Thus if  $J \in F$  and  $\lambda^J(J) > 0$ , then there is  $K \in F$  for which  $K \subseteq J$  and  $\lambda^J(K) \notin \{0, \lambda^J(J)\}$ . By Theorem 2 of [2],  $\lambda^J$  is not representable, and by 3.c.1. of [1] it follows that  $\mu$  is not representable.

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MURRAY STATE UNIVERSITY  
 MURRAY, KY 42071  
 AND  
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