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A PRIMENESS PROPERTY FOR CENTRAL POLYNOMIALS

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In this note we prove an analog of of Amitsur's theorem for central polynomials.

THEOREM. Let F be an infinite field, $f(x) = f(x_1, \dots, x_r)$, $g(x) = g(x_{r+1}, \dots, x_s)$ two noncommutative polynomials in disjoint sets of variables. Assume that $f(x_1, \dots, x_r) \cdot g(x_{r+1}, \dots, x_s)$ is central but not an identity for F_k . Then both $f(x)$ and $g(x)$ are central polynomials for F_k .

Note. Since $[x, y]^2$ is central for F_2 while $[x, y]$ is not, the assumption of disjointness of the variables cannot be removed.

Central polynomials that are not identities of the $k \times k$ matrices F_k were constructed in [2], [3]. In [1] Amitsur proved the following primeness property of the polynomial identities of F_k :

THEOREM (Amitsur). Let F be an infinite field, $f(x) = f(x_1, \dots, x_n)$, $g(x) = g(x_1, \dots, x_n)$ two noncommutative polynomials over F . If $f(x) \cdot g(x)$ is an identity for F_k , then either $f(x)$ or $g(x)$ is an identity for F_k .

Proof of the theorem. Since F is infinite, by standard arguments we may assume it is algebraically closed. Hence every matrix in F_k is conjugate to its Jordan canonical form. We show (W.L.O.G.) that $f(x)$ is central. By assumption there are $y_1, \dots, y_s \in F_k$ such that

$$f(y_1, \dots, y_r) \cdot g(y_{r+1}, \dots, y_s) = \alpha I \neq 0.$$

Denote $A = g(y_{r+1}, \dots, y_s)$, then $\det A \neq 0$ since $\det \alpha I \neq 0$, so that $A^{-1} = B \in F_k$ exist. Thus deduce the identity

$$(1) \quad f(y_1, \dots, y_r) = \alpha(y_1, \dots, y_r) \cdot B$$

where $\alpha(y)$ is a scalar function on $(F_k)^r$, not identically zero. Conjugate both sides of (1) by a matrix $D \in F_k$ so that DBD^{-1} is in a Jordan canonical form. Since $f(x)$ is a polynomial,

$$Df(y_1, \dots, y_r)D^{-1} = f(Dy_1D^{-1}, \dots, Dy_rD^{-1}) = f(\bar{y}_1, \dots, \bar{y}_r).$$

By (1), $Df(y)D^{-1} = \alpha(y)DBD^{-1}$. Since

$$(y_1, \dots, y_r) = (D^{-1}\bar{y}_1D, \dots, D^{-1}\bar{y}_rD),$$

we can write

$$\alpha(y_1, \dots, y_r) = \bar{\alpha}(\bar{y}_1, \dots, \bar{y}_r),$$

so we may finally assume in (1) that B is in its Jordan canonical form:

$$(1') \quad f(y_1, \dots, y_r) = \alpha(y_1, \dots, y_r) \begin{pmatrix} \beta_1 & \varepsilon_1 & & 0 \\ & \ddots & \ddots & \\ 0 & \beta_2 & & \varepsilon_{k-1} \\ & & \ddots & \\ 0 & & & \beta_k \end{pmatrix}$$

each $\varepsilon_i = 0$ or 1 and $\alpha(y)$ is a scalar function on $(F_k)^r$, not identically zero.

We proceed to show that all $\varepsilon_i = 0$, for example, that $\varepsilon_1 = 0$. Choose $(y_1, \dots, y_r) = (y)$ so that $\alpha(y) \neq 0$. Next, let S_k be the Symmetric group on $1, \dots, k$. If $\eta \in S_k$ and A_η denotes the matrix $(\delta_{\eta(i),j})$, then it is well known that A_η^{-1} exists and for any matrix $(a_{i,j}) \in F_k$, $A_\eta(a_{i,j})A_\eta^{-1} = (a_{\eta(i),\eta(j)})$.

To show $\varepsilon_1 = 0$, choose the transposition $\sigma = (1, 2) \in S_k$ and conjugate (1') by A_σ . Denoting $y'_i = A_\sigma y_i A_\sigma^{-1}$ we obtain the equation

$$(2) \quad \alpha(y'_1, \dots, y'_r) \begin{pmatrix} \beta_1 & \varepsilon_1 \\ 0 & \beta_2 \\ & \ddots \\ & & \ddots \end{pmatrix} = \alpha(y_1, \dots, y_r) \begin{pmatrix} \beta_2 & 0 \\ \varepsilon_1 & \beta_1 \\ & \ddots \\ & & \ddots \end{pmatrix}.$$

Equating the (2, 1) entry on both sides we deduce that $\varepsilon_1 = 0$.

Thus B in (1') is diagonal. Since $\det B \neq 0$, we have $\beta_1, \dots, \beta_k \neq 0$. By equating the (1, 1) and the (2, 2) entries in (2) we get

$$\begin{aligned} \alpha(y')\beta_1 &= \alpha(y)\beta_2 \\ \alpha(y')\beta_2 &= \alpha(y)\beta_1 \end{aligned}$$

and all terms are $\neq 0$. Hence $\beta_2 = \pm\beta_1$. Similarly, $\beta_i = \pm\beta_1$, $2 \leq i \leq k$. We want to show that $\beta_1 = \dots = \beta_k$. If $\text{Char } F = 2$, we are already done. Assume therefore that $\text{Char } F \neq 2$. Assume for example that $\beta_2 = -\beta_1$. Let

$$H = \begin{pmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & 1 \end{pmatrix}. \quad \text{Clearly } H^{-1} = \begin{pmatrix} 1 & -1 & & \\ 0 & 1 & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & 1 \end{pmatrix}.$$

Write $z_i = Hy_iH^{-1}$ and conjugate (1') (with B diagonal) by H to obtain

$$\alpha(z_1, \dots, z_r)B = \alpha(y_1, \dots, y_r) \begin{pmatrix} \beta_1 & -2\beta_1 & & 0 \\ 0 & -\beta_1 & & \\ & & \ddots & \\ 0 & & & \end{pmatrix}.$$

This is contradiction since $-2\beta_1 \neq 0$, hence the right hand side is not diagonal while the left is.

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Richard Neal Ball, <i>Topological lattice-ordered groups</i>	1
Stephen Berman, <i>On the low-dimensional cohomology of some infinite-dimensional simple Lie algebras</i>	27
R. P. Boas and Gerald Thomas Cargo, <i>Level sets of derivatives</i>	37
James K. Deveney and John Nelson Mordeson, <i>Splitting and modularly perfect fields</i>	45
Robert Hugh Gilman and Ronald Mark Solomon, <i>Finite groups with small unbalancing 2-components</i>	55
George Grätzer, Andras Hajnal and David C. Kelly, <i>Chain conditions in free products of lattices with infinitary operations</i>	107
Benjamin Rigler Halpern, <i>Periodic points on tori</i>	117
Dean G. Hoffman and David Anthony Klarner, <i>Sets of integers closed under affine operators—the finite basis theorem</i>	135
Rudolf-Eberhard Hoffmann, <i>On the sobrification remainder ${}^s X - X$</i>	145
Gerald William Johnson and David Lee Skoug, <i>Scale-invariant measurability in Wiener space</i>	157
Michael Keisler, <i>Integral representation for elements of the dual of $ba(S, \Sigma)$</i>	177
Wayne C. Bell and Michael Keisler, <i>A characterization of the representable Lebesgue decomposition projections</i>	185
Wadi Mahfoud, <i>Comparison theorems for delay differential equations</i>	187
R. Daniel Mauldin, <i>The set of continuous nowhere differentiable functions</i>	199
Robert Wilmer Miller and Mark Lawrence Teply, <i>The descending chain condition relative to a torsion theory</i>	207
Yoshiomi Nakagami and Colin Eric Sutherland, <i>Takesaki's duality for regular extensions of von Neumann algebras</i>	221
William Otis Nowell, <i>Tubular neighborhoods of Hilbert cube manifolds</i>	231
Mohan S. Putcha, <i>Generalization of Lentin's theory of principal solutions of word equations in free semigroups to free product of copies of positive reals under addition</i>	253
Amitai Regev, <i>A primeness property for central polynomials</i>	269
Saburoou Saitoh, <i>The Rudin kernels on an arbitrary domain</i>	273
Heinrich Steinlein, <i>Some abstract generalizations of the Ljusternik-Schnirelmann-Borsuk covering theorem</i>	285