Pacific Journal of Mathematics

TAKESAKI'S DUALITY FOR REGULAR EXTENSIONS OF VON NEUMANN ALGEBRAS

YOSHIOMI NAKAGAMI AND COLIN ERIC SUTHERLAND

Vol. 83, No. 1 March 1979

PACIFIC JOURNAL OF MATHEMATICS
Vol. 83, No. 1, 1979

TAKESAKI'S DUALITY FOR REGULAR EXTENSIONS OF VON NEUMANN ALGEBRAS

YOSHIOMI NAKAGAMI AND COLIN SUTHERLAND

We extend Takesaki's duality to regular extensions, and hence twisted crossed products, of von Neumann algebras by locally compact groups.

Introduction. For a von Neumann algebra M, ε denotes the canonical map of the automorphism group $\mathrm{Aut}(M)$ of M to the quotient $\mathrm{Aut}(M)/\mathrm{Int}(M) = \mathrm{Out}(M)$ of $\mathrm{Aut}(M)$ by the normal subgroup of inner automorphisms. When M_* is separable, and G is a separable locally compact group (always endowed with a right Haar measure and modular function Δ), we can associate to certain Borel mappings $\alpha_{(\cdot)}: t \mapsto \alpha_t \in \mathrm{Aut}(M)$ with $t \mapsto \varepsilon(\alpha_t)$ a homomorphism, a family of extensions of M by G, known as regular extensions, or, in special cases, twisted crossed products, [7, 10, 12, 13, 15]. Indeed, since $\varepsilon(\alpha_s)\varepsilon(\alpha_t) = \varepsilon(\alpha_{st})$ there is a Borel family $(s, t) \in G \times G \mapsto u(s, t) \in M$ of unitaries such that

$$(1)\qquad \begin{cases} \alpha_s \circ \alpha_t = \mathrm{Ad}\, u(s, t) \circ \alpha_{st} \\ (\text{or } (\alpha \otimes \iota)\circ\alpha = \mathrm{Ad}\, u \circ (\iota \otimes \delta)\circ \alpha) \end{cases}$$

where δ is the isomorphism of $L^\infty(G)$ into $L^\infty(G) \otimes L^\infty(G)$ determined by $(\delta f)(s, t) \equiv f(st)$, $f \in L^\infty(G)$; $\alpha: M \to M \otimes L^\infty(G)$ is given by $(\alpha(x))(t) \equiv \alpha_t(x)$, $x \in M$ and $(u\xi)(s, t) \equiv u(s, t)\xi(s, t)$ for $\xi \in \mathscr{H} \otimes L^2(G) \otimes L^2(G)$ (where M acts on $\mathscr{H}$).

Since $t \mapsto \varepsilon(\alpha_t)$ is a homomorphism, we see

$$\alpha_r(u(s, t))u(r, st) = f_u(r, s, t)u(r, s)u(rs, t)$$

for some Borel map $f_u: G \times G \times G \to M$ with unitary values in the center of M. Also, f_u is a 3-cocycle for the natural action of G on the center of M. If f_u cobounds, we may assume, by modifying by unitaries in the center of M, that

$$(2)\qquad \alpha_r(u(s, t))u(r, st) = u(r, s)u(rs, t)$$

on $G \times G \times G$. Hence we may construct the regular extension $M \otimes_{\alpha,u} G$ of M by G, as the von Neumann algebra on $\mathscr{H} \otimes L^2(G)$ generated by the operators

$$(\alpha(x)\xi)(t) \equiv \alpha_t(x)\xi(t)\ , \quad (\lambda^u(r)\xi)(t) \equiv u(t, r)\xi(tr)$$

for $x \in M$, $r \in G$ and $\xi \in \mathscr{H} \otimes L^2(G)$. (See [13, Theorem 3.1.6] for further details on regular extensions and the significance of f_u cobounding.)

In order to formulate Takesaki's duality for a general locally compact group, we introduce the concept of a dual action of G on a von Neumann algebra N; this is an isomorphism β of N into $N \otimes R(G)$ satisfying

$$(\beta \otimes \iota)\circ\beta = (\iota \otimes \gamma)\circ\beta$$

where $R(G)$ is the von Neumann algebra generated by the right regular representation λ of G and γ is the isomorphism of $R(G)$ into $R(G) \otimes R(G)$ determined by $\gamma(\lambda(t)) = \lambda(t) \otimes \lambda(t)$, $t \in G$. The crossed dual product N by G, $N \bigotimes_\beta^d G$, is the von Neumann algebra generated by $\beta(N)$ and $1 \otimes L^\infty(G)$, [3, 6, 8, 9, 11, 14]. Our main result, Theorem 2 extends Takesaki's duality to regular extensions, thus answering a question raised in [13, §1].

Duality for regular extensions. Before beginning our discussion, we define unitaries U, V, V' and W on $L^2(G) \otimes L^2(G)$ by

$$(U\xi)(s, t) \equiv \xi(t, s)\ , \quad (V\xi)(s, t) \equiv \xi(st, t)\ , \quad (V'\xi)(s, t) \equiv \Delta(t)^{1/2}\xi(t^{-1}s, t)\ ,$$

and $W \equiv UVU$, so $(W\xi)(s, t) = \xi(s, ts)$. Note that $\operatorname{Ad} U$ is the symmetry $\sigma\colon x \otimes y \mapsto y \otimes x$, $\delta f = \operatorname{Ad} V(f \otimes 1_G)$, $f \in L^\infty(G)$, and

$$\gamma(\lambda(t)) = \operatorname{Ad} W^*(\lambda(t) \otimes 1_G)\ .$$

LEMMA 1. *If $\hat\alpha$ is defined on $M \bigotimes_{\alpha,u} G$ by*

$$\hat\alpha(y) \equiv \operatorname{Ad} 1 \otimes W^*(y \otimes 1_G)\ ,$$

then it is a dual action of G on $M \bigotimes_{\alpha,u} G$.

Proof. Direct computations easily show

$$(4)\qquad \begin{cases} \operatorname{Ad} 1 \otimes W^*(\alpha(x) \otimes 1_G) = \alpha(x) \otimes 1_G \\ \operatorname{Ad} 1 \otimes W^*(\lambda^u(r) \otimes 1_G) = \lambda^u(r) \otimes \lambda(r)\ . \end{cases}$$

The identity $(\hat\alpha \otimes \iota)\circ\hat\alpha = (\iota \otimes \gamma)\circ\hat\alpha$ now follows trivially on the generators of $M \bigotimes_{\alpha,u} G$, and hence on all of $M \bigotimes_{\alpha,u} G$.

Following [6, 8], we say that actions[1] α^j of a group G on von Neumann algebras M_j, $j = 1, 2$ are equivalent if

$$(\rho \otimes \iota)\circ\alpha^1 = \alpha^2\circ\rho$$

---

[1] An action α of G on M means a homomorphism of G into $\operatorname{Aut}(M)$ such that $t \mapsto \alpha_t(x)$ is σ-weakly continuous for each $x \in M$.

for some isomorphism ρ of M_1 onto M_2; we denote this relation by $\{M_1, \alpha^1\} \sim \{M_2, \alpha^2\}$.

THEOREM 2. *Let* $\tilde{\alpha} \equiv \mathrm{Ad}\, 1 \otimes V' \circ (\iota \otimes \sigma) \circ \mathrm{Ad}\, u^* \circ (\alpha \otimes \iota)$, and

$$\hat{\hat{\alpha}}(x) \equiv \mathrm{Ad}\, 1 \otimes 1_G \otimes V'(x \otimes 1_G) \qquad \left(x \in \left(M \underset{\alpha\ u}{\bigotimes} G\right) \overset{d}{\underset{\hat{\alpha}}{\bigotimes}} G\right),$$

so that $\hat{\hat{\alpha}}$ *is the action*[2] *of* G *on* $(M \bigotimes_{\alpha,u} G) \bigotimes^d_{\hat{\alpha}} G$ *dual to* $\hat{\alpha}$. *Then* $\tilde{\alpha}$ *is an action of* G *on* $M \otimes B(L^2(G))$ *and we have*

$$\left\{\left(M \underset{\alpha, u}{\bigotimes} G\right) \overset{d}{\underset{\hat{\alpha}}{\bigotimes}} G, \hat{\hat{\alpha}}\right\} \sim \{M \otimes B(L^2(G)), \tilde{\alpha}\} .$$

Proof. We note first that the operators $\alpha(x)$, $x \in M$, $\lambda^u(r)$, $r \in G$ and $1 \otimes f$, $f \in L^\infty(G)$ generate $M \otimes B(L^2(G))$. Indeed, if N is the von Neumann algebra generated by the above operators, then $N' \subset B(\mathscr{H}) \otimes L^\infty(G)$. If $x \in N'$, then for all $y \in M$ we see that

$$\alpha_t(y)x(t)\xi(t) = (\alpha(y)x\xi)(t) = (x\alpha(y)\xi)(t) = x(t)\alpha_t(y)\xi(t)$$

a.e. on G, so that $x(t) \in M'$ a.e. Since also $\lambda^u(r)x = x\lambda^u(r)$ for all $r \in G$, we obtain $x(t)u(t, r) = u(t, r)x(tr)$ a.e. in t for each $r \in G$. A routine argument now shows $x \in M' \otimes 1_G$, and $N = M \otimes B(L^2(G))$. Note that in fact we have shown that $\alpha(x)$, $x \in M$ and $1 \otimes L^\infty(G)$ generate $M \otimes L^\infty(G)$.

Now define a map $\rho\colon M \otimes B(L^2(G)) \to M \otimes B(L^2(G)) \otimes B(L^2(G))$ by $\rho \equiv \mathrm{Ad}\, 1 \otimes V^* \circ \mathrm{Ad}\, u^* \circ (\alpha \otimes \iota)$. We have then

$$(5) \qquad \begin{cases} \rho(\alpha(x)) = \alpha(x) \otimes 1_G \\ \rho(\lambda^u(r)) = \lambda^u(r) \otimes \lambda(r) \\ \rho(1 \otimes f) = 1 \otimes 1_G \otimes f . \end{cases}$$

Of these, the last is trivial, the first follows from (1), and the second is checked as follows. Since, from (2),

$$\alpha_{st^{-1}}(u(t, r))u(st^{-1}, tr) = u(st^{-1}, t)u(s, r) ,$$

we have, for $\xi \in \mathscr{H} \otimes L^2(G) \otimes L^2(G)$,

$$\begin{aligned} &((1 \otimes V^*)u^*(\alpha \otimes \iota(\lambda^u(r)))u(1 \otimes V)\xi)(s, t) \\ &\quad = u(st^{-1}, t)^*(\alpha \otimes \iota(\lambda^u(r))u(1 \otimes V)\xi)(st^{-1}, t) \\ &\quad = u(st^{-1}, t)^*\alpha_{st^{-1}}(u(t, r))u(st^{-1}, tr)((1 \otimes V)\xi)(st^{-1}, tr) \\ &\quad = u(st^{-1}, t)^*\alpha_{st^{-1}}(u(t, r))u(st^{-1}, tr)\xi(sr, tr) \end{aligned}$$

[2] α is an action of G on M if and only if α is a normal isomorphism of M into $M \otimes L^\infty(G)$ with $(\alpha \otimes \iota)\circ\alpha = (\iota \otimes \delta)\circ\alpha$, [**8**, Theorem 2.1].

$$= u(s, r)\xi(sr, tr)$$
$$= ((\lambda^u(r) \otimes \lambda(r))\xi)(s, t) .$$

Since, from (4), the right hand sides of (5) generate

$$\left(M \underset{\alpha,u}{\otimes} G\right) \overset{d}{\underset{\hat{\alpha}}{\otimes}} G ,$$

ρ is an isomorphism of $M \otimes B(L^2(G))$ onto $(M \otimes_{\alpha,u} G) \otimes^d_{\hat{\alpha}} G$.

It remains to check the identity $(\rho \otimes \iota)\circ\tilde{\alpha} = \hat{\hat{\alpha}}\circ\rho$. Notice that $\tilde{\alpha} = \mathrm{Ad}\,(1 \otimes V'UV)\circ\rho$, and that

$$(V'UV\xi)(s, t) = \Delta(t)^{1/2}\xi(s, t^{-1}s) , \quad ((V'UV)^*\xi)(s, t) = \Delta(ts^{-1})^{1/2}\xi(s, st^{-1}) .$$

Thus we obtain

$$\begin{aligned}(\rho \otimes \iota)\circ\tilde{\alpha}(\alpha(x)) &= (\rho \otimes \iota)\circ\mathrm{Ad}(1 \otimes V'UV)(\alpha(x) \otimes 1_G)\\ &= (\rho \otimes \iota)(\alpha(x) \otimes 1_G)\\ &= \alpha(x) \otimes 1_G \otimes 1_G ,\end{aligned}$$

and

$$\begin{aligned}(\rho \otimes \iota)\circ\tilde{\alpha}(\lambda^u(r)) &= (\rho \otimes \iota)\circ\mathrm{Ad}(1 \otimes V'UV)(\lambda^u(r) \otimes \lambda(r))\\ &= (\rho \otimes \iota)(\lambda^u(r) \otimes 1_G)\\ &= \lambda^u(r) \otimes \lambda(r) \otimes 1_G .\end{aligned}$$

Also

$$\begin{aligned}\tilde{\alpha}(1 \otimes f) &= \mathrm{Ad}\,(1 \otimes V')\circ(\iota \otimes \sigma)\circ\mathrm{Ad}\,u^*(1 \otimes 1_G \otimes f)\\ &= \mathrm{Ad}\,(1 \otimes V')(1 \otimes f \otimes 1_G) = 1 \otimes \kappa f ,\end{aligned}$$

where $(\kappa f)(s, t) = f(t^{-1}s)$, by direct computation.

Finally, noticing that $\mathrm{Ad}\,V'(\lambda(r) \otimes 1_G) = \lambda(r) \otimes 1_G$, and that $\mathrm{Ad}\,V'(f \otimes 1_G) = \kappa f$, we obtain also

$$\begin{aligned}\hat{\hat{\alpha}}\circ\rho(\alpha(x)) &= \alpha(x) \otimes 1_G \otimes 1_G ,\\ \hat{\hat{\alpha}}\circ\rho(\lambda^u(r)) &= \hat{\hat{\alpha}}(\lambda^u(r) \otimes \lambda(r))\\ &= \mathrm{Ad}(1 \otimes 1_G \otimes V')(\lambda^u(r) \otimes \lambda(r) \otimes 1_G)\\ &= \lambda^u(r) \otimes \lambda(r) \otimes 1_G ,\end{aligned}$$

and

$$\begin{aligned}\hat{\hat{\alpha}}\circ\rho(1 \otimes f) &= \hat{\hat{\alpha}}(1 \otimes 1_G \otimes f)\\ &= \mathrm{Ad}(1 \otimes 1_G \otimes V')(1 \otimes 1_G \otimes f \otimes 1_G)\\ &= 1 \otimes 1_G \otimes \kappa f ;\end{aligned}$$

the equality $(\rho \otimes \iota)\circ\tilde{\alpha} = \hat{\hat{\alpha}}\circ\rho$ is verified.

COROLLARY 3. *If* ${}^u\lambda(r)$ *is defined on* $\mathscr{H} \otimes L^2(G)$ *by*

$$({}^u\lambda(r)\xi)(s) \equiv \Delta(r)^{1/2}u(r, r^{-1}s)^*\xi(r^{-1}s) ,$$

then $\tilde{\alpha}_t = \mathrm{Ad}\, {}^u\lambda(t)\circ(\alpha_t \otimes \iota)$.

Proof. It suffices to show the indicated equality on the generators $\alpha(x)$, $\lambda^u(r)$ and $1 \otimes f$ of $M \otimes B(L^2(G))$. We compute

$$\begin{aligned}&({}^u\lambda(t)\alpha_t \otimes \iota(\alpha(x)){}^u\lambda(t)^*\xi)(s)\\ &= \Delta(t)^{1/2}u(t, t^{-1}s)^*(\alpha_t \otimes \iota(\alpha(x)){}^u\lambda(t)^*\xi)(t^{-1}s)\\ &= u(t, t^{-1}s)^*\alpha_t(\alpha_{t^{-1}s}(x))u(t, t^{-1}s)\xi(s)\\ &= \alpha_s(x)\xi(s)\end{aligned}$$

for $\xi \in \mathscr{H} \otimes L^2(G)$ and

$$(\tilde{\alpha}(\alpha(x))\xi)(s, t) = ((\alpha(x) \otimes 1_G)\xi)(s, t) = \alpha_s(x) \otimes 1_G\xi(s, t) ,$$

for $\xi \in \mathscr{H} \otimes L^2(G) \otimes L^2(G)$. Similarly, we have

$$\begin{aligned}&(\mathrm{Ad}\, {}^u\lambda(t)\circ(\alpha_t \otimes \iota)(\lambda^u(r))\xi)(s)\\ &= \Delta(t)^{1/2}u(t, t^{-1}s)^*\alpha_t(u(t^{-1}s, r))({}^u\lambda(t)^*\xi)(t^{-1}sr)\\ &= u(t, t^{-1}s)^*\alpha_t(u(t^{-1}s, r))u(t, t^{-1}sr)\xi(sr)\\ &= u(s, r)\xi(sr) \qquad\qquad \text{(by (2))}\\ &= (\lambda^u(r)\xi)(s) ,\end{aligned}$$

and

$$\begin{aligned}(\mathrm{Ad}^u\, \lambda(t)\circ(\alpha_t \otimes \iota)(1 \otimes f)\xi)(s) &= (\mathrm{Ad}^u\, \lambda(t)(1 \otimes f)\xi)(s)\\ &= u(t, t^{-1}s)^*f(t^{-1}s)u(t, t^{-1}s)\xi(s)\\ &= f(t^{-1}s)\xi(s)\end{aligned}$$

for $\xi \in \mathscr{H} \otimes L^2(G)$. Since

$$(\tilde{\alpha}(\lambda^u(r))\xi)(s, t) = ((\lambda^u(r) \otimes 1_G)\xi)(s, t)$$

and

$$\begin{aligned}(\tilde{\alpha}(1 \otimes f)\xi)(s, t) &= ((1 \otimes \kappa f)\xi)(s, t)\\ &= f(t^{-1}s)\xi(s, t)\end{aligned}$$

for $\xi \in \mathscr{H} \otimes L^2(G) \otimes L^2(G)$, the verification is complete.

This result is a partial clarification of [13, Proposition 2.1.3] asserting that the 2-cocycle $u \otimes 1_G$ cobounds with respect to $\alpha_t \otimes \iota$ in $M \otimes B(L^2(G))$. Indeed, it is trivially checked that

$$u(s, t) \otimes 1_G = (\alpha_s \otimes \iota)({}^u\lambda(t)^*){}^u\lambda(s)^{*}{}^u\lambda(st)$$

as required.

For a given action θ of G on a von Neumann algebra N, we write $N^\theta \equiv \{x \in N: \theta_t(x) = x, \forall t \in G\}$, the fixed point subalgebra of N.

COROLLARY 4. $M \bigotimes_{\alpha,u} G = (M \otimes B(L^2(G)))^{\tilde{\alpha}}$.

Proof. Since $((M \bigotimes_{\alpha,u} G) \bigotimes^d_{\hat{\alpha}} G)^{\hat{\hat{\alpha}}} = \hat{\alpha}(M \bigotimes_{\alpha,u} G)$ by [8, Proposition 6.4], Takesaki's duality (Theorem 2) tells us that

$$\hat{\alpha}\Big(M \bigotimes_{\alpha,u} G\Big) = \rho((M \otimes B(L^2(G)))^{\tilde{\alpha}}) \, .$$

From (4) and (5), we see that $\hat{\alpha}$ and ρ agree on $M \bigotimes_{\alpha,u} G$, so that $M \bigotimes_{\alpha,u} G = (M \otimes B(L^2(G)))^{\tilde{\alpha}}$ as claimed.

Corollary 4 gives some information on when regular extensions $M \bigotimes_{\alpha^1,u} G$ and $M \bigotimes_{\alpha^2,v} G$ of M by G, with $\varepsilon \circ \alpha^1 = \varepsilon \circ \alpha^2$, are isomorphic. For if $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ denote the actions of G on $\bar{M} \equiv M \otimes B(L^2(G))$ with fixed point algebras $M \bigotimes_{\alpha^1,u} G$ and $M \bigotimes_{\alpha^2,v} G$ respectively, then $\bar{M} \bigotimes_{\tilde{\alpha}^1} G$ and $\bar{M} \bigotimes_{\tilde{\alpha}^2} G$ will be isomorphic whenever there is a Borel map $t \in G \mapsto u_t$ with $\tilde{\alpha}^1_t = \mathrm{Ad}\, u_t \circ \tilde{\alpha}^2_t$ and $u_t \tilde{\alpha}^2_t(u_s) = u_{ts}$ for $t, s \in G$, [14]. On the other hand these crossed products are isomorphic respectively to $(M \bigotimes_{\alpha^1,u} G) \otimes B(L^2(G))$ and $(M \bigotimes_{\alpha^2,v} G) \otimes B(L^2(G))$, [8].

Also, note that $\varepsilon \circ \tilde{\alpha}^1 = \varepsilon \cdot \tilde{\alpha}^2$ whenever $\varepsilon \circ \alpha^1 = \varepsilon \circ \alpha^2$, so it is necessary only to provide conditions under which the "comparison cocycle" $\omega_{\tilde{\alpha}^1, \tilde{\alpha}^2}$ associated to $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ is trivial, [13]. The hypothesis of the next result are two situations in which this is known to happen, [1, 4].

COROLLARY 5. *Let $M \bigotimes_{\alpha^1,u} G$ and $M \bigotimes_{\alpha^2,v} G$ be regular extensions of M by G with $\varepsilon \circ \alpha^1 = \varepsilon \circ \alpha^2$. If either*

(1) *G is discrete, acts freely on the center of M, and is a locally finite extension of a solvable group; or*

(2) *G is a compact, abelian and connected group K, or $K \times \boldsymbol{R}$, and acts trivially on the center of M,*

then $(M \bigotimes_{\alpha^1,u} G) \otimes B(L^2(G))$ and $(M \bigotimes_{\alpha^2,v} G) \otimes B(L^2(G))$ are isomorphic.

Just as in the case of ordinary crossed products, regular extensions may be characterized by the existence of a dual action and of a distinguished family of unitaries.

THEOREM 6. *Let N be a von Neumann algebra with N_* separable and β a dual action of G on N. Then the following two conditions are equivalent:*

(i) *there is* $\{M, \alpha\}$ *with* M_* *separable such that* $\{N, \beta\} \sim \{M \otimes_{\alpha,u} G, \hat{\alpha}\}$ *for some* u; *and*

(ii) *there is a Borel map* $t = G \mapsto v(t) \in N$ *with unitary values such that* $\beta(v(t)) = v(t) \otimes \lambda(t)$, $t \in G$.

The proof goes the same way as in the proof [5, 8, 11] except the following lemma.

LEMMA 7. *Assume the condition* (ii) *in Theorem* 6. *Then,* N *is generated by* $N^\beta \equiv \{y \in N: \beta(y) = y \otimes 1_G\}$ *and* $v(t)$, $t \in G$.

Proof (Takesaki). Let $\bar{N} \equiv N \otimes F_\infty$, $\bar{\beta} \equiv (\iota \otimes \sigma)\circ(\beta \otimes \iota)$ and $\bar{v}(t) \equiv v(t) \otimes 1$, where F_∞ is a factor of type I_∞. Then $\bar{\beta}$ is a dual action of G on $\bar{N}$, $\bar{N}^{\bar\beta} = N^\beta \otimes F_\infty$ is properly infinite and $\bar{\beta}(\bar{v}(t)) = \bar{v}(t) \otimes \lambda(t)$ for all t. Therefore $\bar{\beta}$ is dominant[3], because $\bar{\beta}(v) = (v \otimes 1_G)(1 \otimes W)$ for a unitary v in $N \otimes L^\infty(G)$ defined by $(v\xi)(t) \equiv v(t)\xi(t)$, [2, 9]. Therefore there exists a strongly continuous unitary representation u of G in $\bar{N}$ such that $\bar{\beta}(u(t)) = u(t) \otimes \lambda(t)$ by [5, 8, 11]. In this case $\bar{N}$ is generated by $N^\beta \otimes F_\infty$ and $u(t)$, $t \in G$. If e is a projection in $\bar{N}$ of the form $1 \otimes p$ with $\dim p = 1$, then $\{N, \beta\}$ is identified with $\{\bar{N}_e, \bar{\beta}^e\}$. Since $\bar{\beta}(v(t)) = v(t) \otimes \lambda(t)$, $t \in G$, $v(t)u(t)^* \in N^\beta \otimes F_\infty$ and hence $v(t) = ew(t)u(t)e$ for some $w(t) \in N^\beta \otimes F_\infty$. Here we may assume that $w(t) = ew(t)\alpha_t(e)$. So, $w(t)$ is a partial isometry. If x is an arbitrary element in $N^\beta \otimes F_\infty$, then

$$exu(t)e = exw(t)^*w(t)u(t)e = exw(t)^*v(t)$$

and hence $e(N^\beta \otimes F_\infty)u(t)e = N^\beta v(t)$. It remains to show that $e(N^\beta \otimes F_\infty)u(t)e$, $t \in G$ generate $e\bar{N}e = N$. Since the set L of all finite linear conbinations of $xu(t)$ with $x \in N^\beta \otimes F_\infty$ and $t \in G$ is a σ-weakly dense *-subalgebra of $\bar{N}$, eLe is σ-weakly dense in $e\bar{N}e = N$. Consequently, $N^\beta v(t)$, $t \in G$ generate N.

Proof of Theorem 6. That (i) $\Rightarrow$ (ii) has already verified in Lemma 1.

(ii) $\Rightarrow$ (i). Let $M \equiv N^\beta$. Since $\beta(v(t)xv(t)^*) = v(t)xv(t)^* \otimes 1_G$ for $x \in M$, $v(t)$ normalizes M. Also with $u(s, t) \equiv v(s)v(t)v(st)^*$, we see $\beta(u(s, t)) = u(s, t) \otimes 1_G$, so $u(s, t) \in M$ for $s, t \in G$.

Set $\alpha_s \equiv \mathrm{Ad}\, v(s) \restriction M$. Then $\alpha_s\circ\alpha_t = \mathrm{Ad}\, u(s, t)\circ\alpha_{st}$ and $\alpha_r(u(s, t))u(r, st) = u(t, s)u(rs, t)$. Then α and u determine a regular extension $M \otimes_{\alpha,u} G$ of M by G, with generators $\alpha(M)$ and $\lambda^u(s)$, $s \in G$. Define a unitary v in $N \otimes L^\infty(G)$ by $(v\xi)(t) = v(t)\xi(t)$. Then, by di-

[3] A dual action β of G on N is said to be *dominant*, if N^β is properly infinite and $\{\bar{N}, \bar{\beta}\} \sim \{\bar{N}, \tilde{\beta}\}$, where $\bar{N} = N \otimes B(L^2(G))$, $\bar{\beta} = (\iota \otimes \sigma)\circ(\beta \otimes \iota)$ and $\tilde{\beta} = (\mathrm{Ad}\, 1 \otimes W)\circ\bar{\beta}$. If β is dominant, then $\{N, \beta\} \sim \{\bar{N}, \beta\} \sim \{(N \otimes^d_\beta G) \otimes_{\hat\beta} G, \tilde{\tilde{\beta}}\}$.

rect computation,

$$v^*\lambda^u(s)v = \beta(v(s)) \quad \text{and} \quad v^*\alpha(x)v = \beta(x)$$

for $s \in G$ and $x \in M$. Thus $v^*(M \otimes_{\alpha,u} G)v = \beta(N)$ by Lemma 7.

According to the above theorem we know the relation between [2, Theorem III. 3.1] and [5, Theorem].

ACKNOWLEDGMENT. The authors want to express their sincere gratitude to Professors H. Dye and M. Takesaki for their kind hospitality at UCLA and discussion. The first named author (Y. N.) also wants to express his heartfelt thanks to Mr. A. Ikunishi for the discussion on regular extensions.

REFERENCES

1. A. Connes and W. Krieger, *Measure space automorphisms, the normalizers of their full groups and approximate finiteness*, J. Functional Analysis, **24** (1977), 336-352.
2. A. Connes and M. Takesaki, *The flow of weights on factors of type III*, Tôhoku Math. J., **29** (1977), 473-575.
3. M. Enock, *Produit croise d'une algebre de von Neumann par une algebre de Kac*, J. Functional Analysis, **26** (1977), 16-47.
4. F. Hansen and D. Olesen, *Perturbations of center-fixing dynamical systems*, Math. Scand., **41** (1977), 295-307.
5. M. B. Landstad, *Duality theory for covariant system*, Trans. Amer. Math. Soc., **248** (1979), 223-267.
6. M. B. Landstad, *Duality for dual covariance algebra*, Commun. Math. Phys., **52** (1977), 191-202.
7. H. Leptin, *Verallgemeinerte L^1-Algebren und projective Darstellungen lokal kompacter Gruppen*, Inventions Math., **3** (1967), 257-281, ibid., **4** (1967), 68-81.
8. Y. Nakagami, *Dual action on a von Neumann algebra and Takesaki's duality for a locally compact group*, Publ. RIMS, Kyoto Univ., **12** (1977), 727-775.
9. ———, *Essential spectrum $\Gamma(\beta)$ of a dual action on a von Neumann algebra*, Pacific J. Math., **70** (1977), 437-479.
10. M. Nakamura and Z. Takeda, *On the extensions of finite factors*, Proc. Japan Acad., **35** (1959), 149-154.
11. S. Strătilă, D. Voiculescue and L. Szidó, *On crossed products, I*, Rev. Roumaine Math., **21** (1976), 1411-1449; II, ibid., **22** (1977), 83-117.
12. C. E. Sutherland, *Cohomology and extensions of von Neumann algebras I*, to appear in RIMS, Kyoto.
13. ———, *Cohomology and extensions of von Neumann algebras II*, to appear in RIMS, Kyoto.
14. M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type III*, Acta Math., **131** (1973), 249-310.
15. G. Zeller-Meier, *Produit croisés d'une C*-algebre par un groupe d'automorphisms*, J. Math. Pures et appliques, Series 9, **47** (1968), 101-239.

Received March 3, 1978 and in revised form July 3, 1978. The first author's work was supported in part by the NSF.

KYUSHU UNIVERSITY
FUKUOKA
812 JAPAN
and
UNIVERSITY of OREGON
EUGENE, OR 97403

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 83, No. 1 March, 1979