Pacific Journal of Mathematics

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ON A THEOREM OF HAYMAN CONCERNING THE DERIVATIVE OF A FUNCTION OF BOUNDED CHARACTERISTIC

PATRICK AHERN

W. Hayman [On Nevanlinna's second theorem and extensions, Rend. Circ. Mat. Palermo, Ser. II, II (1953).] has given sufficient conditions on a function, f, of bounded characteristic in the unit disc, in order that f' also have bounded characteristic. In this paper it is shown that one of these conditions is also necessary for the conclusion of the theorem to hold.

Let U be the open unit disc in the complex plane and let T be its boundary. It is well known that there are functions f, that are bounded and holomorphic in U, such that $f' \notin N(U)$. Here $N(U)$ is the Nevanlinna class. In fact, O. Frostman, [1, Théoreme IX], has shown that there are Blaschke products with some degree of "smoothness" whose derivatives fail to lie in $N(U)$. More precisely, he shows that there is a Blaschke product B, whose zeros $\{a_n\}$ satisfy the condition,

$$\sum_n (1 - |a_n|)^\alpha < \infty\ , \quad \text{for all} \quad \alpha > \frac{1}{2}\ ,$$

but $B' \notin N(U)$. In Frostman's example, every point of T is a limit point of the sequence $\{a_n\}$.

W. Hayman, [2, Theorem IV], has proved a result in the positive direction. A function f, that is holomorphic in a bounded domain D, is said to be of order K if, for every complex number a, the number of solutions of the equation, $f(z) = a$, that are at a distance of at least ε from the boundary of D is at most $C\varepsilon^{-K}$, for some constant C. C may depend on a but not on ε. We say f has finite order if it has order K for some K. Now let D be a bounded open set such that $U \subseteqq D$, and let $D \cap T = \bigcup_n I_n$, where $I_n = \{e^{i\theta}: \alpha_n < \theta < \beta_n\}$.

THEOREM A (*Hayman*). *Suppose that*

(i) (a) $\sum_n (\beta_n - \alpha_n) = 2\pi$

(b) $\sum_n (\beta_n - \alpha_n) \log 1/(\beta_n - \alpha_n) < \infty$.

(ii) *there are constants* $\varepsilon, C > 0$ *such that if* $\alpha_n < \theta < \beta_n$, *then*

$$\operatorname{dist}(e^{i\theta}, \partial D) \geqq \varepsilon(|\theta - \alpha_n|\,|\theta - \beta_n|)^C\ .$$

(iii) *f is holomorphic and of finite order in D and $f \in N(U)$.*

Then $f^{(k)} \in N(U)$ *for* $k = 1, 2, 3, \cdots$.

The conditions (i)(a) and (i)(b) just mean that the set $E = T \backslash \bigcup_n I_n$ is what is usually called a Carleson set.

In [4], P. Kennedy investigates the necessity of condition (i)(b). He shows that if (i)(a) holds but

$$(*) \qquad \overline{\lim_{n\to\infty}} \left(\sum_{j=n}^{\infty} (\beta_j - \alpha_j) \right) \log \frac{1}{(\beta_n - \alpha_n)} = \infty ,$$

then there is a bounded open set $D \supseteqq U$ such that $D \cap T = \bigcup_n I_n$, $I_n = \{e^{i\theta}: \alpha_n < \theta < \beta_n\}$, and a function f that is bounded and holomorphic in D such that $f' \notin N(U)$. He observes that condition (*) does not follow from the condition

$$\sum_n (\beta_n - \alpha_n) \log \frac{1}{\beta_n - \alpha_n} = \infty ,$$

and writes that "there is still a gap between the positive information given by Hayman's theorem and the negative information" given by his example.

In this note we close the gap by showing that condition (i)(b) is the right one. Our example is a Blaschke product that retains the same degree of smoothness as the one of Frostman's example.

THEOREM. *To each sequence of arcs* $\{I_n\}$, $I_n = \{e^{i\theta}: \alpha_n < \theta < \beta_n\}$, *that satisfies* (i)(a) *but not* (i)(b), *there corresponds a Blaschke product, B, whose zero sequence,* $\{a_n\}$, *clusters only on* $T\backslash \bigcup_n I_n$, *such that* $B' \notin N(U)$ *and* $\sum (1 - |a_n|)^{\alpha} < \infty$ *for all* $\alpha > 1/2$. *Moreover, there is a bounded open set D, such that* $D \supseteqq U$, $D \cap T = \bigcup_n I_n$, *D satisfies condition* (i)(c) *with* $C = 2$, *and B extends to be bounded and of order* 1 *in D.*

Proof. Let $\varepsilon_n = \beta_n - \alpha_n$. We are assuming that $\sum_n \varepsilon_n \log(1/\varepsilon_n) = \infty$. We may choose numbers δ_n, $0 < \delta_n < 1$, such that $\lim_{n\to\infty} \delta_n = 0$, and $\sum_n \delta_n \varepsilon_n \log 1/\varepsilon_n = \infty$. Now define $d_n = \varepsilon_n^{2-\delta_n}$ and $c_n = (1 - d_n)e^{i\alpha_n}$ and $\gamma_n = (1 - d_n)e^{i\beta_n}$. Let B be the Blascke product whose zeros are $\{c_n\} \cup \{\gamma_n\}$. The zeros of B cluster only on the set $E = T\backslash \bigcup_n I_n$ so B is holomorphic on I_n for every n. We calculate that

$$B'(z) = B(z)\left\{\sum_n \frac{1 - |c_n|^2}{(z - c_n)(1 - \bar{c}_n z)} + \sum_n \frac{1 - |\gamma_n|^2}{(z - \gamma_n)(1 - \bar{\gamma}_n z)}\right\}$$

so that when $e^{i\theta} \in I_n$ we get

$$e^{i\theta} B'(e^{i\theta}) = B(e^{i\theta})\left\{\sum_k \frac{1 - |c_k|^2}{|e^{i\theta} - c_k|^2} + \sum_k \frac{1 - |\gamma_k|^2}{|e^{i\theta} - \gamma_k|^2}\right\} .$$

If B' were in $N(U)$ it would follow that

$$\sum_n \int_{I_n} \log^+\Big(\sum_k \frac{1-|c_k|^2}{|e^{i\theta}-c_k|^2}\Big)d\theta < \infty .$$

Now,

$$\begin{aligned}|e^{i\theta}-c_n|^2 &= (1-|c_n|)^2 + 4|c_n|\sin^2\Big(\frac{\theta-\alpha_n}{2}\Big)\\ &\leqq d_n^2 + (\theta-\alpha_n)^2\end{aligned}$$

and hence

$$\frac{1-|c_n|^2}{|e^{i\theta}-c_n|^2} \geqq \frac{d_n}{d_n^2+(\theta-\alpha_n)^2} .$$

If $e^{i\theta}\in I_n$, then

$$\begin{aligned}\log^+\Big(\sum_k \frac{1-|c_k|^2}{|e^{i\theta}-c_k|^2}\Big) &\geqq \log\Big(\sum_k \frac{1-|c_k|^2}{|e^{i\theta}-c_k|^2}\Big)\\ &\geqq \log\frac{1-|c_n|}{|e^{i\theta}-c_n|^2} \geqq \log\frac{d_n}{d_n^2+(\theta-\alpha_n)^2} .\end{aligned}$$

So we see that

$$\begin{aligned}\sum_n \int_{I_n} \log^+\Big(\sum_k \frac{1-|c_k|^2}{|e^{i\theta}-c_k|^2}\Big) &\geqq \sum_n \int_{I_n} \log\frac{d_n}{d_n^2+(\theta-\alpha_n)^2}d\theta\\ &\geqq \sum_n \varepsilon_n \log\frac{d_n}{d_n^2+\varepsilon_n^2} .\end{aligned}$$

Since $\delta_n < 1$, we see that $d_n = \varepsilon_n^{2-\delta_n} \leqq \varepsilon_n$ (assuming $\varepsilon_n < 1$), so

$$\log\frac{d_n}{d_n^2+\varepsilon_n^2} \geqq \log\frac{d_n}{2\varepsilon_n^2} = \log\frac{1}{2\varepsilon_n^{\delta_n}} = \log\frac{1}{2} + \delta_n\log\frac{1}{\varepsilon_n} .$$

Hence,

$$\sum_n \int_{I_n} \log^+\Big\{\sum_k \frac{1-|c_k|^2}{|e^{i\theta}-c_k|^2}\Big\}d\theta \geqq 2\pi\log\frac{1}{2} + \sum_n \delta_n\varepsilon_n\log\frac{1}{\varepsilon_n} = \infty .$$

So $B'\notin N(U)$. Also we see that

$$\sum_n (1-|a_n|)^\alpha = 2\sum_n d_n^\alpha = \sum \varepsilon_n^{(2-\delta_n)\alpha} < \infty$$

if $\alpha > 1/2$ because $(2-\delta_n)\alpha \geqq 1$ for all sufficiently large n.

It remains to construct the domain D. We have the inequality,

$$|B(re^{i\theta})|^2 \geqq 1 - (1 - r^2)\left\{\sum_k \frac{1 - |c_k|^2}{|1 - re^{i\theta}\bar{c}_k|^2} + \sum_k \frac{1 - |\gamma_k|^2}{|1 - re^{i\theta}\bar{\gamma}_k|^2}\right\}$$
$$\geqq 1 - 4(1 - r^2)\left\{\sum_k \frac{1 - |c_k|^2}{\left|re^{i\theta} - \dfrac{1}{\bar{c}_k}\right|^2} + \sum_k \frac{1 - |\gamma_k|^2}{\left|re^{i\theta} - \dfrac{1}{\bar{\gamma}_k}\right|^2}\right\} .$$

(We may assume $|c_k| \geqq 1/2$, $|\gamma_k| \geqq 1/2$.)
Now suppose $\alpha_n \leqq \theta \leqq (\alpha_n + \beta_n)/2$ and $|z| \leqq 1$, then

$$|B(re^{i\theta})|^2 \geqq 1 - \frac{4(1 - r^2)}{|re^{i\theta} - e^{i\alpha_n}|^2}\Big(\sum_k (1 - |c_k|^2) + \sum_k (1 - |\gamma_k|^2)\Big) .$$

So, $|B(re^{i\theta})|^2 \geqq 1/4$ if

$$\frac{1 - r^2}{|re^{i\theta} - e^{i\alpha_n}|^2} \leqq \frac{3}{16} \frac{1}{\sum (1 - |c_k|^2) + \sum (1 - |\gamma_k|^2)} = C .$$

Note that C is independent of θ and n. Similarly we see that if $(\alpha_n + \beta_n)/2 \leqq \theta \leqq \beta_n$ and

$$\frac{1 - r^2}{|re^{i\theta} - e^{i\beta_n}|^2} \leqq \frac{3}{16} \frac{1}{\sum_k (1 - |c_k|^2) + \sum (1 - |\gamma_k|^2)} = C ,$$

then $|B(re^{i\theta})|^2 \geqq 1/4$. We may calculate that, for $C > 0$,

$$\left\{re^{i\theta}: \frac{1 - r^2}{|re^{i\theta} - e^{i\lambda}|^2} < C\right\} = \{re^{i\theta}: |re^{i\theta} - \rho e^{i\lambda}| > 1 - \rho\} ,$$

where $\rho = C/(1 + C)$.

So, if

$\Delta_n = \{re^{i\theta}: r \leqq 1, \alpha_n < \theta < \beta_n, |re^{i\theta} - \rho e^{i\alpha_n}| > 1 - \rho$, and $|re^{i\theta} - \rho e^{i\beta_n}| > 1 - \rho\}$ and $\Delta = \bigcup_n \Delta_n$, then $|B(z)| \geqq 1/2$, $z \in \Delta$.

Now for $|z| > 1$, $B(z) = 1/\overline{B(1/\bar{z})}$, so $|B(z)| \leqq 2$ if $1/\bar{z} \in \Delta$. Assuming, as we may, that $C < 1$, we see that $\Gamma_n = \{z: 1/\bar{z} \in \Delta_n\} = \{z: |z| \geqq 1$, $|z + \delta e^{i\alpha_n}| < 1 + \delta$ and $|z + \delta e^{i\beta_n}| < 1 + \delta\}$, where $\delta = C/(1 - C)$. Finally, if we let $\mathscr{O} = U \cup \bigcup_n \Gamma_n$ then $\mathscr{O}$ is an open set and $|B(z)| \leqq 2$ for $z \in \mathscr{O}$.

Now we define a function

$$\psi(\theta) = \begin{cases} (\theta - \alpha_n)^2(\theta - \beta_n)^2 & \text{if } \alpha_n < \theta < \beta_n \text{ for some } n \\ 0 & \text{otherwise} . \end{cases}$$

We check that $\psi'(\theta)$ exists for all θ and that there is a constant K such that

$$|\psi'(\theta_1) - \psi'(\theta_2)| \leqq K|\theta_1 - \theta_2| \,.$$

(See [4, Lemma 1] for a similar calculation.) For $\varepsilon > 0$ we define $D_\varepsilon = \{re^{i\theta}: r < e^{\varepsilon\psi(\theta)}\}$. Then D_ε satisfies condition (ii). of Theorem A with $C = 2$. (Again, see [4, Lemma 2], for a similar calculation.) Also, it is not hard to that $D_\varepsilon \subsetneqq \mathcal{O}$ for all sufficiently small $\varepsilon > 0$. So we fix some $\varepsilon > 0$ such that $D_\varepsilon \subsetneqq \mathcal{O}$ and let $D = D_\varepsilon$. Since $D \subsetneqq \mathcal{O}$, B is bounded in D. It remains to show that B has order 1 in D. Let $\varphi: D \to U$ be a conformal map. Since ψ' satisfies a Lipschitz condition it follows from a theorem of Kellogg [3], that φ' extends to be continuous and nonvanishing on $\bar{D}$. From this we can conclude that there is a $\delta > 0$ such that $1 - |\varphi(z)| \geqq \delta \operatorname{dist}(z, \partial D)$ for all $z \in D$. Fix $a \in \boldsymbol{C}$ and let $f = B - a$ and let $\{a_n\}$ be the zero sequence of f. Then $\{\varphi(a_n)\}$ is the zero sequence of the bounded function $f \circ \varphi^{-1}$ so $\sum_n (1 - |\varphi(a_n)|) < \infty$ and hence $\sum_n \operatorname{dist}(a_n, \partial D) < \infty$. From this we may conclude that B has order 1 in D.

As a final remark we point out that we may choose the arcs I_n in such a way that $E = T \backslash \bigcup_n I_n$ is a countable set with only one limit point, and such that (i)(b) fails. If we apply the theorem to this situation we get a Blaschke product B whose zeros converge to a single point such that $B' \notin N(U)$, while the zeros sequence, $\{a_n\}$, satisfies $\sum (1 - |a_n|)^\alpha < \infty$ for all $\alpha > 1/2$.

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FINITENESS OF LOWER SPECTRA OF A CLASS OF HIGHER ORDER ELLIPTIC OPERATORS

W. ALLEGRETTO

Finiteness criteria are established for the lower spectrum of a class of higher order elliptic operators. The results are obtained by the introduction and consideration of a suitable second order operator. Examples are given to show that the method can yield optimal results.

Let G denote a domain of Euclidean m-space E^m. We always consider the topology of one point compactification of E^m, so that if G is unbounded, then ∞ is a point of ∂G, the boundary of G. This note deals with the spectrum of the Friedrich's extension L of the operation ℓ defined on $C_0^\infty(G)$ by:

$$\ell u = (-1)^n \Delta^n u - qu \; . \tag{1}$$

Here we denote by Δ^n the n-times iterated Laplacian and we assume that q is a real function defined in G.

In the case $n = 1$ there is a well known connection between the spectrum $S(L)$ of L and its oscillation properties, [1], [2], [6], [7], [8]. Basically, it is shown that, under suitable regularity conditions, the oscillation constant of L is the least point μ of the essential spectrum of L and that $(-\infty, \mu) \wedge S(L)$ is finite iff $L - \mu$ is nonoscillatory. It is our purpose to obtain conditions, based on oscillation theory, which guarantee that $(-\infty, \delta) \wedge S(L)$ is a finite set, where δ is a constant which is assumed hereafter to be zero. We observe that, given the monotonic dependence of the least eigenvalue as a domain function, the same proof as in [6], for the case $n = 1$, shows that if L is oscillatory then $(-\infty, 0) \wedge S(L)$ is infinite. It does not appear known, however, whether there is a higher order version of the arguments used in [7] to show that if L is nonoscillatory then $(-\infty, 0) \wedge S(L)$ is finite. This observation is the main reason behind our attempt to relate L to a second order operator.

Basically, our method consists in introducing a second order expression ℓ_1, related to ℓ, and in then obtaining finiteness conditions for $(-\infty, 0) \wedge S(L)$ by examining the nonoscillation properties of ℓ_1. It may intuitively appear that the introduction of a second order expression implies that the results obtainable in this way are not optimal. This indeed can happen, but we show by example that our method may yield best possible results in the sense that the constants appearing in the expressions can not be improved.

After some preliminary results we shall consider (1) only for the case $n = 2$, and merely indicate how the formulas are to be modified for the cases $n > 2$. We do this because our method remains unchanged in the general case, while the expressions involved can become quite lengthy and complicated (depending on n, m, G).

We now state our assumptions on ℓ. We shall assume that:

(i) $q \in C^{\alpha}_{\mathrm{loc}}$ (i.e., q is locally Holder continuous) in a neighborhood of ∂G and $q \in L^2_{\mathrm{loc}}(G)$;

(ii) ℓ is bounded below on $C_0^\infty(G)$ so that L is well defined. Consider a real second order elliptic expression ℓ_1 given by

$$\ell_1 u = -\sum D_i(a_{ij}D_j u) ,$$

with $a_{ij} = a_{ji}$. We shall say that ℓ_1 is admissible iff the following condition is satisfied:

(iii) if G_1 is any bounded smooth subdomain of G with $\bar{G}_1 \subset G$ and $\sigma \in L^\infty(\partial G_1)$, then the form $B(u, v)$ given by:

$$B(u, v) = \int_{G_1} \{\sum a_{ij} D_i u D_j \bar{v} - q u \bar{v}\} + \int_{\partial G_1} \sigma u \bar{v}$$

on $C^1(\bar{G}_1)$ gives rise, by extension, to a self-adjoint operator in $L^2(G_1)$ with finite negative spectrum.

Explicit conditions on q, a_{ij} which are sufficient in special cases for (ii), (iii) to hold may be found in [9], [11]. We observe that our assumptions allow the possibility that q become singular on parts of (possibly all of) ∂G.

We also recall the following definition of nonoscillation at ∂G (see [1]); The operator L (or the expression ℓ) is nonoscillatory at ∂G iff there exists a neighborhood N of ∂G (i.e., N is open in $E^m \cup \{\infty\}$ and $\partial G \subset N$) such that if F is a bounded domain in $N \wedge G$ then $(-\infty, 0] \wedge S(L(F)) = \phi$. Here $L(F)$ denotes the extension of ℓ defined on $C_0^\infty(F)$. The definition of L oscillatory at parts of ∂G is analogous.

Finally, we shall say that G satisfies condition (A) iff: there exists a family of nested bounded smooth closed surfaces $\{S_i\}_{i=0}^\infty$ and associated domains $\{G_i^j\}$ $(j > i)$ such that: $\bar{G}_i^j \subset G$, $\partial G_i^j = S_i \cup S_j$, $j = i+1, \cdots, \infty$; $\{\bigcup_{j=i+1}^\infty G_i^j\}_{i=1}^\infty$ is a deleted neighborhood base of ∂G (in the induced topology on $\bar{G}$). Condition (A) is usually satisfied by the regular domains considered in oscillation theory.

THEOREM 1. *Assume that G satisfies condition* (A) *and that there exists an admissible second order expression ℓ_1 with C^∞ coefficients such that*:

$$(\phi, (-1)^n \Delta^n \phi) \geqq (\phi, \ell_1 \phi) , \tag{2}$$

for all $\phi \in C_0^\infty(G)$. *Assume further that* $\ell_1 - q$ *is nonoscillatory at* ∂G. *Then* $S(L) \wedge (-\infty, 0)$ *is a finite set.*

Proof. Since $\ell_1 - q$ is nonoscillatory at ∂G, it follows from our assumptions that there exists a positive solution v of $(\ell_1 - q)v = 0$ in a neighborhood N of ∂G. A suitable form B, as given in (iii), may then be constructed using v so that if $\phi \in C_0^\infty(G)$ is perpendicular (in L^2) to a finite dimensional subspace (determined by B) of L^2 we then have:

$$(\phi, \ell_1, \phi) - (q\phi, \phi) \geqq 0 .$$

Detailed proofs of the above statements follow by trivially modifying the arguments given in [6 – 9]. The conclusion now follows from inequality (2) and the spectral theorem.

We remark that if G is an exterior domain with smooth boundary then Theorem 1 remains valid if "nonoscillatory at ∞" is substituted for "nonoscillatory at ∂G". Furthermore it is now sufficient that $q \in C_{\text{loc}}^\alpha$ near ∞. In the definition of admissible we substitute here for the form B of (iii) the form B' defined on $\{u \mid u \in C^1(\bar{G} \wedge \{|x| \leqq R\}), u = 0 \text{ near } \partial G - \{\infty\}\}$ by:

$$B'(u, v) = \int_{G \wedge \{|x| < R\}} \{\sum a_{ij} D_i u D_j \bar{v} - q u \bar{v}\} + \int_{|x| = R} \sigma u \bar{v} .$$

The proof of this remark is essentially identical to that of Theorem 1. We remark that an essential requirement is that ∞ be an isolated point of ∂G. Analogous results are possible for problems on bounded domains G with singularities on isolated parts of ∂G.

COROLLARY 1. *Assume that for some function* $w > 0$, $w \in C^\infty(G)$ *we have* $(\phi, (-1)^{n-1}\Delta^{n-1}\phi) \geqq (w\phi, \phi)$ *for all* $\phi \in C_0^\infty(G)$, *and let* ℓ_1 *denote the expression:* $\ell_1\phi = -\sum_{k=1}^m D_k(wD_k\phi)$. *If* ℓ_1 *is admissible and* $\ell_1 - q$ *is nonoscillatory at* ∂G *then* $S(L) \wedge (-\infty, 0)$ *is a finite set.*

COROLLARY 2. *Let* G *be contained in an exterior domain. Then there exists constant* C, α, β *(which depend on* n, m*) such that for any* $\phi \in C_0^\infty(G)$ *we have* $(\phi, (-1)^{n-1}\Delta^{n-1}\phi) \geqq (w\phi, \phi)$, *where* $w = C|x|^\alpha(\ell n|x|)^\beta$.

The proof of Corollary 1 is immediate from the observation:

$$(\phi, (-1)^n\Delta^n\phi) = \sum_k (D_k\phi, (-1)^{n-1}\Delta^{n-1}D_k\phi) \geqq \sum_k (wD_k\phi, D_k\phi) .$$

Corollary 2 is a summary of results found in [3], [4] where explicit, but often lengthy, expressions are given for suitable C, α, β in terms of n, m.

The general operator L may now be considered by using Corollaries 1 and 2. As mentioned above, however, we proceed by explicitly considering only the case $n = 2$, and by showing that in this case Theorem 1 can lead to optimal results. We do this by first obtaining a lemma which gives better results than those obtainable from Corollaries 1 and 2.

LEMMA 1. *Let G be an exterior domain, $m > 4$ and let $\phi \in C_0^\infty(G)$. It follows that*:

$$(\Delta\phi, \Delta\phi) \geqq \frac{m^2}{4}\int \frac{1}{|x|^2} \sum (D_i\phi)^2 \, . \tag{3}$$

Proof. We adopt the procedure used in [3], [10] for similar estimates. Let Y_i denote a system of complete orthonormal spherical harmonics and let $k = k(i)$ denote the order of Y_i. For a given $\phi \in C_0^\infty(G)$ we set $f_i = \int_\Phi \phi Y_i dw$ where Φ is the full range of the angular variables and dw denotes the angular component of the volume element in polar coordinates. It follows that:

$$\int (\Delta\phi)^2 = \sum_{i=0}^{\infty} \int_0^\infty r^{m-1}\Big(f_i'' + \frac{(m-1)f_i'}{r} - \frac{k(k+m-2)}{r^2}f_i\Big)^2 dr \, ,$$

and:

$$\int \frac{1}{r^2} \sum (D_i\phi)^2 = \sum_{i=0}^{\infty} \int_0^\infty \{r^{m-3}(f_i')^2 + f_i^2 r^{m-5} k(k+m-2)\} dr \, .$$

Consequently, (3) will be satisfied if we can show that for all k:

$$\begin{aligned}&\int_0^\infty r^{m-1}\Big(f'' + \frac{(m-1)f'}{r} - \frac{k(k+m-2)}{r^2}f\Big)^2 \\ &\qquad \geqq \frac{m^2}{4}\int_0^\infty \{r^{m-3}(f')^2 + f^2 r^{m-5}k(k+m-2)\}dr \, ,\end{aligned} \tag{4}$$

where we have set $f_i = f$. We first expand and integrate by parts the left hand side of (4) and then estimate the $(f'')^2$ term by Formula (9) of [5, p. 83]. This procedure shows that for (4) to hold it is sufficient that:

$$\begin{aligned}&\int_0^\infty \Big\{r^{m-3}(f')^2 2k(k+m-2) + r^{m-5} \\ &\qquad \times f^2\Big[k^2(k+m-2)^2 + k(k+m-2)\Big(2m-8-\frac{m^2}{4}\Big)\Big]\Big\} \geqq 0 \, .\end{aligned} \tag{5}$$

Estimating the $(f')^2$ term by the results of [3] reduces (5) to showing that, for each possible value of k, we have:

$$\int_0^\infty r^{m-5}f^2\left\{2k(k+m-2)\frac{(m-4)^2}{4}+k^2(k+m-2)^2+k(k+m-2)\left(2m-8-\frac{m^2}{4}\right)\right\}$$
$$\geqq 0 .$$

But this inequality is easily seen to be valid by direct examination, and the result follows.

We remark that if $m \leqq 4$ the above procedure apparently leads to worse constants than $m^2/4$.

To apply the lemma we first recall that, by [3], [10], the operator L generated by

$$\ell u = \Delta^2 u - qu$$

in $G \subset E^m$, $m > 4$, is oscillatory (resp. nonoscillatory) if $16|x|^2 q \geqq m^2(m-4)^2 + \delta$ (resp. $\leqq m^2(m-4)^2$) near infinity, where $\delta > 0$.

COROLLARY 3. *Let $n = 2$, $m > 4$ and let G be an exterior domain with smooth boundary. Assume that $-4^{-1}\sum D_i(m^2|x|^{-2}D_i\phi)$ is admissible and that for all $|x|$ sufficiently large we have $16|x|^4 q(x) \leqq m^2(m-4)^2$. Then $S(L) \wedge (-\infty, 0)$ is finite. Furthermore $m^2(m-4)^2$ is the largest possible constant.*

Proof. By the remark following Theorem 1 and by Lemma 1 it is sufficient to show that the operator generated by:

$$\ell_1\phi = -4^{-1}\sum D_i(m^2|x|^{-2}D_i\phi) - q\phi$$

is nonoscillatory at $\{\infty\}$. Since $16|x|^4 q(x) \leqq m^2(m-4)^2$ near ∞, this is the case by the results in [3]. Finally that $m^2(m-4)^2$ is optimal follows from the above remarks.

As another simple example where "optimal" results are obtained, let us consider the case where G is the 1/2 plane in E^2 given by $x_2 > 0$ and q has singularities on $x_2 \equiv 0$. In this case the analogue of Corollary 3 is:

COROLLARY 4. *Let $-\sum D_k((1/4x_2^2)D_k\phi)$ be admissible. Assume further that near ∂G we have $x_2^4 q(x) \leqq 9/16$. Then $S(L) \wedge (-\infty, 0)$ is finite. Furthermore 9/16 is the optimal constant.*

Proof. In this case we have (see [1])

$$(\phi, -\Delta\phi) \geqq \left(\frac{1}{4x_2^2}\phi, \phi\right),$$

and it is therefore sufficient to show that the operator generated by the expression:

$$-\sum_{i=1}^{2} D_i\left[\frac{1}{4x_2^2}D_i\phi\right] - q\phi$$

is nonoscillatory at ∂G. Again from [1] it follows that the condition $x_2^4 q(x) \leqq 9/16$ is sufficient for nonoscillation at ∂G. That this constant is best possible follows from a separation of variables argument which makes use of the observation that 9/16 is optimal in one dimension (by a theorem of Leighton and Nehari [12, p. 143]).

In conclusion we remark that other second order nonoscillation theorems (for example those involving integral and/or logarithmic estimates, which are explicitly given in [1], [3], [4], [12]) could be used in place of the simple criteria we employed. It is also evident that other regions could be substituted for the exterior domains and 1/2 plane case which we explicitly considered. By these means, several variants of our results can easily be stated.

Finally, we note that the regularity requirement "$q \in C^{\alpha}_{\mathrm{loc}}$" of condition (1) can be modified. It is also sufficient, by the spectral theorem, that the expression $\Delta_1 u + qu$ "majorize" (in the sense of forms) a nonoscillating second order expression with regular coefficients.

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SUPERHARMONIC INTERPOLATION IN SUBSPACES OF $C_c(X)$

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Let E be a closed subset of the compact Hausdorff X and let A be a closed separating subspace of $C_C(X)$. Let ρ be a dominator (strictly positive, *l.s.c.*) defined on $X\times T$, T the unit circle in C. Conditions, formulated in terms of boundary measures, are discussed for approximate and exact solutions to the problem of finding ρ-dominated extensions in A of functions $g\in(A|_E)^-$ satisfying re tg $(x)\leqq\rho(x,t)$ on $E\times T$. Various interpolation theorems of Rudin-Carleson type for superharmonic dominators are incorporated into this framework.

We do not assume that A contains the constant functions. We denote $M(X)=C(X)^*$, the space of regular Borel measures on X.

We consider $N=M(E)$ as situated in $M(X)$ as the range of the projection $\pi_1\mu=\mu|_E$ and denote the complementary projection $\pi_2\mu=\mu|_{X\setminus E}$. Thus $(A|_E)^\perp$ is identified with the subspace $A^\perp\cap N$ in $M(X)$.

We call $\mu\in M(X)$ a *boundary measure* if $|\mu|$ is maximal with respect to the Choquet ordering as a measure of X (embedded by evaluation) in the w^* compact unit ball A_1^*. If $1\in A$ then this is the same as $|\mu|$ being maximal on the state space S_A, as $X\subset S_A$, a w^* closed face of A_1^*.

For brevity we denote the boundary measures by $\partial_A M(X)$, or $\partial M(X)$, if A is understood, and in general, adopt the convention of writing $\partial_A S$ for $S\cap\partial_A M(X)$. Thus, $\partial_A A^\perp$ refers to the boundary measures annihilating A. The space A^* is the quotient space $M(X)/A^\perp$ and images under the quotient map are denoted $\hat{\mu}$ for $\mu\in M(X)$. A subset $S\subset M(X)$ is called *A-stable* if $\hat{S}=(\partial_A S)^\wedge$.

We call E an *interpolation set* if $A|_E$ is closed in $C(E)$. Gamelin [8] shows that E is an interpolation set if and only if there is a k; $0\leqq k<\infty$, such that for each $m\in A^\perp$,

$$\|\pi_1 m+A^\perp\cap N\|\leqq k\|\pi_2 m\|\,. \tag{1}$$

The best value of k is called the *extension constant*, $e(A,E)$.

In [10] Roth introduces a general framework for interpolation problems by means of a *dominator*, ρ, defined as a strictly positive *l.s.c.* extended real-valued function on $X\times T$ (T the unit circle in C). We let

$$U=\{f\in C(X)\colon re\ \mathrm{tf}\,(x)/\rho(x,t)\leqq 1 \text{ for all } (x,t)\in X\times T\}$$

and write

$$\|f\|_\rho = \sup\{re\, tf(x)/\rho(x, t): (x, t) \in X \times T\}$$

for the Minkowski functional of U. Thus $\|f\|_\rho \leqq 1$ if and only if $re\, tf(x) \leqq \rho(x, t)$, $(x, t) \in X \times T$. Then $\|\mu\|_\rho$, $\mu \in M(X)$, refers to the polar functional given by

$$\|\mu\|_\rho = \sup\{re(f, \mu): f \in U\} .$$

Since ρ is *l.s.c* and positive there is a constant c such that $\|f\|_\rho \leqq c\|f\|$ (the uniform norm corresponding to $\rho \equiv 1$) and if ρ is bounded above the two are equivalent.

We say E is an *approximate ρ-interpolation set* for A if E is an interpolation set and for each $g \in (A|_E)^-$ and $\varepsilon > 0$ there is an $f \in A$ such that $f|_E = g$ and $\|f\|_\rho < \|g\|_\rho + \varepsilon$. We say E is an *exact ρ-interpolation set* if f can be chosen with $\|f\|_\rho = \|g\|_\rho$. It is shown in [5] that for bounded ρ, E is an approximate ρ-interpolation set for A if and only if for each $m \in A^\perp$,

$$\|\pi_1 m + A^\perp \cap N\|_\rho \leqq \|-\pi_2 m\|_\rho . \tag{2}$$

If, in addition, the image $\hat{U}$ of U^0 under the quotient map is *decomposable* by $\hat{N}$ then E is an exact ρ-interpolation set. If there is an s, $0 \leqq s < 1$, such that for each $m \in A^\perp$,

$$\|\pi_1 m + A^\perp \cap N\|_\rho \leqq s\|-\pi_2 m\|_\rho \tag{3}$$

then the above holds and E is ρ-exact for A. Gamelin's results [8] can be phrased as follows: Let G be a compact set in $X \backslash E$ and let

$$\rho(G, k)(x, t) = \begin{cases} 1 \text{ for } (x, t) \in E \times T \\ k \text{ for } (x, t) \in G \times T \\ 1 \vee k \text{ otherwise.} \end{cases}$$

Then E is an approximate $\rho(G, k)$-interpolation set for all such G if and only if (1) holds and if, in addition, $e(A, E) < 1$ then E is an exact ρ-interpolation set for any continuous T-invariant ρ such that $\rho > e(A, E)$ on $X \times T$. This was obtained in abstract form using polar techniques by Ando [3].

In [6] Briem shows that if E is a subset of the Choquet boundary, $\partial_A X$, then E is an interpolation set if and only if (1) holds only for $m \in \partial_A A^\perp$. Further, if X is metrizable then (1) holds for $\partial_A A^\perp$ if and only if E is an approximate $\rho(G, k)$-interpolation set for each compact $G \subset \partial_A X \backslash E$. The A-stability of the unit ball $M_1(X)$ (Hustad's theorem [9]) and of $N = M(E)$ (since $E \subset \partial_A X$) are essential here. If (1) holds for $\tilde{e}(A, E) < 1$ (again, $\tilde{e}$ is the smallest k such that (1) holds for all $m \in \partial A^{\perp}$) then E is $\rho(G, k)$ exact for any $G \subset \partial_A X \backslash E$ and $k > \tilde{e}$.

If (1) holds for all $m \in \partial_A A^{\perp}$ with $k = 0$ this can be expressed as

$$(4) \qquad m \in \partial_A A^{\perp} \text{ imples } \pi_1 m \in A .$$

The set E is called an M-set if $M(E)$ is A-stable and (4) holds. Roth [10] shows that if E is an M-set and ρ is a bounded A-*superharmonic* (if $1 \in A$ this means $\rho(x, t) \geqq \int \rho(\cdot, t) d\mu$ for any $\mu \in M_1^+(X)$ and $\hat{\mu} = x \in X \subset A_1^*$) dominator then E is an exact ρ-interpolation set for A. This generalizes the Alfsen-Hirsberg theorem [2] which deals with T-invariant ρ and $E \subset \partial_A X$.

In this note we consolidate these results by showing that for E an interpolation set with $M(E)$ A-stable and ρ A-superharmonic then E is an approximate ρ-interpolation set if and only if (2) holds for $m \in \partial_A A^{\perp}$. If in addition $\hat{U}$ is decomposable by $\hat{N}$ in A^* then the interpolation is exact. This is the case if ρ is bounded and (3) holds for $m \in \partial_A A^{\perp}$. (If ρ is bounded and (2) or (3) holds then E is already an interpolation set.) We give a measure theoretic condition for the decomposability of $\hat{U}$ and show by means of simple examples of $A(K)$ spaces that exactness of interpolation can be deduced in this way even though equality holds in (2) which, of course, precludes the use of (3).

1. Hustad-Roth stability theorems. Let A be a closed separating subspace of $C(X)$. Define $\Phi: C(X) \to C(X \times T)$ by $\Phi f(x, t) = tf(x)$. By separating we shall mean that the range of $\Phi|_A$ separates the points of $X \times T$. This assumption can be avoided, as is shown in Fuhr-Phelps [7], but at the expense of additional technicalities. If $\nu \in M(X \times T)$ then the Hustad map is given by

$$\mu = \Phi^* \nu \in M(X); \mu(f) = \int_{X \times T} tf(x) d\nu(x, t) .$$

If $\Phi = \Phi|_A$ has range $B \subset C(X \times T)$ and ν is a maximal probability measure on $X \times T \subset B_1^*$ representing $\tilde{L} \in B_1^*$ then Hustad's theorem says $\mu = \Phi^* \nu$ belongs to $\partial_A M(X)_1$ with $\hat{\mu} = L = \phi^* \tilde{L}$. We combine this with the following observations concerning T-invariant A-superharmonic dominators to obtain a general stability theorem due to Roth [11].

Thus let ρ be a strictly positive *l.s.c.* extended real-valued function on X such that for each $x \in X$ and $\mu \in M_1^+(X)$ with $\hat{\mu} = x \in A^*$, we have $\rho(x) \geqq \int_X \rho d\mu$, that is, ρ is A-superharmonic. If $U = \{f \in C(X): ref/\rho \leqq 1\}$ then U^0 is a w^* compact convex subset of the positive cone $M^+(X)$, and we let $\hat{U}$ be the quotient image in A^*. Take $\bar{R}^+$ to be the one-point-compactification of R^+ and

$$X_0 = \{(x, s) \in X \times \bar{R}^+ : \rho(x) \leqq s \leqq +\infty\} ,$$
$$Y_0 = \{(x, \rho(x)) \in X_0 : \rho(x) < \infty\} ,$$
$$Y_\infty = \{(x, \rho(x)) \in X_0 : \rho(x) = +\infty\} .$$

Since ρ is *l.s.c.*, $Y_0 \cup Y_\infty$ and Y_∞ are both G_δ subsets of X_0 so that Y_0 is a Borel set. Define

$$\psi : C(X) \longrightarrow C(X_0);\ \psi f(x, s) = f(x)/s ,$$

and let $\theta = \psi|_A$ with (not necessarily closed) range $B \subset C(X_0)$. Since ρ is strictly positive ψ is bounded and θ^* is one-to-one from B^* into A^*. Let

$$\phi_0 : X_0 \longrightarrow B_1^*$$

be the evaluation map and let $\hat{V} = w^* - \overline{co}\phi_0(X_0)$.

PROPOSITION 1.1. *Let ρ be a T-invariant A-superharmonic dominator on X as above.*

(1) *ϕ_0 is one-to-one on $X_0 \backslash (X \times \{\infty\})$, $X \times \{\infty\} = \phi_0^{-1}(0)$, and $\theta^* \hat{V} = \hat{U}$.*

(2) *If ν is a maximal probability measure on $\hat{V}$ then $\nu[\phi_0(Y_0) \cup \{0\}] = 1$ and ν may be identified with the measure on Y_0 given by $\nu \circ \phi_0$.*

(3) If *ν is as in* (2) *and $\mu = \psi^* \nu$ then for any bounded Borel function h on X*

$$\int_X h d\mu = \int_{Y_0} (h(x)/\rho(x) d\nu(x, \rho(x)) .$$

In particular, $\mu \in U^0$.

(4) *Let $\mu_0 \in M_1^+(X)$ with $\hat{\mu}_0 = x_0 \in X \subset A_1^*$ and define $\tilde{\mu}_0 \in M(X_0)$ by*

$$\tilde{\mu}_0(F) = (1/\rho(x_0)) \int_X F(x, \rho(x)) \rho(x) d\mu_0(x) .$$

Then for any bounded Borel function h on X

$$\int_{X_0} (h(x)/s) d\tilde{\mu}_0(x, s) = (1/\rho(x_0)) \int_X h d\mu_0 .$$

In particular $\tilde{\mu}_0 \geqq 0$, $\tilde{\mu}_0(X_0) = \tilde{\mu}_0(Y_0) \leqq 1$, and $\tilde{\mu}_0$ represents $(x_0, \rho(x_0)) \in \hat{V}$.

(5) *If ν is maximal on $\hat{V}$ then $\mu = \psi^* \nu$ is maximal on $K = \overline{co} X \subset A^*$.*

Proof. (1) The separation theorem shows $\hat{U} = w^*\overline{co}\{x/s: (x, s) \in X_0\}$. Now

$$\theta^* \circ \phi_0(x, s) = x/s \in A^*$$

so the rest of (1) follows from the fact that A separates points in X. For (2) let $p = 1 - \chi_{\{0\}}$ on $\hat{V}$ and note that the lower envelope $\check{p}$ is the Minkowski functional of $\hat{V}$. Since ν is maximal,

$$1 = \nu[\{x: p(x) = \check{p}(x)\}] = \nu[\{x: \check{p}(x) = 1 \text{ or } 0\}] .$$

Now $\lambda \geqq 1$ implies $\phi_0(x, \lambda s) = (1/\lambda)\phi_0(x, s)$, so that

$$\nu[\phi_0(Y_0) \cup \{0\}] = 1 .$$

If $f \in C(X)$ then $\psi^*\nu(f) = \int_{X_0}(f(x)/s)d\nu(x, s) = \int_{Y_0}(f(x)/\rho(x))d\nu(x, \rho(x))$ and so (3) holds.

(4): If $F \in C(X_0)$ and $0 \leqq F \leqq 1$ then

$$0 \leqq \tilde{\mu}_0(F) \leqq (1/\rho(x_0))\int_X \rho d\mu_0 \leqq 1 .$$

Thus $\tilde{\mu}_0 \geqq 0$, $\tilde{\mu}_0(X_0) \leqq 1$ and $\mu_0[\{x: \rho(x) = +\infty\}] = 0$. For $F = \psi h$,

$$\begin{aligned}\tilde{\mu}_0(F) &= \int_{X_0}(h(x)/s)d\tilde{\mu}_0(x, s) \\ &= (1/\rho(x_0))\int_X hd\mu_0 .\end{aligned}$$

(5): Let f be a continuous convex function of K and denote the upper envelope of f by $\hat{f}(K)$, where [1, I. 3.6]

$$\hat{f}(K)(x_0) = \sup\{\mu(f): \mu \in M_1^+(X) \text{ and } \hat{\mu} = x_0 \in A^*\} .$$

If $g = \psi(f|_X)$ then $g \in C(X_0)$ with $g \equiv 0$ on $X \times \{\infty\}$. If $\tilde{\mu}_0 = x_0$ and $\tilde{\mu}_0$ is as in (4) then $\tilde{\mu}_0$ represents $(x_0, \rho(x_0)) \in \hat{V}$ and the upper envelope, $\hat{g}(\hat{V})$, satisfies

$$\hat{g}(\hat{V})(x_0, \rho(x_0)) \geqq \sup\{\tilde{\mu}_0(g): \hat{\mu}_0 = x_0\} = (1/\rho(x_0))\hat{f}(K)(x_0)$$

by part (4). Thus, using part (3), and [1, I. 4.5],

$$\int_X[\hat{f}(K) - f]d\mu = \int_{Y_0}[\hat{f}(K) - f]/\rho d\nu \leqq \int_{Y_0}[\hat{g}(\hat{V}) - g]d\nu = 0$$

since ν is maximal. Hence, μ is maximal on K.

We now consider the case where ρ is defined on $X \times T$. We say such a ρ is *A-superharmonic* if for each $(x, t) \in X \times T$ and $\mu \in M(X \times T)_1^+$ with

$$\int_{X\times T} sf(y)d\mu(y, s) = tf(x) \text{ for all } f \in A$$

we have $\rho(x, t) \geqq \int_{X\times T} \rho d\mu$.

THEOREM 1.2 (Hustad-Roth). *If ρ is an A-superharmonic dominator then U^0 is A-stable.*

Proof. Let $\Phi: C(X) \to C(X \times T)$; $\Phi f(x, t) = tf(x)$ and let

$$U^1 = \{F \in C(X \times T): reF(x, t)/\rho(x, t) \leqq 1\}$$

and $\phi = \Phi|_A$ with range B.

Let $\Psi: C(X \times T) \to C(X_0)$; $\Psi F(x, t, s) = F(x, t)/s$, where X_0 is the closed epigraph of ρ in $(X\times T)\times \bar{R}^+$. Now $\Phi U \subset U^1$ and $\phi(A \cap U) = B \cap U^1$. Given $L \in \hat{U}$, let $\tilde{L} \in (U^1)^\wedge \subset B^*$ and $L' \in \hat{V}$ (as in Proposition 1.1) with $\theta^* L' = \tilde{L}$ and $\phi^* \tilde{L} = L$. We have $B_1^* = w^*\overline{co}(X \times T)$ and the hypothesis says ρ on $X \times T$ is B-superharmonic. Hence the results of Proposition 1.1 apply. Thus if ν' is maximal on $\hat{V}$ representing L' then 1.1 (3) and (5) show $\nu = \Psi^*\nu'$ is maximal on B_1^* representing $\tilde{L} \in (U^1)^\wedge$. Then $\mu = \phi^*\nu \in U^0$ and $\hat{\mu} = L \in \hat{U}$. Furthermore, Hustad's theorem shows μ is a boundary measure.

If $1 \in A$ then the condition for A-superharmonicity is somewhat simpler.

PROPOSITION 1.3. *If $1 \in A$ then ρ is A-superharmonic if and only if for each $\mu \in M_1^+(X)$ with $\hat{\mu} = x$,*

$$\rho(x, t) \geqq \int_X \rho(\cdot, t)d\mu .$$

Proof. If ρ is A-superharmonic and $\mu \in M_1^+(X)$ with $\hat{\mu} = x$ we can embed X as $X \times \{t\} \subset X \times T$ so that the measure μ satisfies

$$\int_{X\times T} sf(y)d\mu = tf(x)$$

and hence

$$\rho(x, t) \geqq \int_{X\times\{t\}} \rho(x, t)d\mu = \int_X \rho(\cdot, t)d\mu .$$

Conversely, if $\mu \in M_1^+(X \times T)$ and represents tx then, since $1 \in A$, we have $t\overline{co}X = tS_A$(S_A the state space of A) is a face of A_1^*. Hence $\mathrm{supp}\mu \subset X \times \{t\}$ and the measure $\mu_1(f) = \int_{X\times T} f(x)d\mu$ represents x so that

$$\rho(x, t) \geqq \int_X \rho(\cdot, t)d\mu_1 = \int_{X\times T} \rho d\mu .$$

2. **Dominated interpolation.** If E is a compact subset of X we let

$$M = \{f \in C(X): f|_E = 0\}$$

and denote $M \cap A$ by $E^\perp$. It is well known that E is an interpolation set for A if and only if $A + M$ is closed in $C(X)$ and this in turn is equivalent to $\hat{N}$ being w^* (or norm) closed in A^*. The following characterization of approximate ρ-interpolation sets follows from results in [5; 4.2]. We denote $N = M(E) \subset M(X)$.

THEOREM 2.1. *Let ρ be a (strictly positive l.s.c) dominator on X such that either ρ is bounded or E is an interpolation set. The following are equivalent*:

(i) *E is an approximate ρ-interpolation set for A,*

(ii) *$A + M$ is closed in $C(X)$ and*

$$(A + M) \cap (U + M) = (A \cap U + M)^- ,$$

(iii) $\hat{U} \cap \hat{N} = (U^0 \cap N)^\wedge$,

(iv) $\|\mu + A^\perp \cap N\|_\rho = \|\mu + A^\perp\|_\rho$ *for all* $\mu \in N$,

(v) $\|\pi_1 m + A^\perp \cap N\|_\rho \leqq \|-\pi_2 m\|_\rho$ *for all* $m \in A^\perp$.

For $x \in A^*$ we write $\|x\|_\rho$ for the Minkowski functional of $\hat{U}$ so that if $\hat{\mu} = x$

$$\|x\|_\rho = \|\mu + A^\perp\|_\rho .$$

The set U^0 is *split*, that is, $\|\mu\|_\rho = \|\pi_1\mu\|_\rho + \|\pi_2\mu\|_\rho$ [10, 5].

PROPOSITION 2.2. *Let N and U^0 be A-stable sets in $M(X)$. Then for $\mu \in \partial_A M(X)$,*

(1) $\|\mu + A^\perp\|_\rho = \|\mu + \partial A^\perp\|_\rho = \|\hat{\mu}\|_\rho$,

(2) $\|\mu + N + A^\perp\|_\rho = \|\pi_2\mu + \pi_2\partial A^\perp\|_\rho$ $(\pi_2\mu = \mu|_{X\setminus E})$,

(3) If $\|\mu\|_\rho = \|\hat{\mu}\|_\rho$ *then*

$$\|\pi_i\mu\|_\rho = \|(\pi_i\mu)^\wedge\|_\rho \quad (i = 1, 2) .$$

Proof. If $\mu \in \partial M(X)$ and $\|\hat{\mu}\|_\rho \leqq r$ then $\mu = r\nu + m$ with $\nu \in U^0$ and $m \in A^\perp$. The stability of U^0 shows we can assume $\nu \in \partial U^0$, so that $m \in \partial A^\perp$. Then (1) follows. If $\mu = r\nu + \eta + \zeta$ with $\nu \in \partial U^0$, $\eta \in \partial N$, $\zeta \in A^\perp$, then $\zeta \in \partial A^\perp$ and $\pi_2\mu = r\pi_2\nu + \pi_2\zeta \in r\pi_2 U^0 + \pi_2\partial A^\perp$. Conversely, if $\pi_2\mu = r\nu + \pi_2\zeta$, $\nu \in \partial U^0$, $\zeta \in \partial A^\perp$ then

$$\mu = r\nu + (\pi_1\mu - \pi_1\zeta) + \zeta \in rU^0 + \partial N + \partial A^\perp .$$

For (3), we have

$$\begin{aligned}||\pi_1\mu||_\rho &\geqq ||(\pi_1\mu)^\wedge||_\rho = ||\pi_1\mu + A^\perp||_\rho = ||\mu - \pi_2\mu + A^\perp||_\rho \\ &\geqq ||\mu||_\rho - ||\pi_2\mu + A^\perp||_\rho \geqq ||\mu||_\rho - ||\pi_2\mu||_\rho = ||\pi_1\mu||_\rho .\end{aligned}$$

Since we do not assume $1 \in A$, we take the *Choquet boundary*, $\partial_A X$, to be $X \cap ext A_1^*$. There are two main instances where the A-stability of N can be deduced.

PROPOSITION 2.3. *Let E be a closed subset of X such that either*
(a) $E \subset \partial_A X$ *or*
(b) $E = F \cap X$, *F a w^* closed face of A_1^*.*
Then N is A-stable.

Proof. In the case (a) each probability measure on E is maximal and so the result follows since $\overline{co}E$ spans $\hat{N}$. In case (b) each maximal probability measure μ with $\hat{\mu} \in \overline{co}E$ has its support on $(ext\ F)^- \subset E$.

THEOREM 2.4. *Let E be a closed subset of X such that either*
(a) $E \subset \partial_A X$, *or*
(b) $E = F \cap X$, *F a closed face of A_1^*.*
Let ρ be an A-superharmonic dominator such that either ρ is bounded or E is an interpolation set. Then the following are equivalent:
(i) *E is an approximate ρ-interpolation set,*
(ii) $||\mu + A^\perp \cap N||_\rho = ||\mu + \partial A^\perp||_\rho$ *for all* $\mu \in \partial N$,
(iii) $||\pi_1 m + A^\perp \cap N||_\rho \leqq ||-\pi_2 m||_\rho$ *for all* $m \in \partial A^\perp$.

Proof. The hypotheses imply that U^0 and N are A-stable and so 2.2. (1) shows for $\mu \in \partial M$,

$$||\mu + A^\perp||_\rho = ||\mu + \partial A^\perp||_\rho .$$

Thus (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) follows from 2.1. If (ii) holds and $x \in \hat{U} \cap \hat{N}$ then choose $\mu \in \partial N$ with $\hat{\mu} = x$ and $\mu \in U^0 + A^\perp$. Then

$$||\mu + A^\perp \cap N||_\rho = ||\mu + \partial A^\perp||_\rho = ||\mu + A^\perp||_\rho \leqq 1$$

so that $\mu = \nu + m$; $\nu \in U^0$, $m \in A^\perp \cap N$. Hence $\nu \in N$ and $\hat{\mu} = x = \hat{\nu} \in (U^0 \cap N)^\wedge$. Thus 2.1 (iii) holds and hence (i) is shown.

The exactness of ρ-interpolation is characterized by the sum

$A \cap U + E^{\perp}(E^{\perp}$ the ideal of functions in $C(X)$ vanishing on E) being closed in A, a condition which is implied by the decomposability of $\hat{U}$ by $\hat{N}$ in A^* [5; Theorem 3.2]. If E is an interpolation set (so that $\hat{N}$ if w^* closed in A^*) then $\hat{U}$ is said to be *decomposable* by $\hat{N}$ if there is an $\alpha \geqq 1$ such that each $x \in \hat{U}$ is a convex combination of elements y, z with $y \in \hat{U} \cap \hat{N}, z \in \hat{U}$ and $||z|| \leqq \alpha||z + \hat{N}||$ (dual uniform norm).

The condition for decomposability, and hence exact interpolation, can be formulated in terms of representing measures in $M(X)$. We illustrate this for boundary measures in the case where ρ is superharmonic.

THEOREM 2.5. *Let E be a closed subset of X and A a closed separating subspace such that either*

(a) $E \subset \partial_A X$, *or*

(b) $E = F \cap X, F$ *a closed face of* A_1^*,

and let ρ be an A-superharmonic dominator such that either ρ is bounded or E is an interpolation set.

If for each $x \in \hat{U}$ there is a $\mu \in \partial_A U^0$ with $\hat{\mu} = x$ and

$$||\pi_2\mu + \partial A^{\perp}|| \leqq \alpha||\pi_2\mu + \pi_2\partial A^{\perp}||$$

(α a constant independent of μ) then E is an exact ρ-interpolation set.)

Proof. Given $x \in \hat{U}$ choose a boundary measure μ satisfying $\hat{\mu} = x$, $||\hat{\mu}||_\rho = ||\mu||_\rho$ and $||\pi_2\mu + \partial A^{\perp}|| \leqq \alpha||\pi_2\mu + \pi_2\partial A^{\perp}||$. Now $||\mu||_\rho = ||\pi_1\mu||_\rho + ||\pi_2\mu||_\rho$ shows that μ is a convex combination of $\mu_1 \in U^0 \cap N$ and $\mu_2 \in U^0$, scalar multiples of $\pi_1\mu, \pi_2\mu$ respectively. Thus, $||\mu_2 + \partial A^{\perp}|| \leqq \alpha||\mu_2 + \pi_2\partial A^{\perp}||$ and x is a convex combination of $y \in (U^0 \cap N)^\wedge$ and $z \in \hat{U}$ with (using 2.2 (1) and (2))

$$\begin{aligned} ||z|| = ||\mu_2 + \partial A^{\perp}|| &\leqq \alpha||\mu_2 + \pi_2\partial A^{\perp}|| = \alpha||\mu + N + A^{\perp}|| \\ &= \alpha||z + \hat{N}|| . \end{aligned}$$

This shows that $(U^0 \cap N)^\wedge = \hat{U} \cap \hat{N}$ and that $\hat{U}$ is decomposable by $\hat{N}$. Therefore E is an exact ρ-interpolation set.

If E is an M-set then $\pi_2\partial A^{\perp} \subset \partial A^{\perp}$ so that

$$||\pi_2\mu + \pi_2\partial A^{\perp}|| \geqq ||\pi_2\mu + \partial A^{\perp}||$$

and the condition of 2.5 is automatically satisfied (for A-stable U^0). More generally, if U^0 and N are A- stable and, for some $s < 1$

$$||\pi_1 m + A^{\perp} \cap N||_\rho \leqq s||-\pi_2 m||_\rho \text{ for all } m \in \partial A^{\perp}$$

then a computation based on [5; 4.8] shows the condition of Theorem 2.5 holds, so that E is an exact ρ-interpolation set.

COROLLARY 2.6. *If E is an M-set for the closed separating subspace $A \subset C(X)$ then E is an exact ρ-interpolation set for A for any A-superharmonic dominator ρ.*

Proof. If E is an M-set then $\hat{N}$ is the range of a projection in A^* so that E is an interpolation set for A. The conclusion then follows from 2.5.

3. **Examples.** We illustrate the results of §2 with various choices of ρ. First, let X be a compact *metric* space with E a closed subset and $M(E)$ A-stable for the closed separating subspace $A \subset C(X)$. Let G be the collection of compact subsets $G \subset \partial_A X \backslash E$ and let $\rho = \rho(G, k)$ be the dominator mentioned in the introduction. Then (for $k < \infty$)

$$(1) \qquad ||\pi_1 m + A^\perp \cap N|| \leqq k||\pi_2 m|| \text{ for all } m \in \partial A^\perp$$

if and only if E is an approximate $\rho(G, k)$-interpolation set for all $G \in \mathscr{G}$. To see this we note that since $G \subset \partial_A X$, U^0 is A-stable so that the second property holds if and only if

$$(2) \qquad ||\pi_1 m + A^\perp \cap N||_\rho \leqq ||-\pi_2 m||_\rho \text{ for all } m \in \partial A^\perp, G \subset \mathscr{G} .$$

It follows easily from [5; 4.1] that if $Y = X \backslash (E \cap G)$ then

$$||\mu||_\rho = ||\mu|_E|| + k||\mu|_G|| + (1 \vee k)||\mu|_Y||$$

so that

$$||\pi_1 m + A^\perp \cap N|| = ||\pi_1 m + A^\perp \cap N||_\rho$$

and, since for boundary measures μ, the metrizability of X gives

$$|\mu|(X \backslash E) = |\mu|(\partial_A X \backslash E) = \sup\{|\mu|(G) \colon G \in \mathscr{G}\} ,$$

we have

$$k||\pi_2 m|| = \sup\{||\pi_2 m||_\rho \colon \rho = \rho(G, k),\ G \in \mathscr{G}\} .$$

The equivalence of (1) and (2) is now immediate. If (1) holds for $k_0 < 1$ and $k_0 < k \leqq 1$ then for $\rho = \rho(G, k)$

$$\begin{aligned} ||\pi_1 m + A^\perp \cup N||_\rho &= ||\pi_1 m + A^\perp \cap N|| \leqq k_0(||m|_G|| + ||m|_Y||) \\ &\leqq (k_0/k)(k||m|_G|| + ||m|_Y||) = (k_0/k)||\pi_2 m||_\rho \end{aligned}$$

so that E is an exact $\rho(G, k)$-interpolation set for A.

The study of sufficient conditions for the A-convex hull of E to be a generalized peak set (we now assume $1 \in A$) has been shown [4] to be related to an ordering on $C_c(X)$ and $M(X)$ induced by choosing P to be a closed proper convex cone with nonempty interior in $\boldsymbol{C}$. Let α, β be the generators (of modulus one) of the dual cone $P^* = \{z: re az \geqq 0 \text{ for all } a \in P\}$. We denote by e the element of P such that $re e\gamma = 1$ $(\gamma = \alpha, \beta)$. If $f \in C_c(X)$ we say $f \geqq 0(P)$ if $f(X) \subset P$ and $\mu \geqq 0(P^*)$ means $\mu(B) \in P^*$ for all Borel sets $B \subset X$. Then the function $\boldsymbol{e} \equiv e$ becomes an order unit for $C(X)$ in which the order unit norm $\|\cdot\|_e$ (equivalent to the uniform norm) is given by

$$\rho(x, t) = \begin{cases} 1 \text{ for } t = \pm\gamma \\ 1/c \text{ for } t \neq \pm\gamma\,, \end{cases} \quad \gamma = \alpha, \beta$$

where c is a constant such that

$$cz| \leqq |re\alpha z| \vee |re\beta z|\ .$$

This provides an example of a ρ which is not T-invariant.

Let ρ^+ and ρ^- be strictly positive $l.s.c.$ functions on X and take

$$\rho(x, t) = \begin{cases} \rho^+(x) \text{ on } X \times \{1\} \\ \rho^-(x) \text{ on } X \times \{-1\} \\ +\infty \text{ otherwise.} \end{cases}$$

Then $U = \{f \in C(X): -\rho^- \leqq ref \leqq \rho^+\}$. If $\mu \in U^0$ and f is real then $\lambda if \in U$ for all real λ so that

$$1 \geqq re\mu(\lambda if) = -\lambda im\mu(f)$$

and hence $im\mu(f) = 0$. Thus μ is a real measure and $U^0 \subset reM(X)$.

If A_0 is a real subspace of $C(X)$ then we can apply the results of §2 to the self-adjoint space $A_0 + iA_0 = A$. Then $\|f\|_\rho = \|ref\|_\rho$ and $m \in A^\perp$ if and only if $m = m_1 + im_2$ with m_1, m_2 real measures in $A^\perp$. Also m is a boundary measure if and only if m_1, m_2 are boundary. Hence E is an approximate (exact) ρ-interpolation set for A if and only if it is for $A_0 = reA$, and the measure conditions of §2 need only involve real measures in $M(X)$. If X is a compact convex subset of a locally convex space and $A_0 = A(X)$ (real affine continuous functions) then ρ is A-superharmonic if and only if $\rho^+ = (\rho^+)^\wedge$ and $\rho^- = (\rho^-)^\wedge$, that is, if and only if ρ^+ and ρ^- are concave on X.

Let X be a square in R^2 with vertices denoted $\{1, 2, 3, 4\}$ with

$E = \{1, 2\}$ diagonally opposite and $A_0 = A(X)$, ρ^+, $\rho^- \equiv 1$. Then $\partial A^\perp$ is a one-dimensional subspace of the four-dimensional space $\partial M(X)$ spanned by the point-masses $\{\delta_i\}_{i=1}^4$. A generator for $\partial A^\perp$ is $m = \delta_1 + \delta_2 - \delta_3 - \delta_4$. Clearly $A^\perp \cap N = \{0\}$ since coE is a simplex and so

$$||\pi_1 m + A^\perp \cap N|| = ||\pi_1 m|| = ||\pi_2 m|| .$$

This shows E is an approximate ρ-interpolation set for $A(X)$. Obviously E is in fact an exact interpolation set, but this cannot be concluded from a condition such as (3) in the introduction. Nevertheless, the condition of 2.5 holds, since if

$$\mu = \Sigma \lambda_i \delta_i$$

then

$$||\mu|| = \Sigma |\lambda_i|$$

and

$$||\pi_2 \mu + \pi_2 \partial A^\perp|| = inf\{|\lambda_3 - \lambda| + |\lambda_4 - \lambda| : \lambda \in R\} = |\lambda_4 - \lambda_3| .$$

If λ_3 and λ_4 are opposite in sign then

$$||\pi_2 \mu + \partial A^\perp|| \leqq ||\pi_2 \mu|| = |\lambda_3| + |\lambda_4| = |\lambda_4 - \lambda_3| = ||\pi_2 \mu + \pi_2 \partial A^\perp|| .$$

If, say $0 \leqq \lambda_3 \leqq \lambda_4$, consider $\nu = \mu + \lambda_3 m$. Then $\hat{\nu} = \hat{\mu}$ and

$$||\nu|| = \Sigma |\lambda_i - \lambda_3| \leqq (|\lambda_1| + |\lambda_2| + 2|\lambda_3|) + |\lambda_4| - |\lambda_3| = ||\mu||$$

and

$$||\pi_2 \nu + \partial A^\perp|| \leqq ||\pi_2 \nu|| = \lambda_4 - \lambda_3 = ||\pi_2 \mu + \pi_2 \partial A^\perp|| .$$

We conclude with an example of an approximate interpolation set which is not exact. Let X be the unit ball of the sequence space l^1(w^* topology) and let $\rho \equiv 1$. Then take $A = c_0$, the pre-dual of l^1, so that $||a||_\rho = ||a||_\infty = sup\{|a_n|\}$. Let E be the singleton $\{x^0\}$, $x_n^0 = 1/2^n$, $n = 1, 2, \cdots$. If $(a, x^0) = 1$ then $\sum_{n=1}^\infty a_n/2^n = 1$ so that some a_n must be greater than one. Clearly we can find such an a with $||a|| \leqq 1 + \varepsilon$ for any $\varepsilon > 0$. Thus E is an approximate, but not exact, ρ-interpolation set.

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AN ANTI-OPEN MAPPING THEOREM FOR FRÉCHET SPACES

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It is well-known that completeness is necessary for the usual open mapping theorem for Fréchet spaces. In contrast, it is shown that, with the obvious exception of ω, each infinite-dimensional Fréchet space has another distinct complete topology with the same continuous dual.

By a space or subspace, we mean an infinite-dimensional locally convex Hausdorff topological vector space over either the real or the complex scalars. Our notation generally follows Robertson and Robertson [7]. In particular, X' and $\sigma(X, X')$ denote the continuous dual and the weak topology on X, respectively. Denote by ω (respectively, ϕ) the space formed by the product (respectively, direct sum) of countably-many copies of the scalar field. We use c_0, l_1 and l_∞ to denote the Banach sequence spaces (with their usual norms) of, respectively, null sequences, absolutely summable sequences and bounded sequences.

Our main result can be stated as:

THEOREM. *Each Fréchet space* $(X, \zeta) \neq \omega$ *has a topology* η, *so that,* $\sigma(X, X') < \eta < \zeta$ *and the space* (X, η) *is complete.*

By the open mapping theorem, (X, η) is a complete space which is not barrelled. In Section one we prove the theorem for the special cases of $(X, \zeta) = c_0$ (Case I) and (X, ζ) a nuclear space with a continuous norm (Case II). Then in Section two we reduce the theorem to these special cases.

We will have occasion to use Grothendieck's characterization of the completion of the space (X, ζ) as the set of linear functionals on X' which are $\sigma(X', X)$-continuous on ζ-equicontinuous sets (see Robertson and Robertson [7], p. 103). Berezanskii's [4] (see also [2, pp. 61–62]) notion of inductive semi-reflexivity is used in Case II. In particular, complete nuclear spaces are inductive semi-reflexive, and the topology constructed from $\{\mu_n\}$ in Case II is complete in any inductive semi-reflexive space. The only other fact used about nuclear spaces is that their topology can be defined by means of (semi-) inner products (see Case II and Schaefer [7] p. 103).

Perhaps it is worth pointing out, that there are always lots of differently-defined complete topologies on each complete separable space (see Bellenot [1], [2] and with Ostling [3]): the difficulty is in showing that these topologies are really different.

1. Two special cases. First we prove the theorem for the following special cases:

Case I. The Banach space c_0: Let ξ be the norm topology on c_0 and let $\mathscr{U}$ be a free ultrafilter on the set of positive integers (i.e., $\cap \mathscr{U} = \emptyset$). For each $A \in \mathscr{U}$ and $K > 0$ let

$$E(A, K) = \{x = (x_n) \in l_1: \|x\|_1 \leqq K \text{ and } x_m = 0 \text{ for each } m \in A\} .$$

Let η be the topology of uniform convergence of the collection of sets

$$\{E(A, K) \cup \{y^n\}: A \in \mathscr{U}, K > 0, \{y^n\} \text{ a } l_1\text{-norm-null-sequence}\} .$$

Since finite sets are η-equicontinuous and each of the sets above are ξ-equicontinuous, we have $\sigma(c_0, l_1) \leqq \eta \leqq \xi$.

To see that $\eta < \xi$, note that if $\eta = \xi$ there would be a set $E(A, K) \cup \{y^n\}$ whose polar is contained in the unit ball of c_0. Since y^n is a l_1-norm-null-sequence, there is an M, so that $m \geqq M$ implies that $|y^n_m| < 2^{-1}$, for each n. (Where y^n is the sequence $\{y^n_m\}_m$.) Since $\mathscr{U}$ is free, A must be infinite and there is a $k \geqq M$ with $k \in A$. Consider $x \in c_0$, the vector which is the zero sequence, except that it is 2 in the kth position. Clearly x is not in the unit ball of c_0, but it is the polar of $E(A, K) \cup \{y^n\}$, a contradiction.

Consider X, the completion of (c_0, η), as a subspace of the algebraic dual of l_1. Since each l_1-norm-null-sequence is η-equicontinuous, $X \subset l_\infty$. Suppose D is a subset of the positive integers with $D \notin \mathscr{U}$. Then, since $\mathscr{U}$ is an ultrafilter, D^c, the complement of D, is an element of $\mathscr{U}$. Thus the η-topology restricted to the subspace $\{x \in c_0: x_n = 0 \text{ if } n \notin D\}$ is the norm topology. It follows that for each $f = (f_n) \in X$, the subsequence $\{f_n: n \in D\}$ is a null-sequence, since f is $\sigma(l_1, c_0)$-continuous on $E(D^c, 1)$. Let $f = (f_n) \in l_\infty$ with $A = \{n: |f_n| \geqq \delta\}$ infinite, for some $\delta > 0$. If $A \notin \mathscr{U}$, then $f \notin X$ by the above, so assume $A \in \mathscr{U}$. Write $A = B \cup C$, a disjoint union of infinite sets, one of them is not in $\mathscr{U}$, and thus $f \notin X$. Therefore $X = c_0$ and (c_0, η) is complete.

Case II. (X, ξ) is a nuclear Fréchet space with a continuous norm: Let $\|\cdot\|_1 \leqq \|\cdot\|_2 \leqq \cdots$ be a sequence of continuous norms which define the ξ-topology on X. Since X is nuclear, we assume that the unit ball of each $\|\cdot\|_{k+1}$ is precompact in the norm $\|\cdot\|_k$ and that each $\|x\|_k^2 = \langle x, x\rangle_k$, for some continuous inner product $\langle\cdot\, , \cdot\rangle_k$ on $X \oplus X$. Let $\|\cdot\|_k$ also represent (the possibly infinite-valued) dual norm of $\|\cdot\|_k$ on X'. A sequence $\{a'_n\} \subset X'$ is called *k-admissible* if

$\{\|a'_n\|_k\}$ is bounded and the semi-norm, $\rho_a(x) = \sup_n |a'_n(x)|$, defined for $x \in X$, is stronger than $\|\cdot\|_1$. That is, there is a constant K, with

$$(*) \qquad \|x\|_1 \leqq K\rho_a(x) \ , \quad \text{for each} \quad x \in X \ .$$

A nonincreasing null-sequence of positive reals $\{\lambda_n\}$ is said to be *k-discriminating*, if for each k-admissible sequence $\{a'_n\}$,

$$\limsup_n \|a'_n\|_k \lambda_n^{-1} = \infty \ .$$

Note that if $\{\lambda_n\}$ is k-discriminating and $\{\mu_n\}$ is another nonincreasing null-sequence of positive reals so that $\lim_n \mu_n/\lambda_n = 0$, then $\{\mu_n\}$ is also k-discriminating.

First, we prove that for each k, there is a k-discriminating sequence. To see this, let $\{e_n\} \subset X$ be a sequence orthogonal in $\langle \cdot\ ,\ \cdot \rangle_k$ and orthonormal in $\langle \cdot\ ,\ \cdot \rangle_1$. (The $\{e_n\}$ can be chosen inductively, by picking $e_{n+1} \in (\bigcap_1^n \ker f_i) \cap (\bigcap_1^n \ker g_i)$, where f_i and g_i are the continuous linear functionals given by $f_i(x) = \langle e_i,\ x\rangle_1$ and $g_i(x) = \langle e_i,\ x\rangle_k$, $i = 1, 2, \cdots n$.) Re-order $\{e_n\}$ so that the sequence $\{\|e_n\|_k\}_n$ is nondecreasing. We claim that the sequence $\lambda_n = 1/n \|e_{n^2}\|_k$ is k-discriminating. Suppose not, then there is k-admissible $\{a'_n\}$ with

$$(**) \qquad \|a'_n\|_k \leqq \lambda_n \ .$$

Let $\delta = 2^{-1}K^{-1}$, where K is the constant in (*).

Inductively choose $f_n \in X$ and an integer-valued function ϕ, so that

(1) $\|f_n\|_1 = 1$ and $f_n \in \operatorname{span}\{e_j\colon (n-1)^2 < j \leqq n^2\}$;

(2) $a'_{\phi(j)}(f_n) = 0$ for $j < n$; and

(3) $|a'_{\phi(n)}(f_n)| \geqq \delta$.

If f_j and $\phi(j)$ have been chosen for $j < n$, it is possible to choose f_n satisfying (1) and (2) since condition (2) puts $n-1$ constraints on f_n and f_n is chosen from a $(2n-1)$-dimensional space. Thus by (*) we can find a $\phi(n)$ such that $2^{-1}\|f_n\|_1 = 2^{-1} \leqq K|a'_{\phi(n)}(f_n)|$, i.e., that (3) is satisfied.

Let $A(n) = \{j\colon (n-1)^2 < j \leqq n^2\}$ and suppose $f_n = \sum_{i \in A(n)} a_i e_i$. Since $\{e_i\}$ is orthonormal in $\langle \cdot\ ,\ \cdot \rangle_1$, condition (1) implies $\sum_{i \in A(n)} |\alpha_i|^2 = 1$. But $\{e_i\}$ is orthogonal in $\langle \cdot\ ,\ \cdot \rangle_k$ hence

$$\|f_n\|_k = \left[\sum_{i \in A(n)} |\alpha_i|^2 \|e_i\|_k^2\right]^{1/2} \leqq \|e_{n^2}\|_k = n^{-1}\lambda_n^{-1} \ .$$

Thus by condition (3), we have

$$(***) \qquad \delta \leqq |a'_{\varphi(n)}(f_n)| \leqq \|a'_{\phi(n)}\|_k \cdot \|f_n\|_k \leqq n^{-1}\lambda_n^{-1}\|a'_{\phi(n)}\|_k \ .$$

On the other hand, condition (2) implies that ϕ is $1-1$ and hence $\phi(n) \geqq n$, infinitely often. Thus (**) implies $\|a'_{\phi(n)}\|_k \leqq \lambda_{\phi(n)} \leqq \lambda_n$, infinitely often. Combining with (***) yields

$$0 < \delta \leqq n^{-1}\lambda_n^{-1}\lambda_n = \frac{1}{n}\ , \quad \text{for infinitely many} \quad n\ ,$$

a contradiction.

For $k \geqq 1$, let $\{\lambda_n^k\}$ be a k-discriminating. The sequence $\lambda_n = n^{-1}\min\{\lambda_n^j: j \leqq n\}$ is thus k-discriminating for each $k \geqq 1$. Let $\mu_n = \lambda_{n(n+1)}$, and let η be the topology of uniform convergence on sequences $\{a'_n\} \subset X'$ with the property that there is an integer k and constant K with $||a'_n||_k \leqq K\mu_n$. It is easy to check that $\sigma(X, X') < \eta \leqq \xi$. Note that if U is any ξ-neighborhood of the origin and if ρ_{U^0} is the gauge functional of U-polar in X', then there is an integer k and a constant K so that $a' \in X'$ implies $||a'||_k \leqq K\rho_{U^0}(a')$. Thus by Bellenot [2, p. 62 and Th. 4.1, p. 64], η is a ξ-rotor topology and (X, η) is complete.

To show that $\eta < \xi$, we will prove that $||\cdot||_1$ is not η-continuous on X. By Robertson and Robertson [7, p. 46], the η-neighborhoods of the origin are polars of finite unions of the above sequences (as sets of values in X'). (Note that it is possible for $\lim(\mu_n/\mu_{2n}) = \infty$, and so we must consider finite unions.) Suppose $||\cdot||_1$ is η-continuous, then there is a finite number of sequences $\{b'_{n,i}\}_n$, $1 \leqq i \leqq j$, used to define η, so that $||x||_1 \leqq \sup\{|b'_{n,i}(x)|: 1 \leqq i \leqq j,\ n = 1, 2, \cdots\}$ for each $x \in X$. Let k and K be so that $||b'_{n,i}||_k \leqq K\mu_n$, for $1 \leqq i \leqq j$ and each n. Let $\{a'_n\}$ be a listing of values in X' contained in the sequences $\{b'_{n,i}\}_n$, $1 \leqq i \leqq j$, so that $\{||a'_n||_k\}$ is nonincreasing. It follows that $\{a'_n\}$ is k-admissible. Since $n > j \geqq i \geqq 1$ implies $(n+1)(n+2) \geqq nj + i$, and $\{\lambda_n\}$ is nonincreasing, $\lambda_{(n+1)(n+2)} \leqq \lambda_{nj+i}$. Thus if $m \geqq j^2 + j + 1$, then $m = nj + i$ with $n > j \geqq i \geqq 1$, and

$$||a'_m||_k \leqq K\mu_{n+1} = K\lambda_{(n+1)(n+2)} \leqq K\lambda_m\ .$$

Hence $\limsup_m(||a'_m||_k/\lambda_m) < \infty$ and since $\{\lambda_m\}$ is k-discriminating, $\{a'_n\}$ is not k-admissible. This contradiction completes the proof of the theorem for this case.

2. The general case. The following two lemmas are of a general nature. The first lemma shows that completeness is a "three space property" while the second is used often in the proof of the theorem. The referee has pointed out that Lemma 1 is known, we include a proof for completeness.

LEMMA 1. *Let X be a space, Y a closed subspace of X and $Z = X/Y$, the quotient. If Y and Z are complete, then X is complete.*

Proof. Let $\phi: X \to Z$ be the quotient map and let $j: X \to \hat{X}$ be the injection of X into its completion $\hat{X}$. Since Z is complete, ϕ

extends to a map $\hat{\phi}: \hat{X} \to Z$ so that $\hat{\phi} \circ j = \phi$. Furthermore, since Y is complete, $j(Y)$ is closed in $\hat{X}$, and thus we can construct the quotient $W = \hat{X}/j(Y)$ with quotient map $\psi: \hat{X} \to W$. Since $\ker \psi \circ j = Y$, there is a map $\theta: Z \to W$ so that $\theta \circ \phi = \psi \circ j: X \to W$. Thus $\theta \circ \hat{\phi} \circ j = \psi \circ j$, but since $j(X)$ is dense in $\hat{X}$ and since $\theta \circ \hat{\phi}$ and ψ are continuous, we have $\theta \circ \hat{\phi} = \psi$. Therefore θ and thus j are surjective maps, so that X is complete.

LEMMA 2. *A Fréchet space X satisfies the conclusions of the theorem if X has a closed subspace Y which satisfies the conclusions of the theorem.*

Proof. Let ξ be the topology on X with neighborhood basis of the origin $\mathscr{U}$. Let η be a topology on Y with $\sigma(Y, Y') \leqq \eta < \xi|_Y$ and so that the space (Y, η) is complete. Let $\mathscr{V}$ be the neighborhood basis of the origin for (Y, η). Let $\mathscr{W} = \{V + U: V \in \mathscr{V}, U \in \mathscr{U}\}$. It is straightforward to check that $\mathscr{W}$ is a neighborhood basis of the origin for a topology ζ on X with the properties:

(i) $\sigma(X, X') \leqq \zeta < \xi$,

(ii) $\zeta|_Y = \eta$, and

(iii) $(X, \zeta)/Y \equiv (X, \xi)/Y$.

Thus by (ii), (iii) and Lemma 1, (X, ζ) is complete and by (i) it satisfies the conclusion of the theorem.

Proof of the theorem. Let (X, ξ) be a Fréchet space $\neq \omega$. It follows that ξ is not the weak topology on X. First, we show there is a separable closed subspace Y of X, so that ξ, restricted to Y has a continuous norm. Since ξ is strictly stronger than $\sigma(X, X')$, there exists a continuous semi-norm on (X, ξ) which is not a linear combination of semi-norms $x \to |\langle x, x' \rangle|$ with $x' \in X'$, and thus from Schaefer [8], corollary on p. 124 it follows that X has a continuous semi-norm ρ so that the dimension of $X/\ker \rho$ is infinite. Let E be the normed space $X/\ker \rho$ with ρ norm and let $\psi: X \to E$ be the quotient map. Let $\{e_n\} \subset E$ be a linearly independent sequence. Let $\{x_n\} \subset X$ be so that $\psi(x_n) = e_n$, and let Y be the closed linear span of $\{x_n\}$ in (X, ξ). Since $\rho\,(\sum_1^n \alpha_i x_i) = \rho(\sum_1^n \alpha_i e_i)$, for all scalar sequences $\{\alpha_i\}$, ψ, restricted to Y, is an isometry of Y with semi-norm ρ into a subspace of E with norm ρ. Thus by Lemma 2, we assume that (X, ξ) is separable and has a continuous norm.

Suppose (X, ξ) is a Banach space. In the notation of Bellenot and Ostling [3], since X is separable and complete, we have $\xi = \xi_M$. Furthermore, Theorem 3.1 of that same paper shows (X, ξ_{SW}) is complete, where ξ_{SW} is the topology of uniform convergence on ξ-equicontinuous $\sigma(X', X)$-null sequences. Clearly, $\sigma(X, X') < \xi_{SW} \leqq \xi$, and if $\xi_{SW} < \xi$, then we are done. If $\xi_{SW} = \xi$ and since (X, ξ) is a Banach space, there must be a $\sigma(X', X)$-null sequence $\{a'_n\} \subset X'$, whose polar in X is contained in the unit ball of X. It is easy to check that the map, $T: X \to c_0$, which sends $x \in X$ to the sequence $\{a'_n(x)\} \in c_0$, is an isomorphism of X onto a closed subspace of c_0. (These results are known, see the author [1].) A classical result of Banach (see Lindenstrauss and Tzafriri [6, p. 53]) says that X must have a subspace isomorphic to c_0. An application of Lemma 2 and Case I completes the proof if (X, ξ) is a Banach space.

If (X, ξ) is not a Banach space, then X is not a subspace of $B \oplus \omega$, for any Banach space B. Thus a result of Bessaga, Pelczyński and Rolewicz [5] show that (X, ξ) has a nuclear subspace Y. Thus Case II and Lemma 2 completes the proof of the theorem.

REMARKS. It is possible that the following statement is true:

(*) Each complete space (X, ξ) with $\xi \neq \sigma(X, X')$, has another complete topology η with $\sigma(X, X') < \eta < \xi$.

There are three places in the proof of the theorem where metrizability was used. The most subtle use of the metric was in Lemma 2. If (X, ξ) is not Fréchet, it is possible that X/Y is not complete (Schaefer [6, Ex. 11, p. 192]) and hence Lemma 1 cannot be used to show (X, ζ) is complete. (The author thanks E. G. Ostling for pointing this out to the author.) Thus it is possible that (*) could be true for separable X, but false in general.

If (X, ξ) is separable and complete, then, as in the proof of the theorem (X, ξ_{SW}) is complete (see Bellenot and Ostling [3]). In this case $\xi = \xi_{SW}$ implies that (X, ξ) is a closed subspace of a product of copies of the Banach space c_0. In order to handle this in the manner of Case I, one must extend this case to include each (X, ξ) which is not inductively semi-reflexive, but for which $\xi = \xi_{SW}$. Examples of spaces which fall into this extended case and which may fail (*) are the spaces (X, ξ_{SW}) where (X, ξ) is any separable nonreflexive Banach space.

The proof that the topology constructed in Case II is complete works for any inductive semi-reflexive space. However, to show that this constructed topology was different from the given topology made strong use of the metrizability. In fact, if $(X, \xi) = \phi$, then for any positive nonincreasing null-sequence, $\{\mu_n\}$, the topology constructed in Case II will be the ξ-topology. It is open question if ϕ is the only such exception among complete separable spaces with a continuous norm. (Weak topologies are also exceptions.) In any case the space ϕ is perhaps the most likely counter-example (among the inductively semi-reflexive spaces) to (*).

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LOCALE GEOMETRY

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We commence with a locale $\mathscr{L}$ (that is, a complete Heyting algebra) and introduce the notion of an $\mathscr{L}$-valued betweenness relation on a set. The concept of an $\mathscr{L}$-valued geometry is then formulated and the relevant versions of the Radon, Helly and Carathéodory theorems are proved.

Introduction. The abstract theory of join systems was developed by W. Prenowitz [8] and [9] as an aid to studying descriptive and spherical geometries. This notion of join system has since been further developed by V. W. Bryant and R. J. Webster [1] to enable the corresponding axiomatic treatment of such results as the Radon, Helly and Carathéodory theorems. It is this aspect of the theory with which the present article is concerned.

We commence this article by extending the notion of a join system so that it is no longer necessarily two-valued. More precisely, given a locale lattice $\mathscr{L}$, we introduce the notion of an $\mathscr{L}$-valued betweenness relation $(-, -, -): X \times X \times X \to \mathscr{L}$ on a set X; if $(x, y, z) = p \in \mathscr{L}$ we might say that the point z lies on the segment (x, y) with "probability p". This loose description is related to theories of multivalued logic which arise in topos theory. Indeed, one can develop join systems in a reasonably complete topos in terms of multivalued join systems over the category of sets; see §4. These notions, in turn, give rise to the forms of the Radon, Helly and Carathéodory theorems dicussed in §3.

We emphasize here that, in this preliminary article, we do not deal with multigroups (after W. Prenowitz) nor do we enter into all aspects of dimension theory (after V. W. Bryant and R. J. Webster). Also we leave the proof of the more basic elementary deductions as simple exercises for the reader; these results are used without reference.

1. $\mathscr{L}$-forms. Let $\mathscr{L}$ be a locale and let X be a set. A *symmetric $\mathscr{L}$-form* on X is a function $X(-, -): X \times X \to \mathscr{L}$ such that $X(x, x) = 1$, $X(x, y) = X(y, x)$, $\sup_y X(x, y) \wedge X(y, z) = X(x, z)$. A *functional* on X is a set map $A: X \to \mathscr{L}$ such that $A = \sup_x A(x) \wedge X(x, -)$. A *singleton*, or *point* is a functional of the form $\bar{x} = X(x, -)$: $X \to \mathscr{L}$. Thus each functional is an "expansion of singletons" or an "internal colimit of points". For notational convenience we shall represent $\bar{x}$ simply by x unless we wish to emphasize the distinction.

The ordered set of functionals on X is denoted $\mathrm{Fnl}(X, \mathscr{L})$; it is a sublocale of $\mathscr{L}^X$. Note that if $A: X \to \mathscr{L}$ is any functional then $A \geqq x$ iff $A(x) = 1$.

A *map* of $\mathscr{L}$-forms $f: (X, X(-, -)) \to (X', X'(-, -))$ is a set map $f: X \to X'$ such that $X'(fx, fy) = X(x, y)$ for all $x, y \in X$.

2. **Convexity spaces.** An *$\mathscr{L}$-preconvexity space* is a set X equipped with a symmetric $\mathscr{L}$-form $X(-, -): X \times X \to \mathscr{L}$ and a map $(-, -, -): X \times X \times X \to \mathscr{L}$ which is functional in each variable separately. A *map* of preconvexity space is a map $f: X \to X'$ of $\mathscr{L}$-forms such that $(fx, fy, fz) = (x, y, z)$ for all $x, y, z \in X$. The resultant category is denoted $\mathscr{L}pc$.

Given $X \in \mathscr{L}pc$ we define the *convolutions*:

$$AB(x) = \sup_{y,z} A(y) \wedge B(z) \wedge (y, z, x)$$

$$A/B(x) = \sup_{y,z} A(y) \wedge B(z) \wedge (z, x, y) .$$

Then $\bar{x}\bar{y} = (x, y, -)$ is the *join* of x to y, while $\bar{x}/\bar{y} = (y, -, x)$ is the *extension* of x by y.

An interesting consequence of these definitions is the following *Kan-extension principle*: If f and g are polynomials of n-variables in the convolution operations AB and A/B, and $f(x_1, \cdots, x_n) \geqq g(x_1, \cdots, x_n)$ for all points $x_1, \cdots, x_n$ then $f(A_1, \cdots, A_n) \geqq g(A_1, \cdots, A_n)$ for all functionals $A_1, \cdots, A_n$.

An *$\mathscr{L}$-convexity space* is an $\mathscr{L}$-preconvexity space which satisfies the following axioms:

C1. (symmetry) $(x, y, z) = (y, x, z)$.

C2. (idempotence) $(a, a, x) = X(a, x)$, $(a, x, a) = X(a, x)$.

C3. (associativity) $\sup_w (y, v, w) \wedge (w, z, x) = \sup_w (v, z, w) \wedge (y, w, x)$.

C4. (transposition) $\sup_w (z, w, y) \wedge (x, w, v) \leqq \sup_w (x, y, w) \wedge (z, v, w)$.

C5. (cancellation) $\sup_w (x, y, w) \wedge (x, z, w) = X(y, z) \vee (x, y, z) \vee (x, z, y)$.

The full subcategory of $\mathscr{L}pc$ comprising the $\mathscr{L}$-convexity spaces is denoted $\mathscr{L}c$.

The following propositions are immediate from the axioms.

PROPOSITION. $xy/xz = y/z \vee xy/z \vee y/xz$.

PROPOSITION. $AB = BA$, $(AB)C = A(BC)$, $A \leqq AA$ *and* $A \leqq A/A$, $(A/B)/C = A/BC$, $A(B/C) \leqq AB/C$, *and* $A/(B/C) \leqq AC/B$.

PROPOSITION. (i) $xA/x = A \vee xA \vee A/x$,
(ii) $xA/xB = A/B \vee xA/B \vee A/xB$,
(iii) $x/xB = x/B$.

The following relations are easily deduced by iterated use of the preceding proposition:

LEMMA 2.1 (*Radon*).

$$\frac{x_0 \cdots x_n}{x_0 \cdots x_n} = \bigvee \left\{ x_{i_0} \cdots x_{i_r}, \frac{x_{i_0} \cdots x_{i_s}}{x_{i_{s+1}} \cdots x_{i_r}};\ i_0, \cdots, i_r \text{ all different} \right\} .$$

LEMMA 2.2 (*Carathéodory*). *For $n \geqq r$*

$$\frac{x_0 \cdots x_n}{x_0 \cdots x_r} = \bigvee \left\{ x_{i_0} \cdots x_{i_p}, \frac{x_{i_0} \cdots x_{i_q}}{x_{i_{q+1}} \cdots x_{i_p}};\ i_0, \cdots, i_p \text{ all different and } p - q \leqq r \right\} .$$

For the remainder of this section we shall suppose that X is a fixed $\mathscr{L}$-convexity space. A functional $A: X \to \mathscr{L}$ is said to be *convex* if $AA = A$; note that singletons are convex (C2). The *convex hull* of a functional A is defined to be $\bigvee_{n=1}^{\infty} A^n$.

PROPOSITION. (i) *If $A_1, \cdots, A_n$ are convex then so are $A_1 \cdots A_n$ and A_1/A_2.*

(ii) *The convex hull of a functional A is the intersection in* $\mathrm{Fnl}(X, \mathscr{L})$ *of all the convex functionals which contain A.*

A functional $A: X \to \mathscr{L}$ is said to be *linear* if it is convex and $A/A = A$. The *linear hull* of a functional A is defined to be $\bigvee_{m,n=1}^{\infty} A^m/A^n$ and is denoted by $[A]$.

PROPOSITION. (i) *The linear hull of a functional A is the intersection in* $\mathrm{Fnl}(X, \mathscr{L})$ *of all the linear functionals which contain A.*

(ii) *If A is convex then A/A is linear.*
(iii) *If A is convex then $[A] = A/A$.*
(iv) $[x_0 \cdots x_n] = x_0 \cdots x_n/x_0 \cdots x_n$.

3. The Radon, Helly and Carathéodory theorems. Henceforth in this section we suppose that X is a fixed $\mathscr{L}$-convexity space. We shall also suppose that whenever we consider a set $\{x_0, \cdots, x_n\}$ then the $\bar{x}_i$'s are distinct (recall that $\bar{x}_i$ is denoted simply by x_i). The product functional of $M = \{x_1, \cdots, x_n\}$ is denoted by $M^* = x_1 \cdots x_n$.

A set $\{x_0, \cdots, x_n\}$ of singletons is said to be *strongly dependent* if there exists an $i(0 \leqq i \leqq n)$ such that $[x_0 \cdots x_{i-1}x_{i+1} \cdots x_n](x_i) = 1$. If every set of $n + 2$ singletons is strongly dependent then we say that X has *dimension* $\leqq n$.

THEOREM 3.1 (*Radon*). *If $\{x_0, \cdots, x_{n+1}\}$ is a set of $n + 2$ singletons in a convexity space of dimension $\leqq n$ then there exist disjoint nonempty subsets M and N of $\{x_0, \cdots, x_{n+1}\}$ such that $M^* \wedge N^* \neq 0$.*

Proof. The $n + 2$ points lie in a space of dimension $\leqq n$ so we may assume, without loss of generality, that $[x_1 \cdots x_{n+1}](x_0) = 1$. By Lemma 2.1 we have either $N^*(x_0) \neq 0$ where N is a subset of $\{x_1, \cdots, x_{n+1}\}$ or $N^*/P^*(x_0) \neq 0$ where N and P are nonempty disjoint subsets of $\{x_1, \cdots, x_{n+1}\}$. Thus the result follows from taking $M = x_0$ in the first case and $M = \{x_0, P\}$ in the second case. In the first case we have $N^*(x_0) \neq 0$ implies $x_0 \wedge N^* \neq 0$ since $x_0 \wedge N^* = 0$ implies $x_0(x_0) \wedge N^*(x_0) = 0$ implies $N^*(x_0) = 0$, and in the second case we have $N^*/P^*(x_0) \neq 0$ implies $\sup_{u,v} N^*(u) \wedge P^*(v) \wedge (v, x_0, u) \neq 0$ implies $\sup_u N^*(u) \wedge x_0P^*(u) \neq 0$ implies there exists a $u \in X$ such that $N^*(u) \wedge x_0P^*(u) \neq 0$.

THEOREM 3.2 (*Helly*). *If $A_0, \cdots, A_{n+1}$ is a family of $n + 2$ convex functionals on a convexity space of dimension $\leqq n$ and any $n + 1$ of these functionals intersect with certainty then all the functionals have a nonzero intersection.*

Proof. For each $i(0 \leqq i \leqq n + 1)$ there exists, by hypothesis, a singleton x_i such that

$$x_i \leqq A_0 \wedge \cdots \wedge A_{i-1} \wedge A_{i+1} \wedge \cdots \wedge A_{n+1}\,.$$

If $x_i = x_j$ for some $i \neq j$ then $x_i \leqq A_0 \wedge \cdots \wedge A_{n+1}$ and the result follows. Otherwise the singletons x_i are distinct so that, by Theorem 3.1, there exist nonempty disjoint subsets M and N of $\{x_0, \cdots, x_{n+1}\}$ such that $M^* \wedge N^* \neq 0$. Because $M^* \wedge N^* \leqq A_0 \wedge \cdots \wedge A_{n+1}$ the result follows.

LEMMA 3.3. *If $x \leqq x_0 \cdots x_n$ and $M^*/N^*(x) \neq 0$ where M and N are nonempty disjoint subsets of $\{x_0, \cdots, x_n\}$ then there exists a proper subset P of $\{x_0, \cdots, x_n\}$ such that $P^*(x) \neq 0$.*

Proof. The proof is by induction on the cardinal of N. Firstly, if $|N| = 1$, assume $N = x_0$ without loss of generality. Let $S = \{x_1, \cdots, x_n\}$. Now $x \leqq x_0 \cdots x_n$ implies $x_0 \leqq x/S^*$. Moreover, if

$M^*/x_0(x) \neq 0$ where M is a nonempty subset of S then $S^*/x_0(x) \neq 0$. Thus $0 \neq S^*/x_0(x) \leqq S^*/(x/S^*) \leqq S^*/x(x)$ since S^* is convex. But $S^*/x(x) \neq 0$ implies $\sup_u S^*(u) \wedge (x, x, u) \neq 0$ implies $S^*(x) \neq 0$ so $x_1 \cdots x_n(x) \neq 0$. Now suppose $|N| = r + 1$ and $x \leqq x_0 \cdots x_n$ and $M^*/N^*(x) \neq 0$. Without loss of generality let $N = \{x_0, \cdots, x_r\}$. The conditions $x \leqq x_0 \cdots x_n$ and $M^*/x_0 \cdots x_r(x) \neq 0$ imply that $x_1 \cdots x_n/x_1 \cdots x_r(x) \neq 0$ since $x \leqq x_0 \cdots x_n$ implies $x_0 \leqq x/x_1 \cdots x_n$. Thus $0 \neq M^*/x_0 \cdots x_r(x) \leqq (x_1 \cdots x_n)/(x/x_1 \cdots x_n)x_1 \cdots x_r(x)$ implies $x_1 \cdots x_n/xx_1 \cdots x_r(x) \neq 0$. But $x_1 \cdots x_n/xx_1 \cdots x_r(x) = ((x_1 \cdots x_n/x_1 \cdots x_r)/x)(x)$ so $x_1 \cdots x_n/x_1 \cdots x_r(x) \neq 0$. Thus, by Lemma 2.2, either $P^*(x) \neq 0$ where P is a nonempty subset of $\{x_1, \cdots, x_n\}$ or $Q^*/R^*(x) \neq 0$ where Q and R are nonempty disjoint subsets of $\{x_1, \cdots, x_n\}$ and $|R| \leqq r$.

THEOREM 3.4 (*Carathéodory*). *If $x \leqq x_0 \cdots x_{n+1}$ for singletons in a convexity space of dimension $\leqq n$ then there exists a proper subset P of $\{x_0, \cdots, x_{n+1}\}$ such that $P^*(x) \neq 0$.*

Proof. Without loss of generality let us assume $x_0 \leqq [x_1 \cdots x_{n+1}]$. Thus, by Lemma 2.2, either $M^*(x_0) \neq 0$ where M is a subset of $\{x_1, \cdots, x_{n+1}\}$ or $M^*/N^*(x_0) \neq 0$ where M and N are nonempty disjoint subsets of $\{x_1, \cdots, x_{n+1}\}$. In the first case $x_1 \cdots x_{n+1}(x) \neq 0$ and in the second case $x_1 \cdots x_{n+1}/N^*(x) \neq 0$. In order to establish these assertions let $S = \{x_1, \cdots, x_{n+1}\}$. In the first case note that $M^*(x_0) \neq 0$ implies $S^*(x_0) \neq 0$. But $x \leqq x_0 S^*$ implies $x_0 \leqq x/S^*$ thus

$$\begin{aligned} 0 \neq S^*(x_0) &= \sup_u S^*(u) \wedge X(x_0, u) = \sup_u S^*(u) \wedge x_0(u) \\ &\leqq \sup_u S^*(u) \wedge x/S^*(u) = \sup_{u\ v\ w} S^*(u) \wedge x(v) \wedge S^*(w) \wedge (w, u, v) \\ &= \sup_{u\ w} S^*(u) \wedge S^*(w) \wedge (w, u, x) = S^*(x) \end{aligned}$$

since S^* is convex. Thus $x_1 \cdots x_{n+1}(x) \neq 0$. In the second case we have to show that $\sup_{u,v} S^*(u) \wedge N^*(v) \wedge (v, x, u) = \sup_u S^*(u) \wedge xN^*(u) \neq 0$. But we have

$$\begin{aligned} 0 \neq \sup_u S^*(u) \wedge x_0 N^*(u) &\leqq \sup_u S^*(u) \wedge (x/S^*)N^*(u) \\ &\leqq \sup_u S^*(u) \wedge xN^*/S^*(u) = \sup_{u\ v\ w} S^*(u) \wedge xN^*(v) \wedge S^*(w) \wedge (w, u, v) \\ &= \sup_v S^*(v) \wedge xN^*(v) \end{aligned}$$

since S^* is convex, as required. Thus either $P^*(x) \neq 0$ where P is a nonempty subset of $\{x_1, \cdots, x_{n+1}\}$ or $M^*/N^*(x) \neq 0$ where M and N are nonempty disjoint subsets of $\{x_1, \cdots, x_{n+1}\}$. The first case is as required while in the second case the result follows from Lemma 3.3.

REMARK. In the case $\mathscr{L} = 2$ these results reduce to the generalizations of Radon, Helly and Carathéodory theorems discussed by Bryant and Webster [1].

4. Examples. Examples of $\mathscr{L}$-convexity spaces can be generated by various different processes. Perhaps the most basic of these arises from the fact that $\mathscr{L}c$ is closed under colimits in $\mathscr{L}pc$ and $\mathscr{L}c$ has a generator (namely the one-point space). Thus, by the special adjoint-functor theorem (Mac Lane [7]), the inclusion $\mathscr{L}c \subset \mathscr{L}pc$ has a right adjoint, so every $\mathscr{L}$-preconvexity space has a canonical associated convexity space.

If X is an $\mathscr{L}$-convexity space then X^Z is an $\mathscr{L}^Z$-convexity space for all sets Z. Thus it is consistent to define, in a topos $\mathscr{E}$ (see Johnstone [6]) for which each $\mathscr{E}(Z, \Omega)$ is complete as a Heyting algebra, an Ω-convexity space as a map $(-, -, -): X \times X \times X \to \Omega$ in $\mathscr{E}$ such that $\mathscr{E}(Z, X)$ is an $\mathscr{E}(Z, \Omega)$-convexity space for all $Z \in \mathscr{E}$.

Another example arises as follows. Call a functional $A: X \to \mathscr{L}$ *left exact* if $A(x) \wedge A(y) = \sup_a A(a) \wedge X(a, x) \wedge X(a, y)$ and $\sup_a A(a) = 1$; a left-exact functional is always linear. Given $X \in \mathscr{L}c$ define $\hat{X}$ to be the set of all left-exact functionals from X to $\mathscr{L}$. On $\hat{X}$ define $\hat{X}(A, B) = \sup_x A(x) \wedge B(x)$ and $(A, B, C) = \sup_{x,y,z} A(x) \wedge B(y) \wedge C(z) \wedge (x, y, z)$. Then $\hat{X}$ is an $\mathscr{L}$-convexity space and $X \to \hat{X}$ is a map of $\mathscr{L}$-convexity spaces.

Finally, if $X \times X \times X \to \mathscr{L}_\lambda$, $\lambda \in \Lambda$, represents a set of convexity space structures on a set X, one for each $\lambda \in \Lambda$, the induced map $X \times X \times X \to \prod_\Lambda \mathscr{L}_\lambda$ is a convexity-space structure. This fact allows the construction of $\mathscr{L}$-valued convexity spaces from families of classical convexity spaces on X (see, for example, quasiconvexities [5]).

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CONTINUOUSLY VARYING PEAKING FUNCTIONS

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Let X be a compact metric space, $A \subseteqq C(X)$ a closed subalgebra. Let $\mathscr{P} \subseteqq X$ be the set of peak points for A. It is shown that there is a continuous function $\Phi: \mathscr{P} \to A$ such that $\Phi(x)$ peaks at x for all $x \in \mathscr{P}$.

0. Let X be a compact Hausdorff space, $C(X)$ the continuous functions on X under the uniform norm, and A a closed subspace of $C(X)$ containing 1. Let $\mathscr{P}$ be the set of peak points for A. Clearly if X has more than one point and $x \in \mathscr{P}$ then there are infinitely many functions in A which peak at x. Can one construct a function

$$\Phi: \mathscr{P} \longrightarrow A$$

so that $\Phi(x)$ peaks at x and Φ has some regularity properties?

In [4], using the von Neumann selection principle, it was shown that for $X = \bar{\mathscr{D}} \subset \subset \boldsymbol{C}^n$ with smooth boundary, $A = A(\mathscr{D})$ (the analytic functions on $\mathscr{D}$ which extend continuously to $\bar{\mathscr{D}}$), one can choose Φ to be measurable. The same argument is valid under much more general circumstances.

In the present note we prove that, for quite general X and for A an algebra, Φ can be chosen to be continuous. This generalizes results in [1, Theorem 3.1] and [2, Proposition 4].

1. Throughout the discussion, X will be a fixed compact metric space with metric d. We let $C(X)$ denote the continuous, complex-valued functions on X with the uniform norm and $A \subseteqq C(X)$ will be a closed complex linear subspace. If $x \in X$, $r > 0$, then $B(x, r) = \{t \in X: d(x, t) < r\}$.

DEFINITION. A point $x \in X$ is said to be a *peak point* for A if there is an $f \in A$ with $f(x) = 1$ and, for all $y \in X \sim \{x\}$, $|f(y)| < 1$. The function f is said to peak at x.

We let $\mathscr{P}(A)$ denote the set of peak points for A.

THEOREM. *Let X be a compact metric space, $A \subseteqq C(X)$ a closed subalgebra (with or without 1). Then there is a continuous map*

$$\Phi: \mathscr{P}(A) \longrightarrow A$$

such that $\Phi(x)$ peaks at x for each $x \in \mathscr{P}(A)$.

The remainder of the paper is devoted to the proof of the theorem. We proceed via a sequence of lemmas. The plan of the proof is as follows.

For each $k \in \{1, 2, \cdots\}$ we will construct a continuous function

$$\Phi_k: \mathscr{P}(A) \longrightarrow A$$

such that for each $x \in \mathscr{P}(A)$ we have

(i) $\|\Phi_k(x)\| = 1$;

(ii) $[\Phi_k(x)](x) = 1$;

(iii) if $t \in X \sim B(x, 1/k)$ then $|[\Phi_k(x)](t)| \leqq 1 - 1/(k+2)$.

Once the $\{\Phi_k\}$ are constructed, the proof is immediate. For let $\Phi = \sum_{l=1}^{\infty} 2^{-l}\Phi_l$. Then Φ is continuous and for each $x \in \mathscr{P}(A)$ we have $\Phi(x) \in A$ and $[\Phi(x)](x) = 1$. Moreover, if $t \neq x$ and $k > 1/d(x, t)$ then

$$\begin{aligned} |[\Phi(x)](t)| &\leqq \sum_{l \neq k} 2^{-l}|[\Phi_l(x)](t)| + |2^{-k}[\Phi_k(x)](t)| \\ &\leqq 1 - 2^{-k} + 2^{-k}(1 - 1/(k+2)) < 1 . \end{aligned}$$

So $\Phi(x)$ peaks at x. Thus it remains to construct the Φ_k.

LEMMA 1. *Let $x_0 \in \mathscr{P}(A)$. Let p be a strictly positive continuous function on X with $p(x_0) = 1$. Then there is an $f \in A$ with $f(x_0) = 1$ and $|f(x)| \leqq p(x)$ for all $x \in X$.*

Proof. This is a special case of Theorem 12.5 of Gamelin [3], p. 58.

COROLLARY 2. *With hypotheses as in Lemma 1, there is a $g \in A$ such that $g(x_0) = 1$, $|g(x)| < p(x)$ for all $x \in X \sim \{x_0\}$.*

Proof. Immediate.

LEMMA 3. *Let $x_0 \in \mathscr{P}(A)$. Let $\psi \in A$ peak at x_0. There is a map*

$$\Psi: \mathscr{P}(A) \cap \{|\psi(x)| > 1/2\} \longrightarrow A$$

so that

(i) *$\Psi(x)$ peaks at x for each $x \in \mathscr{P}(A) \cap \{|\psi(x)| > 1/2\}$,*

(ii) *$\Psi(x_0) = \psi$,*

(iii) *Ψ is continuous at x_0.*

Proof. For each $x \in \mathscr{P}(A) \sim \{x_0\}$ choose, by Corollary 2, a function $\varphi_x \in A$ such that $\varphi_x(x) = 1$ and

$$(*)\qquad |\varphi_x(t)| < \min\{(2-|\psi(x)|-|\psi(t)|)/2(1-|\psi(x)|),\, 1\} \quad \text{for all} \quad t \in X \sim \{x\}\,.$$

Now for each $x \in \mathscr{P}(A) \cap \{|\psi(x)| > 1/2\}$ we define

$$\Psi(x) = \begin{cases} [2(1-|\psi(x)|)\varphi_x + \overline{\operatorname{sgn}\psi(x)}\psi]/[2-|\psi(x)|] & \text{if} \quad x \neq x_0,\ |\psi(x)| > 1/2\,, \\ \psi & \text{if} \quad x = x_0\,. \end{cases}$$

Here $\operatorname{sgn} z \equiv z/|z|$, any $z \in \boldsymbol{C} \sim \{0\}$.

Clearly if $x \neq x_0$ and x is sufficiently close to x_0 then $|\psi(x)| > 1/2$ and we have

$$\begin{aligned}
\|\Psi(x) - \psi\| &\leqq \|\Psi(x) - \overline{\operatorname{sgn}\psi(x)}\cdot\psi\| + \|\overline{\operatorname{sgn}\psi(x)}\cdot\psi - \psi\| \\
&\leqq \|[2(1-|\psi(x)|)\varphi_x + \overline{\operatorname{sgn}\psi(x)}\cdot\psi]/[2-|\psi(x)|] - \overline{\operatorname{sgn}\psi(x)}\cdot\psi\| \\
&\quad + \|\psi(1-\overline{\operatorname{sgn}\psi(x)})\| \\
&\leqq \{[2(1-|\psi(x)|)\|\varphi_x - \overline{\operatorname{sgn}\psi(x)}\cdot\psi\| \\
&\quad + (1-|\psi(x)|)\|\overline{\operatorname{sgn}\psi(x)}\cdot\psi\|]\}/[2-|\psi(x)|] + |1-\overline{\operatorname{sgn}\psi(x)}| \\
&\leqq 5(1-|\psi(x)|) + |1-\overline{\operatorname{sgn}\psi(x)}| \\
&\longrightarrow 0 \quad \text{as} \quad x \longrightarrow x_0\,.
\end{aligned}$$

It remains to verify that $\Psi(x)$ peaks at x when $|\psi(x)| > 1/2$. For such x, we have $[\Psi(x)](x) = 1$. Further, if $t \neq x$ then by (*) we have

$$2(1-|\psi(x)|)|\varphi_x(t)| < 2 - |\psi(x)| - |\psi(t)|$$

or

$$|2(1-|\psi(x)|)\varphi_x(t)| + |\psi(t)| < 2 - |\psi(x)|$$

whence

$$|2(1-|\psi(x)|)\varphi_x(t) + \overline{\operatorname{sgn}\psi(x)}\psi(t)| < 2 - |\psi(x)|$$

or

$$|[\Psi(x)](t)| < 1\,.$$

LEMMA 4. *Fix a positive integer k. There is a sequence $\{\Phi_k^j\}_{j=1}^\infty$ of functions,*

$$\Phi_k^j\colon \mathscr{P}(A) \longrightarrow A$$

satisfying, for each $z \in \mathscr{P}(A)$ and every j,

(i) $\|\Phi_k^j(x)\| = 1$;

(ii) $[\Phi_k^j(x)](x) = 1$;

(iii) $\limsup_{\mathscr{P}(A) \ni y \to x} \|\Phi_k^j(x) - \Phi_k^j(y)\| \leqq 4^{-j}\cdot(1/k)$;

(iv) *for every* $t \in X \sim B(x, (1 - 2^{-j}) \cdot (1/k))$,

$$|[\Phi_k^j(x)](t)| \leqq (1 - 2/(k + 2)) + \sum_{i=1}^{j} 2^{-i} \cdot (1/(k + 2)) ;$$

(v) $\|\Phi_k^j(x) - \Phi_k^{j-1}(x)\| \leqq 2^{-j} \cdot (1/k)$, $j \geqq 2$.

Proof. This lemma is the heart of the matter. We construct the Φ_k^j inductively on j. First consider $j = 1$. For each $x \in \mathscr{P}(A)$ construct, by Lemma 1, a function $\varphi_x \in A$ which satisfies $\varphi_x(x) = 1$ and

$$|\varphi_x(t)| \leqq \min\{1 - 8kd(x, t)/(k + 2), 1 - 2/(k + 2)\} .$$

Using $\psi = \varphi_x$, construct a function

$$(*) \qquad \Psi_x^1 \colon \mathscr{P}(A) \cap \{|\psi(x)| > 1/2\} \longrightarrow A$$

satisfying the conclusions of Lemma 3. Choose r_x^1, $0 < r_x^1 < 1/4k$ so that $t \in B(x, r_x^1)$ implies that $|\varphi_x(t)| > 1/2$ and

$$\|\Psi_x^1(x) - \Psi_x^1(t)\| < 4^{-2} \cdot (1/(k + 2)) .$$

Now observe that if $y \in B(x, r_x^1)$ and $t \notin B(y, 1/2k)$ then

$$d(x, t) \geqq d(y, t) - d(y, x) \geqq 1/4k .$$

Therefore for such y, t we have

$$(**) \qquad \begin{aligned} |[\Psi_x^1(y)](t)| &\leqq |[\Psi_x^1(x)(t)]| + |[\Psi_x^1(x)](t) - [\Psi_x^1(y)](t)| \\ &\leqq |\varphi_x(t)| + 4^{-2} \cdot (1/(k + 2)) \\ &\leqq (1 - 2/(k + 2)) + 2^{-1} \cdot (1/(k + 2)) . \end{aligned}$$

Now since $\mathscr{P}(A)$ is a metric space, it is paracompact ([5], p. 160, Cor. 35). Hence there is a locally finite refinement $\mathscr{U}^1 = \{U_\omega^1\}_{\omega \in \Omega_1}$ of the covering $\{B(x, r_x^1)\}_{x \in \mathscr{P}(A)}$ of $\mathscr{P}(A)$. Let x_ω, $\omega \in \Omega_1$, be chosen so that $U_\omega^1 \subseteqq B(x_\omega, r_\omega^1)$. Let B_ω^1 denote $B(x_\omega, r_{x_\omega}^1)$. We may assume that $\bar{U}_\omega^1 \subseteqq B_\omega^1$. Let $\{\chi_\omega^1\}$ be a continuous partition of unity subordinary to $\mathscr{U}^1$ and define

$$\Phi_k^1 = \sum_{\omega \in \Omega} \chi_\omega^1 \Psi_{x_\omega}^1 .$$

Then conclusions (i) and (ii) are immediate. Conclusion (iv) follows from (**). Conclusion (v) is vacuous for $j = 1$. It remains to verify (iii).

Fix $x \in \mathscr{P}(A)$. Then there is a neighborhood W of x and $\{\omega_1, \cdots, \omega_m\} \subseteqq \Omega_1$ so that $W \cap \operatorname{supp} \chi_\omega \neq 0$ only if $\omega \in \{\omega_1, \cdots, \omega_m\}$. Of course m may depend on x. Letting x_i denote x_{ω_i}, $i = 1, \cdots, m$, we have that

$$\begin{aligned}\limsup_{\mathscr{P}(A)\ni y\to x} \|\Phi_k^1(x) - \Phi_k^1(y)\| &\leqq \sum_{i=1}^{m} \limsup_{\mathscr{P}(A)\cap W\ni y\to x} |\chi_{\omega_i}^1(x) - \chi_{\omega_i}^1(y)|\,\|\Psi_{x_i}(y)\| \\ &\quad + \sum_{i=1}^{m} \chi_{\omega_i}^1(x) \limsup_{\mathscr{P}(A)\cap W\ni y\to x} \|\Psi_{x_i}^1(x) - \Psi_{x_i}^1(y)\| \\ &\leqq 0 + \sum_{i=1}^{m} \chi_{\omega_i}^1(x) \limsup_{\mathscr{P}(A)\cap B_{\omega_i}\ni y\to x} \|\Psi_{x_i}^1(x) - \Psi_{x_i}^1(x_i)\| \\ &\quad + \sum_{i=1}^{m} \chi_{\omega_i}^1(x) \limsup_{\mathscr{P}(A)\cap B_{\omega_i}\ni y\to x} \|\Psi_{x_i}^1(x_i) - \Psi_{x_i}^1(y)\| \\ &\leqq 2\cdot 4^{-2}/(k+2) \leqq 4^{-1}\cdot(1/k)\,.\end{aligned}$$

Now suppose that $\Phi_k^1, \cdots, \Phi_k^j$ have been constructed so that (i)-(v) are satisfied. Let $x \in \mathscr{P}(A)$. Using $\psi = \Phi_k^j(x)$, we construct a function

$$\Psi_x^{j+1}\colon \mathscr{P}(A) \cap \{|\psi(x)| > 1/2\} \longrightarrow A$$

satisfying the conclusions of Lemma 3. Choose r_x^{j+1}, $0 < r_x^{j+1} < 2^{-j-1}\cdot(1/k)$ so that $t \in B(x, r_x^{j+1})$ implies that $|[\Phi_k^j(x)](t)| > 1/2$ and both

$$\|\Psi_x^{j+1}(x) - \Psi_x^{j+1}(t)\| \leqq 4^{-j-2}\cdot(1/(k+2))$$

(***) and

$$\|\Phi_k^j(x) - \Phi_k^j(t)\| \leqq (4/3)\cdot 4^{-j}\cdot(1/k)\,.$$

If now $y \in B(x, r_x^{j+1})$, $t \notin B(y, (1-2^{-j-1})\cdot(1/k))$ then

$$d(x, t) \geqq d(y, t) - d(y, x) \geqq (1-2^{-j})(1/k)\,.$$

Hence for such y, t we have

$$\begin{aligned}|[\Psi_x^{j+1}(y)](t)| &\leqq |[\Psi_x^{j+1}(x)](t)| + |[\Psi_x^{j+1}(x)](t) - [\Psi_x^{j+1}(y)](t)| \\ &\leqq |[\Phi_k^j(x)](t)| + 4^{-j-1}\cdot(1/(k+2)) \\ &\leqq (1 - 2/(k+2)) + \sum_{i=1}^{j} 2^{-i}\cdot(1/(k+2)) + 2^{-j-1}\cdot(1/(k+2)) \\ &= (1 - 2/(k+2)) + \sum_{i=1}^{j+1} 2^{-i}\cdot(1/(k+2))\,.\end{aligned}$$

Choose a locally finite refinement $\mathscr{U}^{j+1} = \{U_\omega^{j+1}\}_{\omega\in\Omega_{j+1}}$ of the covering $\{B(x, r_x^{j+1})\}_{x\in\ (A)}$ of $\mathscr{P}(A)$. Let $\{x_\omega\}_{\omega\in\Omega_{j+1}}$ be chosen so that $U_\omega^{j+1} \subseteqq B(x_\omega, r_{x_\omega}^{j+1}) \equiv B_\omega^{j+1}$, each $\omega \in \Omega_{j+1}$. We may assume that $\bar{U}_\omega^{j+1} \subseteqq B_\omega^{j+1}$. Let $\{\chi_\omega^{j+1}\}$ be a continuous partition of unity subordinate to $\mathscr{U}^{j+1}$. Define

$$\Phi_k^{j+1} = \sum_{\omega\in\Omega_{j+1}} \chi_\omega^{j+1}\Psi_{x_\omega}^{j+1}\,.$$

It follows as in the case $j = 1$ that (i), (ii), (iii), and (iv) hold. To verify (v) fix $x \in \mathscr{P}(A)$. Let $\omega_1, \cdots, \omega_m$ satisfy the property that

$\chi_\omega(x) \neq 0$ iff $\omega \in \{\omega_1, \cdots, \omega_m\}$. Let x_i denote x_{ω_i}, $i = 1, \cdots, m$. Then

$$\begin{aligned}||\Phi_k^{j+1}(x) - \Phi_k^j(x)|| &\leqq \left|\left|\sum_{l=1}^m \chi_{\omega_l}^{j+1}(x)[\Psi_{x_l}^{j+1}(x) - \Psi_{x_l}^{j+1}(x_l)]\right|\right| \\ &+ \left|\left|\sum_{l=1}^m \chi_{\omega_l}^{j+1}(x)[\Psi_{x_l}^{j+1}(x_l) - \Phi_k^j(x_l)]\right|\right| \\ &+ \left|\left|\sum_{l=1}^m \chi_{\omega_l}^{j+1}(x)[\Phi_k^j(x_l) - \Phi_k^j(x)]\right|\right| \\ &\leqq 4^{-j-2}(1/(k+2) + 0 + (4/3)4^{-j}(1/k)) \leqq 2^{-j}(1/k) .\end{aligned}$$

The induction is complete.

LEMMA 5. *For* $k \in \{1, 2, \cdots\}$ *there exist functions*

$$\Phi_k \colon \mathscr{P}(A) \longrightarrow A$$

such that

(i) $||\Phi_k(x)|| = 1$ *for all* $x \in \mathscr{P}(A)$,

(ii) $[\Phi_k(x)](x) = 1$;

(iii) Φ_k *is continuous*;

(iv) $|[\Phi_k(x)](t)| \leqq 1 - 1/(k+2)$ *for all* $x \in \mathscr{P}(A)$, $t \in X \sim B(x, 1/k)$.

Proof. Let Φ_k^j be as in Lemma 4 and define $\Phi_k = \lim_{j\to\infty} \Phi_k^j$. That the limit exists follows from (v) of Lemma 4. The conclusions (i)-(iv) of the present lemma now follow from the corresponding parts of Lemma 4.

By the discussion preceding Lemma 1, the proof of the theorem is complete.

REMARK. Our proof yields something more general. Indeed, instead of assuming X to be metric, one need only assume that the relative topology on $\mathscr{P}$ has a σ-locally finite base. By [5], p. 128, this is equivalent to assuming that $\mathscr{P}$ is metric, hence paracompact, and the proof goes through as before.

The referee has kindly observed that given our Lemma 3, one can use Theorem 3.1″ of [6] to prove that if X is compact Hausdorff and A is separable then the theorem holds. This is a weaker result than the one outlined in the preceding paragraph. Moreover, the proof using [6] is not essentially shorter than the elementary one presented here, and the construction of Φ as the unifiorm limit of discontinuous functions has intrinsic interest.

REMARK. It would be interesting to know whether, in the presence of differentiable structure in X and A, the peaking functions may be chosen to vary differentiably.

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LONG WALKS IN THE PLANE WITH FEW COLLINEAR POINTS

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Let S be a set of vectors in $\boldsymbol{R}^n$. An S-walk is any (finite or infinite) sequence $\{z_i\}$ of vectors in $\boldsymbol{R}^n$ such that $z_{i+1}-z_i \in S$ for all i. We will show that if the elements of S do not all lie on the same line through the origin, then for each integer $K \geqq 2$, there exists an S-walk $W_K=\{z_i\}_{i=1}^{N(K)}$ such that no $K+1$ elements of W_K are collinear and $N(K)$ grows faster than any polynomial function of K.

Specifically, we will prove that

$$\log_2 N(K) > \frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) .$$

We will then show that if the elements of S lie on at least L distinct lines through the origin, then there exists an S-walk of length $N(K, L)$ with no $K+1$ elements collinear, such that $N(K, L) \geqq (1/4)L^*N(K-1)$, where $L-2 \leqq L* \leqq L+1$ and $L* \equiv 0 \bmod 4$. In [3] it was shown that if $S \subset Z^2$, and for all $\boldsymbol{s} \in S$ we have $\|\boldsymbol{s}\| \leqq M$, then there does not exist an S-walk $W = \{z_i\}_{i=1}^{N(K,M)}$ such that no $K+1$ elements of W are collinear and

$$\log_2 N(K, M) > 2^{13}M^4K^4 + \log_2 K .$$

Before proving these theorems we introduce some notation. If $A = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ and $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m)$ are ordered sets of vectors, we let $RA = (\boldsymbol{a}_n, \cdots, \boldsymbol{a}_1)$ and we let $(A, B) = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_m)$. We let $2A = (A, A)$ and, for every positive integer k, we let $(k+1)A = (kA, A)$. If J is a vector operator, we let $JA = (J\boldsymbol{a}_1, \cdots, J\boldsymbol{a}_n)$.

THEOREM 1. *Let S contain two vectors independent over $\boldsymbol{R}$, and let K be an integer greater than or exual to 2. There exists an S-walk $W_K = \{z_p\}_{p=1}^{N(K)}$ such that no $K+1$ elements of W_K are collinear and such that*

$$\log_2 N(K) > \frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) .$$

Proof. If we let $(\log_2 K - 1)^2/9 - (\log_2 K - 1)/6 = \log_2 K$, then $\log_2 K = (25 + 3\sqrt{65})/4 > 12$ or $(25 - 3\sqrt{65})/4 < 1$. Therefore if $1 \leqq \log_2 K \leqq 12$, and $2 \leqq K \leqq 4096$, then

$$\frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) < \log_2 K .$$

Since W_K cannot have more than $N(K)$ collinear points, we need only consider $K > 4096$.

We may let $S = \{i, j\}$ without loss of generality, where i and j are orthonormal unit vectors.

For every positive integer m and nonnegative integer n, let $A_0^m = i$, and let

$$A_{n+1}^m = (mA_n^m, 2^n RJA_n^m) ,$$

where $Ji = j$ and $Jj = i$. Let $V = \{v_p\}_{p=1}^N = \mu A_\nu^\mu$, where μ is the greatest integer less than or equal to $((7/9)K)^{1/3}$, and ν is the least integer greater than or equal to $\log_2 \mu - 3/2$. Note that since $K >$ 4096, we have $\mu \geqq 14$, and $\nu \geqq 3$. Let $z_p = \sum_{q=1}^p v_q$ for each p, and let $W = \{z_p\}_{p=1}^N$. We maintain that W has no more than K collinear points and that $\log_2 N > (\log_2 K - 1)^2/9 - (\log_2 K - 1)/6$.

Let $b_0 = 1$ and let $b_{n+1} = (\mu + 2^n)b_n$. Then b_n is the cardinality of A_n^μ, and $N = \mu b_\nu$. Clearly $b_n \geqq \mu^n$, so $N \geqq \mu^{\nu+1}$ and $\log_2 N \geqq (\nu + 1)\log_2 \mu \geqq (\log_2 \mu - 1/2)\log_2 \mu$. Since μ is the greatest integer less than or equal to $((7/9)K)^{1/3}$, and $((7/9)K)^{1/3} > 14$, we have $\mu > (14/15)((7/9)K)^{1/3} > ((1/2)K)^{1/3}$. It follows that $\log_2 N > 1/9[\log_2((1/2)K)]^2 - \log_2((1/2)K)/6 = (\log_2 K - 1)^2/9 - (\log_2 K - 1)/6$.

We now prove that W has no more than K collinear points.

Let $C_n^\alpha = \{z_p : \alpha b_n \leqq p \leqq (\alpha + 1)b_n\}$. For each n, all C_n^α are congruent; specifically one can get from any one to any other by a translation plus, possibly, a reflection about the major diagonal (i.e., a reflection about the line passing through the vector $i + j$, which interchanges i and j), followed by a rotation about the origin of 180°. This reflection plus rotation is equivalent to a reflection about the line perpendicular to the major diagonal (i.e., the line passing through the vector $i - j$). We will refer to this latter line as the minor diagonal. Let

$$U_n^\beta = \{C_n^\alpha : \beta(\mu + 2^n) \leqq \alpha < (\beta + 1)(\mu + 2^n)$$
$$\text{if } n \neq \nu \text{ and } U_\nu^0 = \{C_\nu^\alpha : 0 \leqq \alpha \leqq \mu\} .$$

Note that $C_{n+1}^\beta = \{z_p : \beta(\mu + 2^n)b_n \leqq p \leqq (\beta + 1)(\mu + 2^n)b_n\}$, so U_n^β is a partition of C_{n+1}^β and U_ν^0 is a partition of W. We now consider a line with slope m and determine for each n, the maximum number of elements of U_n^β which the line can intersect (the maximum number cannot depend on β, since all C_{n+1}^β are congruent). Let r_n be this maximum number. Then the line cannot intersect more than $r = \prod_{n=0}^\nu r_n$ points of W.

Let s_n be the slope of z_{b_n}; i.e., $s_n = y_n/x_n$ where $z_{b_n} = x_n\boldsymbol{i} + y_n\boldsymbol{j}$. The slope of $z_{(\alpha+1)b_n} - z_{\alpha b_n}$ is then either s_n or s_n^{-1}, depending on whether C_n^α is a simple translation of C_n^0, or a translation of the reflection of C_n^0 about the minor diagonal. We wish to find a lower bound on s_n/s_{n-1}.

Now $x_0 = 1$, $y_0 = 0$, $x_{n+1} = \mu x_n + 2^n y_n$, and $y_{n+1} = \mu y_n + 2^n x_n$. It follows that x_n, y_n, and s_n are strictly positive for all $n \geqq 1$. We now prove by induction that $s_n < 2^n/\mu$. Clearly $s_0 = 0 < 2^0/\mu$ and $s_1 = 1/\mu < 2^1/\mu$. Suppose $s_n < 2^n/\mu$. Let $t_n = 2^n/s_n\mu$. Then $t_n > 1$. Now

$$\begin{aligned} s_{n+1} &= (\mu y_n + 2^n x_n)/(\mu x_n + 2^n y_n) \\ &= (\mu s_n + 2^n)/(\mu + 2^n s_n) \\ &= (\mu s_n + \mu s_n t_n)/(\mu + \mu s_n^2 t_n) \\ &= (s_n + s_n t_n)/(1 + s_n^2 t_n)\ . \end{aligned}$$

Thus

$$\begin{aligned} t_{n+1} &= 2^{n+1}/s_{n+1}\mu = 2s_n t_n/s_{n+1} \\ &= 2s_n t_n(1 + s_n^2 t_n)/(s_n + s_n t_n) \\ &= 2t_n(1 + s_n^2 t_n)/(t_n + 1)\ . \end{aligned}$$

We now view t_{n+1} as a function of the real variables t_n and s_n, and compute its partial derivatives:

$$\partial t_{n+1}/\partial t_n = 2(s_n^2 t_n^2 + 2s_n^2 t_n + 1)/(t_n + 1) > 0$$

and

$$\partial t_{n+1}/\partial s_n = 4t_n^2 s_n/(t_n + 1) > 0\ .$$

Since t_{n+1} has the value 1 when $s_n = 0$ and $t_n = 1$, it follows that $t_{n+1} > 1$ when $s_n \geqq 0$ and $t_n > 1$, as is the case here. Therefore $s_{n+1} < 2^{n+1}/\mu$.

Next, recall that $\nu - 1 < \log_2 \mu - 3/2$, so if $n \leqq \nu - 1$, then $2^n \leqq 2^{\nu-1} < 2^{-3/2}\mu$. Since $2^n > s_n\mu$, it follows firstly that $s_n < 2^{-3/2}$, and secondly that

$$\begin{aligned} s_{n+1}/s_n &= (\mu s_n + 2^n)/(\mu s_n + 2^n s_n^2) \\ &> 2\mu s_n/(\mu s_n + 2^{-3/2}\mu s_n^2) \\ &= 2/(1 + 2^{-3/2}s_n) > 2\Big/\left(1 + \frac{1}{8}\right) = \frac{16}{9}\ . \end{aligned}$$

It follows that, given m, there is at most one n such that $(3/4)s_n \leqq m \leqq (4/3)s_n$. Suppose there exists λ such that $(3/4)s_\lambda \leqq m \leqq (4/3)s_\lambda$. Then $m < (3/4)s_{\lambda+1}$ and $m > (4/3)s_{\lambda-1}$. Moreover, for all $n > \lambda + 1$, we have $m < (27/64)s_n < (1/2)s_n$, and for all $n < \lambda - 1$, we have $m > (64/27)s_n > 2s_n$. All of the above also holds if we replace s_n by s_n^{-1}, except that some of the inequalities are reversed and constants replaced by their reciprocals in the obvious way.

We now calculate for each of the five cases, $n = \lambda$, $n = \lambda + 1$, $n = \lambda - 1$, $n > \lambda + 1$, and $n < \lambda - 1$, the maximum number r_n of elements of U_n^β which a line of slope m can intersect. We can assume without loss of generality that C_{n+1}^β is a simple translation of C_{n+1}^0; if C_{n+1}^β is a translation of the reflection of C_{n+1}^0 about the minor diagonal, then we can apply the same argument, replacing s_n by s_n^{-1}. Then C_n^α is a simple translation of C_n^0 for $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$, and a translation of the reflection of C_n^0 for $\beta(\mu + 2^n) + \mu \leqq \alpha < (\beta + 1)(\mu + 2^n)$. For each α, the first point of $C_n^{\alpha+1}$ coincides with the last point of C_n^α. It is easy to prove by induction on n that C_n^0 (and therefore C_n^α for all α) lies entirely within a right triangle, with sides x_n and y_n adjacent to the right angle, and with the first and last points of C_n^0 at opposite ends of the hypotenuse. Therefore the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$ lie within congruent right triangles, whose hypotenuses are adjacent segments of a line with slope s_n (see Fig. 1). It follows

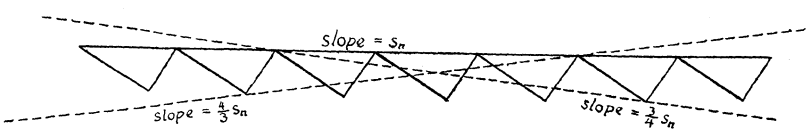

FIGURE 1

that a line with slope $m > s_n q/(q-1)$ or $m < s_n(q-1)/q$ can intersect at most q of the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$ at distinct points (i.e., assign the last point of each set C_n^α to the set $C_n^{\alpha+1}$, and do not count the line as intersecting C_n^α if it only intersects this last point). Suppose $m \leqq 1$. Then $m < (1/2)s_\lambda^{-1}$, and a line of slope m can intersect no more than two of the sets C_n^α: $\beta(\mu + 2^n) + \mu \leqq \alpha < (\beta + 1)(\mu + 2^n)$. If $n = \lambda$, then a line of slope m can intersect all μ of the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$ for a total of $\mu + 2$. If $n = \lambda + 1$ or $\lambda - 1$, the line can intersect at most 4 of the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$, for a total of 6, while if $n > \lambda + 1$ or $n < \lambda - 1$, the line can intersect at most two of the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$ for a total of 4. If $m > 1$, then we obtain essentially the same results by redefining λ so that $(3/4)s_\lambda^{-1} \leqq m \leqq (4/3)s_\lambda^{-1}$, the only difference being that μ is replaced by 2^n, which in any case is less than μ. Therefore we have $r_n \leqq \mu + 2$ if $n = \lambda$, $r_n \leqq 6$ if $n = \lambda - 1$ or $\lambda + 1$, and $r_n \leqq 4$ for all other n. Finally, we have

$$r = \prod_{n=0}^{\nu} r_n \leqq (\mu + 2)\cdot 6^2 \cdot 4^{\nu-2} < 36(\mu + 2)\cdot 4^{\log_2 \mu - 5/2}$$
$$= \frac{36}{32}\mu^2(\mu + 2) \leqq \frac{9}{7}\mu^3 \leqq K .$$

If λ does not exist, then there are at most two values of n for which $(27/64)s_n \leqq m \leqq (64/27)s_n$, and these two values can take the place of $\lambda - 1$ and $\lambda + 1$ in our argument.

REMARK. We can use this method to get slightly better results as follows: The method works by partitioning W into a heiarchy of sets, each set of order $n + 1$ being partitioned into $\mu + 2^n$ sets of order n, and showing that for almost all n, a given line can intersect at most four sets of order n within a given set of order $n + 1$. Suppose that instead of using the partition based on the sets C_n^α, we modify this partition slightly by splitting each C_n^α into two sets of order n, namely $\{z_p: \alpha b_n \leqq p \leqq \alpha b_n + \mu b_{n-1}\}$ and $\{z_p: \alpha b_n + \mu b_{n-1} \leqq p \leqq (\alpha + 1)b_n\}$. Then each set of order $n + 1$ would have either 2μ or 2^{n+1} sets of order n, and it should not be hard to show that for almost all n, a given line can intersect at most three sets of order n within a given set of order $n + 1$. We would then have $r = c\mu\cdot 3^\nu = c\mu^{1+\log_2 3}$, where c is a constant which does not depend on K, and finally

$$\log_2 N = (1 + \log_2 3)^{-2}(\log_2 K)^2 + O(\log_2 K) .$$

However, it seems impossible to push this method any further.

THEOREM 2. *Suppose that S contains L elements which are pairwise independent over $\boldsymbol{R}$. Then there exists an S-walk $\Omega = \{\boldsymbol{u}_i\}_{i=1}^N$ containing no set of $K + 1$ collinear points, such that*

$$\log_2 N > \frac{1}{9}[\log_2 (K - 1) - 1]^2 - \frac{1}{6}[\log_2 (K - 1) - 1] + \log_2 L^* - 2 ,$$

where $L - 2 \leqq L^* \leqq L + 1$ and $L^* \equiv 0 \bmod 4$.

Proof. The L elements of S with distinct arguments must include $L/2$ elements (if L is even) or $(L + 1)/2$ elements (if L is odd) in the same half-plane. Label these elements $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \cdots$ in order of their arguments. For $1 \leqq n \leqq (1/4)L^*$, let $W_n = \varphi_n W$ where W is defined as in the proof of Theorem 1, and φ_n is the linear vector operator which maps $\boldsymbol{i}$ to $\boldsymbol{s}_{2n-1}$ and $\boldsymbol{j}$ to $\boldsymbol{s}_{2n}$. Let N_0 be the cardinality of W and let $w_n = x\boldsymbol{s}_{2n-1} + y\boldsymbol{s}_{2n}$ be the final element of W_n. For $1 \leqq i \leqq N_0$, let z_i be defined as in the proof of Theorem 1, and let $\boldsymbol{u}_i = \varphi_1 z_1$. Let $\boldsymbol{u}_{N_0 n+i} = \sum_{j=1}^n \boldsymbol{w}_j + \varphi_{n+1} z_i$ for

$1 \leqq n \leqq (1/4)L^* - 1$. Finally, let $N = (1/4)L^* N_0$ and let $\Omega = \{\boldsymbol{u}_i\}_{i=1}^{N}$. Note that Ω is constructed by placing the W_n end to end in sequence.

By Theorem 1,

$$\log_2 N > \frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) + \log_2 L^* - 2 .$$

We will now prove that no $K + 2$ points of Ω are collinear. Substituting $K - 1$ for the bound variable K then gives us Theorem 2 for the case $K \geqq 3$. For the case $K = 2$, we simply let $\boldsymbol{u}_i = \sum_{j=1}^{i} \boldsymbol{s}_j$. The resulting set $\{\boldsymbol{u}_i\}$, which contains at least $(1/2)L^*$ elements, is the set of vertices of a convex polygon; hence no three elements are collinear.

Let $T_n = \{u_i\}_{i=N_0(n-1)+1}^{N_0 n}$ and let $\boldsymbol{t}_n = \sum_{j=1}^{n} \boldsymbol{w}_j$, so that $\boldsymbol{t}_n$ is the final element of T_n. Let $\boldsymbol{t}_0 = 0$ and let $\boldsymbol{r}_n = \boldsymbol{t}_{n-1} + x\boldsymbol{s}_{2n-1}$ for $n \geqq 1$. Note that $\boldsymbol{t}_n = \boldsymbol{r}_n + y\boldsymbol{s}_{2n}$. Note also that from results proved previously, the set T_n must lie entirely on or in the interior of the triangle Δ_n with vertices $\boldsymbol{t}_{n-1}$, $\boldsymbol{r}_n$, and $\boldsymbol{t}_n$. Consequently any line which intersects T_n must intersect Δ_n. Now consider the polygon P with vertices $\boldsymbol{t}_0, \boldsymbol{r}_1, \boldsymbol{t}_1, \boldsymbol{r}_2, \boldsymbol{t}_2, \cdots, \boldsymbol{r}_{L^*/4}, \boldsymbol{t}_{L^*/4}$ in that order. The (directed) edges of this polygon are the vectors $x\boldsymbol{s}_1, y\boldsymbol{s}_2, x\boldsymbol{s}_3, \cdots, y\boldsymbol{s}_{L^*/2}$, and $-x\sum_{n=1}^{L^*/4} \boldsymbol{s}_{2n-1} - y\sum_{n=1}^{L^*/4} \boldsymbol{s}_{2n}$. Since the vectors $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \cdots$ are listed in order of increasing argument, and the range of all their arguments is less than $180°$, it follows that the interior angles of P are all less than $180°$, so P is convex. Now any line intersecting Δ_n, and in particular any line intersecting T_n, must intersect at least two sides of Δ_n (including each vertex in its two adjacent sides), and therefore must intersect P. Since P is convex, a line can only intersect P at one or two points, or along an edge. Therefore no line can intersect more than two of the T_n. Unless the slope of a line is between that of $\boldsymbol{s}_{2n-1}$ and $\boldsymbol{s}_{2n}$ inclusive, it can only intersect one point of T_n. By Theorem 1, no line can intersect more than K points of T_n. Therefore, no line can contain more than $K + 1$ points of Ω.

REMARK. In order to compare these results with the upper bound in [3], we can consider the case where $S = \{\boldsymbol{s} \in Z^2 \colon \|\boldsymbol{s}\| \leqq M\}$. Since the number of lattice points in a disc of radius R is $\pi R^2 + O(R)$ [2], we know that the number of lattice points with both coordinates divisible by q, in a disc of radius M, is $\pi M^2/q^2 + O(M/q)$. Therefore the number L of lattice points with relatively prime coordinates is

$$\pi M^2 \sum_{n=0}^{\infty} (-1)^n \sum_{q \in Q_n} q^{-2} + O(M \sum_{q \in Q} q^{-1}) ,$$

where Q is the set of square free positive integers less than or equal to M, and Q_n is the set of integers in Q with n distinct prime factors. It follows [1] that

$$L = 6M^2/\pi + O(M \log M) .$$

Finally, if we let $N(K, M)$ be the length of the longest S-walk with no more than K collinear points, and we choose any constants $c_1 < (9 \log 2)^{-1}$ and $c_2 > 2^{13} \log 2$, then we have

$$M^2 \exp [c_1(\log K)^2] < N(K, M) < \exp [c_2 M^4 K^4]$$

for all M and all but a finite number of K.

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ON CERTAIN SEQUENCES OF LATTICE POINTS

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Let S be a finite subset of R^n. A sequence $\{z_i\}$ is an S-walk if and only if $z_{i+1}-z_i$ is an element of S for all i. In an effective manner it is shown that long S-walks in Z^2 must have an increasing number of collinear points. In Z^3, however, an infinite S-walk may have a bounded number of collinear points.

1. Introduction.

Let S be a finite subset of R^n.

DEFINITION. An *S-walk* is any (finite or infinite) sequence of vectors in R^n, say $\{z_i\}$, such that $z_{i+1} - z_i \in S$, for all i.

Given S, let M be the maximum of the Euclidean norms of the vectors in S. In [5] the following theorem is proved (see also [3] for the case $M = \sqrt{2}$):

THEOREM. *Let $S \subset Z^2$, and let K be any positive integer. There exists $N = N(K, M)$ such that any S-walk of length at least N must have K collinear points.*

With Theorem 1 of this paper we provide an effective bound on $N(K, M)$. With Theorem 2 we show that the situation of $S \subset Z^3$ is quite different, i.e., an infinite S-walk in Z^3 may have a bounded number of collinear points. In Theorem 3 we show that there are still some restrictions in Z^3, namely that if S has only three elements, then a sufficiently long S-walk must have three collinear points.

2. The Planar case.

THEOREM 1. *Let $S \subset Z^2$, let K be any positive integer, and let N be a positive integer such that*

$$\log_2 N \geqq 2^{13} M^4 (K-1)^4 + \log_2 (K-1) .$$

Then, for every S-walk $\{z_i\}_{i=0}^{N}$, there is some line L, and K choices for i, such that $z_i \in L$.

Proof. We suppose that the theorem is false for some K and derive a contradiction. Let $Q = 8 \cdot 2^{1/2} \cdot M(K-1)$. Let T denote the set of (positive and negative) Farey fractions of order no greater than Q. Let A be the set of all lines through the origin with slopes in T. Let B be the mirror image of A reflected through the line $y = x$. Enumerate the lines in the two sets A and B in order of increasing slope: $L_1, L_2, L_3, \cdots$. Let $\{z_i\}$ be a counterexample to the theorem for K. We may assume that z_0 is the origin.

Let z_J be an arbitrary point of the counterexample sequence. There are lines in the set $A \cup B$, L_n and L_{n+1}, such that z_J is on or between these lines; that is, the slope of the line through the origin and z_J is between or equal to the slopes of L_n and L_{n+1}, respectively a and b.

Dirichlet's theorem [2, page 1] gives us for $x = (a + b)/2$, integers p and q, $0 < q < Q$, such that

$$|qx - p| \leqq Q^{-1} .$$

We have either $p/q \geqq b \geqq a$, or $b \geqq a \geqq p/q$. Note that $ab \geqq 0$. We may therefore choose p/q to be the same sign as a and b. Let H_0 be the line through the origin with slope p/q and let U be the larger of the two angles between H_0 and L_n and between H_0 and L_{n+1}. Clearly, since a, b, and p/q have the same sign (viewing zero as positive and negative), the tangent of U is at most $2Q^{-1}q^{-1}$.

Enumerate the lines parallel to H_0 through points of Z^2 as $\cdots H_{-2}, H_{-1}, H_0, H_1, H_2, \cdots$ so that the distance from H_0 to H_i is $|id|$, where d is the minimum distance between such translates of H_0.

We now return to z_J. Among $z_J, z_{J+1}, \cdots, z_{J+(2P-1)(K-1)}$ at least one point is on some H_i with $|i| > P - 1$. Otherwise one of the H_i, with $|i| \leqq P - 1$, would contain K points of our S-walk, contrary to hypothesis. Let z_f be on a line H_i, with $|i| > P - 1$, and $J \leqq f \leqq J + (2P - 1)(K - 1)$. This point z_f is at least distance Pd from H_0. The component of z_f parallel to H_0 is at most fM. Thus, if V is the angle between z_f and H_0, we have

$$|\tan V| \geqq Pd/fM .$$

By taking P so that $(2P - 1)(K - 1) \geqq J$, we can write that

$$\begin{aligned} |\tan V| &\geqq Pd/M[J + (2P - 1)(K - 1)] \\ &\geqq Pd/2M(2P - 1)(K - 1) \\ &> d/4M(K - 1) . \end{aligned}$$

We now estimate d. We may assume that both L_n and L_{n+1} are in A, since otherwise they are both in B and the mirror image of the forthcoming analysis applies. With this assumption both a and b are in T. We may also assume that p/q is in T, for if not either $p/q \geqq 1$ or $p/q \leqq -1$. In the first case $1/1$ will play the role of p/q and in the second, $-1/1$. Thus

$$d \geqq (p^2 + q^2)^{-1/2} \geqq (2^{1/2}q)^{-1} .$$

Thus

$$|\tan V| > [4 \cdot 2^{1/2}Mq(K-1)]^{-1} .$$

It is now clear that the choice of Q as $8 \cdot 2^{1/2}M(K-1)$ gives us

$$|\tan V| > |\tan U| .$$

It is clear that the broken line path from z_J to z_f has crossed either L_n or L_{n+1}. In summary, given z_J on or between L_n and L_{n+1}, there is some integer t such that $0 < t \leqq (2P-1)(K-1)$ and

(i) P is the first integer such that $(2P-1)(K-1) \geqq J$ and

(ii) z_{J+t} is within M of either L_n or L_{n+1}.

By induction we choose a subsequence $\{z_{t_i}\}$ of $\{z_i\}$ such that

(i) each z_{t_i} is within M of some line in $A \cup B$ and

(ii) $t_i < t_{i+1} \leqq t_i + (2P-1)(K-1)$, where P is the first integer such that $(2P-1)(K-1) \geqq t_i$.

Note that we may choose $t_0 = 0$ and $t_1 = 1$. In general, if $t_i \leqq j_i(K-1)$, then the P for t_{i+1} satisfies

$$2P - 1 \leqq j_i + 1 .$$

Thus, $t_{i+1} \leqq (2j_i + 1)(K-1)$. Thus, if $j_i \leqq 2^i - 1$, we have $j_{i+1} \leqq 2^{i+1} - 1$.

We now count the number of lines in $A \cup B$. It is less than $2Q^2$. For any given line in $A \cup B$, the number of translates of it through points of Z^2 which are within distance M of it is at most $2M/d$, where d is the minimum distance between such translates. If their common slope is p/q in T, we have

$$d \geqq (p^2 + q^2)^{-1/2} \geqq (2^{1/2}Q)^{-1} .$$

If their common slope with respect to the y-axis is in T, the mirror image analysis applies. Thus, in all cases, $2M/d \leqq 2 \cdot 2^{1/2}MQ$. Finally, $(2Q^2)(2 \cdot 2^{1/2}MQ) = 4 \cdot 2^{1/2}MQ^3$ is an upper bound on the number of lines which the subsequence $\{z_{t_i}\}$ can occupy. If the index i on t_i is at least $(K-1)(4 \cdot 2^{1/2}MQ^3)$, one of these lines will have K points of $\{z_{t_i}\}$. All that is required is that $t_i \leqq N$. Since $t_i \leqq (K-1)(2^i - 1)$, it suffices to have

$$\log_2 (K-1) + 4 \cdot 2^{1/2}MQ^3(K-1) \leqq \log_2 N .$$

Since $Q = 8 \cdot 2^{1/2}M(K-1)$, we have $4 \cdot 2^{1/2}MQ^3(K-1) = 2^{13}M^4(K-1)^4$. By our choice of N this is satisfied. This contradiction establishes the theorem.

REMARK 1. Theorem 1 remains true in n-dimensional space with the same relations between N, M and K if we use $n-1$ dimensional hyperplanes for L instead of lines. The proof consists of projecting the S-walk onto Z^2, finding a line there and taking its pre-image under the projection.

REMARK 2. Professor Carl Pomerance of the University of Georgia [4] has extended this theorem by considering walks whose average step size is bounded. His theorem is stated below. Let $d(V) = \sum_{i=0}^{m-1} ||z_{i+1} - z_i||$ for a finite sequece $V = \{z_i\}_{i=0}^{m} \subset Z^2$.

THEOREM. *For every positive integer K and every positive real number M, there exists $m_0 = m_0(M, K)$ such that if $m > m_0$ and $d(V)/m \leqq M$, then there are K points of V which are collinear.*

An effective bound on m_0 is not known for Pomerance's theorem.

III. Three dimensional case.

THEOREM 2. *If S is a set of vectors which do not all lie in the same plane, then there exists an infinite S-walk in which no $5^{11}+1$ vectors are collinear.*

NOTATION. If $A = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ and $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m)$ are ordered sets of vectors, and β is a vector operator, we let $RA = (\boldsymbol{a}_n, \cdots, \boldsymbol{a}_1)$, $(A, B) = (a_1, \cdots, a_n, b_1, \cdots, b_m)$, and $\beta A = (\beta a_1, \cdots, \beta a_n)$. Let $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ be the three orthonormal unit vectors. For a vector $z = z_1\boldsymbol{i} + z_2\boldsymbol{j} + z_3\boldsymbol{k}$, let $||z||^{\shortparallel} = z_1 + z_2 + z_3$ and $||z||^{\perp} = (z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1)^{1/2}$. Note that $||z||^{\shortparallel}$ and $||z||^{\perp}$ are proportional to the components of z parallel and perpendicular respectively to the vector $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$. Let γ be the length of the component of $\boldsymbol{i}$, $\boldsymbol{J}$, or $\boldsymbol{k}$ perpendicular to $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$. Then $\gamma = (2/3)^{1/2}$ and in general the perpendicular component of z has length $\gamma||z||^{\perp}$.

Proof. It suffices to prove Theorem 2 for the case where $S = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. Let α and β be vector operators such that $\alpha\boldsymbol{i} = \boldsymbol{j}$, $\alpha\boldsymbol{j} = \boldsymbol{i}$, $\alpha\boldsymbol{k} = \boldsymbol{k}$, $\beta\boldsymbol{i} = \boldsymbol{i}$, $\beta\boldsymbol{j} = \boldsymbol{k}$, and $\beta\boldsymbol{k} = \boldsymbol{j}$. We define inductively ordered sets of vectors A_n. Let $A_0 = (\boldsymbol{i})$, and let $A_{n+1} = (A_n, \alpha A_n, R\beta A_n, A_n, R\beta\alpha A_n, R\beta A_n, A_n)$. Note that A_n has 7^n elements and that the sequence A_{n+1} begins with A_n. It follows that there exists a unique infinite sequence of vectors $\{\boldsymbol{v}_p\}$ such that $(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{7^n}) = A_n$ for all n. Let $z_p = \sum_{q=1}^{p} \boldsymbol{v}_q$ for all positive integers p. Then $W = \{z_p\}$ is an S-walk. We claim that no $5^{11}+1$ elements of W are collinear.

For convenience of notation we let z_0 be the zero vector. Let $C_n^0 = \{z_0, z_1, \cdots, z_{7^n}\}$. We prove by induction that the projection of

C_n^0 onto the plane perpendicular to $i + j + k$ lies within a trapezoid with base $4^n\gamma$, base angles $60°$, and adjacent sides $4^n\gamma/3$, with z_0 and z_{7^n} lying at extreme ends of the base. We will refer to such a trapezoid as a trapezoid of order n. The case $n = 0$ is trivial. Assume it is true for n. Note that A_n, αA_n, $R\beta A_n$, and $R\beta\alpha A_n$ are all mirror images of each other, either in space or in time (i.e., one can get from one to the others by permuting the unit vectors, by reversing the order of the sequence, or both). It follows that the set $C_n^\nu = \{z_{7^n\nu}, \cdots, z_{7^n(\nu+1)}\}$ is congruent to C_n^0, or its mirror image, for $0 \leqq \nu \leqq 6$. Therefore the projection of C_n^ν lies within a trapezoid of order n, with $z_{7^n\nu}$ and $z_{7^n(\nu+1)}$ lying at extreme ends of the base. From the definition of A_{n+1}, it follows that the seven trapezoids of order n fit together within a trapezoid of order $n + 1$, as illustrated in Figure 1.

It is straightforward to prove, by induction on n, that for any positive integer ν, the projections of C_{n+1}^ν and $C_{n+1}^{\nu+1}$ can fit together in one of only three possible configurations (ignoring rotations, reflections, and reversals of the sequence), namely those illustrated in Figure 2.

It follows that the distance between two points lying in non-adjacent trapezoids of order n must be at least $3^{-1/2}\cdot 4^n\gamma$, and that the distance between two points lying in adjacent trapezoids, or

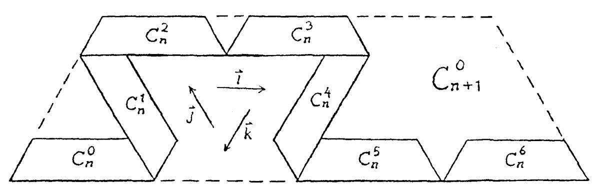

FIGURE 1

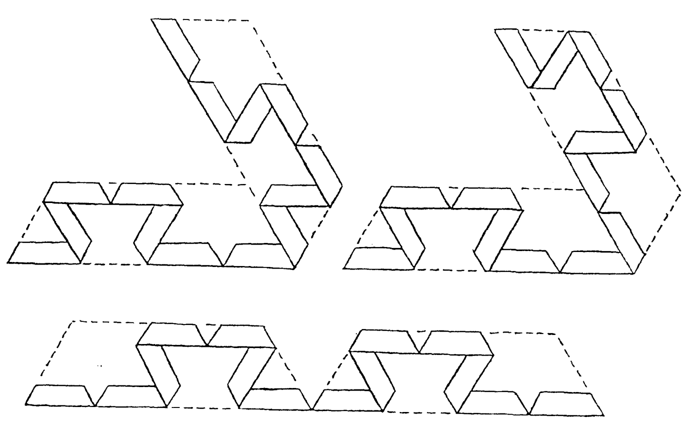

FIGURE 2

the same trapezoid, of order n can be at most $2\cdot 4^n\gamma$.

Now let p and q be positive integers such that $7^n \leqq |p-q| < 7^{n+1}$. Then, if $n \geqq 1$, z_p and z_q cannot lie in adjacent trapezoids of order $n-1$, so $\|z_p - z_q\|^{\perp} \geqq 3^{-1/2}\cdot 4^{n-1}$; if $n = 0$, this inequality is trivially satisfied. Likewise, z_p and z_q must lie in adjacent trapezoids, or the same trapezoid, of order $n+1$, so $\|z_p - z_q\|^{\perp} \leqq 2\cdot 4^{n+1}$. Since $\|z_p - z_q\|^{\shortparallel} = |p-q|$, we have

$$3^{-1/2}\cdot 4^{n-1}\cdot 7^{-(n+1)} < \|z_p - z_q\|^{\perp}/\|z_p - z_q\|^{\shortparallel} \leqq 2\cdot 4^{n+1}\cdot 7^{-n} .$$

Now let r and s be positive integers such that $7^m \leqq |r-s| < 7^{m+1}$, with $m \geqq n$, so that

$$3^{-1/2}\cdot 4^{m-1}\cdot 7^{-(m+1)} < \|z_r - z_s\|^{\perp}/\|z_r - z_s\|^{\shortparallel} \leqq 2\cdot 4^{m+1}\cdot 7^{-m} .$$

If z_p, z_q, z_r, and z_s are collinear, then

$$\|z_p - z_q\|^{\perp}/\|z_p - z_q\|^{\shortparallel} = \|z_r - z_s\|^{\perp}/\|z_r - z_s\|^{\shortparallel}$$

so $3^{-1/2}\cdot 4^{n-1}\cdot 7^{-(n+1)} < 2\cdot 4^{m+1}\cdot 7^{-m}$. It follows that $(7/4)^{m-n} < 224\sqrt{3}$, and $m - n < (\log 224\sqrt{3})/(\log 7/4) < 11$, i.e., $m - n \leqq 10$. Therefore $|r-s|/|p-q| < 7^{11}$, and there are at most 7^{11} collinear points in W.

Furthermore, if X is a set of collinear points in W which all lie within the same trapezoid of order n, but not within the same trapezoid of order $n-1$, then no two points of X can lie within the same trapezoid of order $n-11$. However, no line can intersect more than five trapezoids of order $n-1$ within a trapezoid of order n. For suppose a line intersected six of the trapezoids $C_n^0, C_n^1, \cdots, C_n^6$ in Figure 1. If C_n^0 where excluded, then the line would have to intersect C_n^3 and C_n^5, in which case C_n^1 would be missed. If C_n^2 were excluded, then the line would intersect C_n^0 and C_n^6, missing C_n^3. But a line intersecting C_n^0 and C_n^2 would miss C_n^6. Therefore, there are at most 5^{11} collinear points in W, and the theorem is proved.

It is obvious that this result can be sharpened considerably without changing the method of proof. For example it is not hard to convince oneself, by studying Figure 2, that in fact $4^{n-1} \leqq \|z_p - z_q\|^{\perp} \leqq 4^{n+1}$ if $7^n \leqq |p-q| < 7^{n+1}$. Also, there is no need to lump together all values of $|p-q|$ between 7^n and 7^{n+1}. By using a finer partition it ought to be possible to show that for a given value of $|p-q|$, the possible values of $\|z_p - z_q\|^{\perp}/\|z_p - z_q\|^{\shortparallel}$ range over a factor no greater than 4. Since $4 < (7/4)^3$, this would imply that W can have no more than 7^3 collinear points, all lying in the same trapezoid of order n, and no two lying in the same trapezoid of order $n-4$. Finally, one could examine the 7^4 trapezoids of order $n-4$ within a trapezoid of order n, preferably with the aid of a computer, and find an upper bound on the number which can be collinear, not only in the plane, but in 3-space. To clinch the argument, it might be necessary to descend to order $n-5$.

One would hope that by this method a sufficiently clever and persistent mathematician could determine the true maximum number of collinear points in W, which undoubtedly is three. However, there is no hope of sharpening Theorem 2 further than this, for we have the following theorem:

THEOREM 3. *If S has exactly three elements, then every S-walk of length nine has three collinear vectors; in fact three equally spaced collinear vectors.*

Proof. This result follows from the theorem of T. C. Brown [1] that any sequence of length nine on three symbols contains two adjacent segments which are permutations of each other. Brown's theorem can be verified in about one hour by direct computation.

An S-walk of length eight with no three collinear points is obtained by summing the sequence $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{i}, \boldsymbol{k}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{i}$.

REMARK 3. Theorem 2 also holds in the case where $S \subset \boldsymbol{R}^2$, provided that there are three elements $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_3$ of S, such that $\boldsymbol{e}_1 \times \boldsymbol{e}_2, \boldsymbol{e}_2 \times \boldsymbol{e}_3$, and $\boldsymbol{e}_3 \times \boldsymbol{e}_1$ are linearly independent over the rationals. In other words, the condition that the elements of S be lattice points is necessary for Theorem 1.

The above theorems leave unanswered the question of whether it is possible to have an infinite S-walk with no three collinear points for some $S \subset Z^n$ (in particular, can $n = 3$?).

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ON THE NONOSCILLATION OF PERTURBED FUNCTIONAL DIFFERENTIAL EQUATIONS

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We study the behavior of the solutions of the second order nonlinear functional differential equation

$$(1) \qquad (a(t)x')' = f(t, x(t), x(g(t)))$$

where a, g: $[t_0, \infty) \to R$ **and** f: $[t_0, \infty) \times R^2 \to R$ **are continuous,** $a(t) > 0$, **and** $g(t) \to \infty$ **as** $t \to \infty$. **We are primarily interested in obtaining conditions which ensure that certain types of solutions of (1) are nonoscillatory. Conditions which guarantee that oscillatory solutions of (1) converge to zero as** $t \to \infty$ **are also given. We apply these results to the equation**

$$(2) \qquad (a(t)x')' + q(t)r(x(g(t))) = e(t, x)$$

where q: $[t_0, \infty) \to R$, r: $R \to R$, e: $[t_0, \infty) \times R \to R$ **are continuous and** a **and** g **are as above. We compare our results to those obtained by others. Specific examples are included.**

In the case of nonlinear ordinary equations, the search for sufficient conditions for all solutions to be nonoscillatory has been successful; see, for example, the papers of Graef and Spikes [4-7], Singh [11], Staikos and Philos [14], and the references contained therein. The only such results known for functional equations to date are due to Graef [3], Kusano and Onose [9], and Singh [13]. Moreover, none of the results in [3], [9], or [13] apply to equation (2) if $e(t, x) \equiv 0$ or if r is superlinear, e.g., $r(x) = x^\gamma$, $\gamma > 1$. We refer the reader to the recent paper of Kartsatos [8] for a survey of known results on the oscillatory and asymptotic behavior of solutions of (1) and (2).

In view of a recent paper by Brands [1], it does not appear to be possible to obtain integral conditions on $q(t)$ which will guarantee that all solutions of (2) with $e(t, x) \equiv 0$ are nonoscillatory and which are similar to those usually encountered in the study of ordinary equations. (We will return to this point again in §2.) So too our main results in this direction when applied to equation (2) require that $e(t, x) \not\equiv 0$ (cf. conditions (27) and (28)). Although all the results presented here hold if $r(x)$ is sublinear, we are especially interested in the superlinear case.

2. Main results. The results in this paper pertain only to the continuable solutions of (1). A solution $x(t)$ of (1) will be called oscillatory if its set of zeros is unbounded, and it will be called nonoscillatory otherwise. Some of the results which follow concern solutions of (1) which satisfy growth estimates of the form

$$|x(t)| = O(m(t)) \quad \text{as} \quad t \longrightarrow \infty\,, \tag{3}$$

where $m: [t_0, \infty) \to R$ is continuous and positive. Other authors, for example Staikos and Sficas [15], have studied the asymptotic behavior of nonoscillatory solutions which satisfy estimates of this type with $m(t) = t^k$.

We will assume in the remainder of this paper that the function f satisfies an estimate of the form

$$|f(t, x, y)| \leqq F(t, |x|, |y|) \tag{4}$$

where $F: [t_0, \infty) \times R_+^2 \to R_+$ is continuous and such that

$$F(t, u, v) \leqq F(t, u', v') \quad \text{for} \quad 0 \leqq u \leqq u', 0 \leqq v \leqq v'\,.$$

THEOREM 1. *Suppose that*

$$\int^\infty [1/a(s)] \int_s^\infty F(u, cm(u), cm(g(u)))duds < \infty \tag{5}$$

for all $c > 0$. If $x(t)$ is an oscillatory solution of (1) *satisfying* (3), *then $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $x(t)$ be an oscillatory solution of (1) satisfying (3); then $|x(t)| \leqq cm(t)$, $|x(g(t))| \leqq cm(g(t))$ for all $t \geqq t_1 \geqq t_0$ and some $c > 0$. Suppose that $\limsup_{t\to\infty} |x(t)| > 2M$ for some $M > 0$. Then there exist sequences $\{a_n\}$ and $\{b_n\}$ of zeros of $x(t)$ such that $a_n < b_n$, $a_n, b_n \to \infty$ as $n \to \infty$, $|x(t)| > 0$ on (a_n, b_n), and $M_n = \max\{|x(t)|: a_n \leqq t \leqq b_n\} > M$ for $n = 1, 2, \cdots$. Now choose t_n in (a_n, b_n) so that $|x(t_n)| = M_n$ for $n = 1, 2, \cdots$. Integrating equation (1) from t in $[a_n, t_n]$ to t_n, we have

$$a(t)x'(t) = -\int_t^{t_n} f(s, x(s), x(g(s)))ds\,.$$

A second integration yields

$$x(t_n) = -\int_{a_n}^{t_n} [1/a(s)] \int_s^{t_n} f(u, x(u), x(g(u)))duds\,.$$

Thus

$$M_n = |x(t_n)| \leqq \int_{a_n}^{t_n} [1/a(s)] \int_s^{t_n} F(u, cm(u), cm(g(u)))duds\,.$$

Condition (5) implies that the ri ghthand side of the above inequality converges to zero as $n \to \infty$. This contradicts $|x(t_n)| = M_n > M$ for $n = 1, 2, \cdots$ and completes the proof of the theorem.

The following corollary is an immediate consequence.

COROLLARY 2. *If condition* (5) *holds with* $m(t) \equiv K$ *for every constant* $K > 0$, *then all bounded oscillatory solutions of* (1) *converge to zero as* $t \to \infty$.

In our next theorem the following sublinearity type condition will be used. There exists a continuous function $H: [t_0, \infty) \to R$ such that

$$\limsup_{v \to \infty} F(t, v, v)/v \leqq H(t) . \tag{6}$$

THEOREM 3. *In addition to* (6) *assume that condition* (5) *holds with* $m(t) \equiv K$ *for any constant* $K > 0$,

$$g(t) \leqq t \tag{7}$$

and

$$\int^{\infty} [1/a(s)] \int_s^{\infty} H(u) du ds < \infty . \tag{8}$$

Then every oscillatory solution of (1) *converges to zero as* $t \to \infty$.

Proof. We will first show that all oscillatory solutions are bounded. Suppose that $x(t)$ is an oscillatory solution of (1) and $\limsup_{t\to\infty} |x(t)| = \infty$. Then there exists a sequence of intervals $\{(a_n, b_n)\}$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$, $x(a_n) = x(b_n) = 0$, $|x(t)| > 0$ on (a_n, b_n), and $M_n = \max\{|x(t)|: t \leqq b_n\} = \max\{|x(t)|: a_n \leqq t \leqq b_n\}$ and M_n increases to infinity as $n \to \infty$ with $M_1 \geqq K$. As in the proof of Theorem 1 we obtain

$$M_n = |x(t_n)| \leqq \int_{a_n}^{t_n} [1/a(s)] \int_s^{t_n} F(u, M_n, M_n) du ds$$

where $t_n \in (a_n, b_n)$. Hence

$$1 \leqq \int_{a_n}^{t_n} [1/a(s)] \int_s^{t_n} H(u) du ds$$

which contradicts (8) as $n \to \infty$.

Since $x(t)$ is bounded the conclusion of the theorem then follows from Corollary 2.

THEOREM 4. *Suppose that there exist continuous functions* $G: [t_0, \infty) \times R_+^2 \to R_+$ *and* $h: [t_0, \infty) \to R$ *such that*

$$G(t, u, v) \leqq G(t, u', v') \quad for \ 0 \leqq u \leqq u', \quad 0 \leqq v \leqq v', \tag{9}$$

$$|f(t, x, y) - h(t)| \leqq G(t, |x|, |y|) \quad for \quad x, y \in R, \tag{10}$$

$$\int^{\infty}[1/a(s)]\int_s^{\infty}|h(u)|duds < \infty, \tag{11}$$

and

$$\int^{\infty}[1/a(s)]\int_s^{\infty}G(u, cm(u), cm(g(u)))duds < \infty \tag{12}$$

for all $c > 0$. *If there exists* $c_0 > 0$ *such that either*

$$\lim_{t\to\infty}\int_T^t[1/a(s)]\int_T^s\{h(u) + G(u, c_0, c_0)\}duds = -\infty \tag{13}$$

or

$$\lim_{t\to\infty}\int_T^t[1/a(s)]\int_T^s\{h(u) - G(u, c_0, c_0)\}duds = +\infty \tag{14}$$

for all large T, *then any solwtion* $x(t)$ *of* (1) *satisfying* (3) *is nonoscillatory.*

Proof. Let $x(t)$ be an oscillatory solution of (1) satisfying (3). In view of (11) and (12) all the hypotheses of Theorem 1 are satisfied with $F(t, u, v) = |h(t)| + G(t, u, v)$ and so $x(t) \to 0$ as $t \to \infty$. Thus there exists $T \geqq t_0$ such that $x'(T) = 0$, $|x(t)| \leqq c_0$, and $|x(g(t))| \leqq c_0$ for $t \geqq T$. Hence

$$h(t) - G(t, c_0, c_0) \leqq f(t, x(t), x(g(t))) \leqq h(t) + G(t, c_0, c_0) \tag{15}$$

for $t \geqq T$. Integrating twice we have

$$\begin{aligned}\int_T^t[1/a(s)]\int_T^s\{h(u) - G(u, c_0, c_0)\}duds &\leqq x(t) - x(T)\\ &\leqq \int_T^t[1/a(s)]\int_T^s\{h(u) + G(u, c_0, c_0)\}duds.\end{aligned}$$

If either (13) or (14) holds, then $x(t)$ cannot have arbitrarily large zeros.

REMARK. An alternate form of Theorem 4 can be obtained by replacing conditions (13) and (14) by

$$\limsup_{t\to\infty}\int_T^t\{h(u) + G(u, c_0, c_0)\}du < 0 \tag{16}$$

and

$$\liminf_{t\to\infty} \int_T^t \{h(u) - G(u, c_0, c_0)\}du > 0 \tag{17}$$

respectively. The proof in this case would follow from inequality (15) by noting that (16) or (17) implies that $x'(t)$ would have fixed sign. Condition (16) or (17) may be satisfied when (13) and (14) are not, for example, when $\int_{t_0}^{\infty}[1/a(s)]ds < \infty$. Similarly (13) or (14) may hold with neither (16) nor (17) being satisfied when $\int_{t_0}^{\infty}[1/a(s)]ds = \infty$.

THEOREM 5. *Assume that* (7) *and* (9)-(11) *hold, G is sublinear in the sense of condition* (6), i.e., *there exists* $H_G\colon [t_0, \infty) \to R$ *such that* $\limsup_{v\to\infty} G(t, v, v)/v \leqq H_G(t)$,

$$\int^{\infty}[1/a(s)]\int_s^{\infty} H_G(u)duds < \infty\ , \tag{18}$$

and condition (12) *holds with* $m(t) \equiv K$ *for any constant* $K > 0$. *If either* (13) *or* (14) *holds, then all solutions of* (1) *are nonoscillatory.*

Proof. Let $x(t)$ be an oscillatory solution of (1). If we let $F(t, u, v) = G(t, u, v) + |h(t)|$, then clearly (6) holds and moreover (11) and (18) imply that (8) holds with $H(t) = H_G(t) + |h(t)|$. Hence $x(t) \to 0$ as $t \to \infty$ by Theorem 3. Proceeding exactly as in the proof of Theorem 4 we again obtain a contradiction.

REMARK. Once again an alternate version of Theorem 5 can be obtained by replacing conditions (13)-(14) by (16)-(17).

3. Applications and discussion. We will now apply the results in the previous section to equation (4):

$$(a(t)x')' + q(t)r(x(g(t))) = e(t, x)\ .$$

Assume that

$$|e(t, u)| \leqq |e(t, v)| \quad \text{if} \quad |u| \leqq |v|\ , \tag{19}$$

and there are nonnegative constants A, B and p such that

$$|r(x)| \leqq A|x|^p + B\ . \tag{20}$$

If for some $k \geqq 0$

$$\int_{t_0}^{\infty}[1/a(s)]\int_s^{\infty}[g(u)]^{kp}|q(u)|duds < \infty \tag{21}$$

and

$$(22) \qquad \int_{t_0}^{\infty}[1/a(s)]\int_s^{\infty}|e(u, cu^k)|duds < \infty \quad \text{for all} \quad c \geqq 0\,,$$

then the hypotheses of Theorem 1 are satisfied with $m(t) = t^k$. Hence any oscillatory solution $x(t)$ of (2) satisfying

$$(23) \qquad |x(t)| = 0(t^k) \quad \text{as} \quad t \longrightarrow \infty\,,$$

will converge to zero as $t \to \infty$. If $k = 0$ in conditions (21) and (22) then we obtain the conclusion of Corollary 2 for equation (2). In this case we obtain Theorem 4 of Kusano and Onose [10] as a special case. They required that $r(x)$ be nondecreasing, $xr(x) > 0$ if $x \neq 0$, and $e(t, x) \equiv e(t)$; moreover if $k = 0$, conditions (13) and (14) of [10] imply conditions (21) and (22) above.

Now assume that there exist $w > 0$ and continuous functions h_1, h_2: $[t_0, \infty) \to R$ such that

$$(24) \qquad |e(t, x) - h_1(t)| \leqq h_2(t)|x|^w\,,$$

$$(25) \qquad \int_{t_0}^{\infty}[1/a(s)]\int_s^{\infty}|h_1(u)|duds < \infty\,,$$

and

$$(26) \qquad \int_{t_0}^{\infty}[1/a(s)]\int_s^{\infty}u^{kw}h_2(u)duds < \infty\,.$$

If (7), (19)-(21) and (24)-(26) hold with $p \leqq 1$, $w \leqq 1$, and $k = 0$, then all oscillatory solutions of (2) converge to zero by Theorem 3. Theorem 5 of [10] is a special case of this result. There the authors show that when $r(x)$ is sublinear, i.e., $\limsup_{|x|\to\infty} r(x)/x < \infty$, then the hypotheses of their Theorem 4 insure that all oscillatory solutions are bounded and hence converge to zero. In so doing they generalized Theorems 1, 2, and 3 of Singh [12] who, among other assumptions, required a bounded delay. Under a more restrictive condition on $r(x)$, namely, $0 < r(x)/x \leqq m$ for all x, Singh [13] gives sufficient conditions for all oscillatory solutions of a special case of (2) to bounded above. Under a different set of hypotheses, Kusano and Onose [9] obtained exactly the opposite result. The point to be made here is that while we are primarily interested in the case where $r(x)$ is superlinear, (cf. Theorems 1 and 4 and Corollary 2) i.e., $\limsup_{|x|\to\infty} r(x)/x = +\infty$, our condition (20) includes the sublinear forms of Kusano and Onose [9, 10] and Singh [12, 13] as special cases and, moreover, our integral conditions are similar in form and at times reduce exactly to those used in [9, 10, 12, and 13].

Relative to Theorem 4, if in addition to conditions (19)-(21) and (24)-(26), we ask that $r(0) = 0$ and there exists $N > 0$ such that either

$$\lim_{t\to\infty}\int_T^t [1/a(s)]\int_T^s \{h_1(u) + N[h_2(u) + |q(u)|]\}duds = -\infty \tag{27}$$

or,

$$\lim_{t\to\infty}\int_T^t [1/a(s)]\int_T^s \{h_1(u) - N[h_2(u) + |q(u)|]\}duds = +\infty \tag{28}$$

for all large T, then any solution $x(t)$ of (1) satisfying (23) is nonoscillatory. The alternate forms of (27) and (28) corresponding to (16) and (17) are respectively

$$\limsup_{t\to\infty}\int_T^t \{h_1(u) + N[h_2(u) + |q(u)|]\}du < 0 \tag{29}$$

and

$$\liminf_{t\to\infty}\int_T^t \{h_1(u) - N[h_2(u) + |q(u)|]\}du > 0\ . \tag{30}$$

We will now give some examples to illustrate our results.

EXAMPLE 1. The equation

$$x'' + x/t^2 = [\sin(\ln t)]/t^2,\ t \geqq 1$$

fails to satisfy condition (21) for $k = 0$ or condition (22). Here $x(t) = \cos(\ln t)$ is a bounded oscillatory solution which does not converge to zero.

EXAMPLE 2. The equation

$$x'' + x^3(t^{1/2})/t^3 = h_1(t),\ t \geqq 1$$

where $h_1(t) = [\sin(\ln t) - 3\cos(\ln t)]/t^3 + [\sin^3(\ln t^{1/2})]/t^{9/2}$ satisfies condition (20) with $p = 3$, condition (21) with $k = 0$, and (25). Here neither (27) nor (28) holds and we see that $x(t) = t^{-1}\sin(\ln t)$ is a bounded oscillatory solution.

EXAMPLE 3. Consider the equation

$$(t^\sigma x')' + t^{-\alpha}x^p(t^\beta) = h_1(t),\ t \geqq 1$$

where $h_1(t) = [4 + 2\cos(6\ln t) + 6\sin(6\ln t)]/t^3 + 1/t^\alpha$, $\alpha > 3$ and $\sigma > -1$. Conditions (20), (21) and (25) are satisfied provided that $\beta kp - \alpha < -1$ and $\beta kp - \alpha - \sigma < -2$. If $\sigma \leqq 1$, then (28) is satisfied while if $\sigma > 1$, then (30) is satisfied. Thus, in either case, if $x(t)$ is a solution such that

$$|x(t)| = O(t^k) \quad \text{as} \quad t \longrightarrow \infty$$

with $k < (\alpha + \sigma - 2)/\beta p$, then $x(t)$ is nonoscillatory. Notice that here the forcing term $h_1(t)$ changes signs.

The best nonoscillation theorem known to date for sublinear delay equations is the theorem of Kusano and Onose in [9]; it includes as a special case the nonoscillation criteria of Singh [13; Theorem 4.1]. There are several similarities between the conditions imposed in [9] and those used here. For example, when $k = 0$ conditions (6)-(7) of [9] imply condition (21) above. In addition, conditions (2)-(3) and (4)-(5) of [9] imply conditions (29)-(30) and (27)-(28) above respectively. On the other hand, even when $p \leqq 1$ our condition (20) on $r(x)$ is less restrictive than those used in [9] or [13]. Nor do we require $q(t) > 0$ as was needed in [9] and [13]. In both [9] and [13] the authors required that their forcing term $e(t, x) \equiv e(t)$ be either nonnegative or nonpositive; this was not done here. Other related results for sublinear equations have been obtained by Staikos and Philos [14] who studied nth order equations. They proved that for unforced advanced equations all bounded solutions are nonoscillatory and for forced delay equations all unbounded solutions are nonoscillatory. When $n = 2$, their integral conditions on $a(t)$, $q(t)$ and $e(t)$ are similar to those used in [9-13] and this paper.

Brands [1] constructed an example of an equations of the type (2) with $a(t) \equiv 1$, $g(t) = t - 1$, and $e(t, x) \equiv 0$ such that $q(t)$ satisfied

$$\int_{t_0}^{\infty} e^{\alpha t^2} q(t)dt < \infty, \ \alpha < 2 \tag{31}$$

and yet the equation possessed an oscillatory solution. This is somewhat of a surprise since many sufficient conditions for oscillation of ordinary equations have analogous counterparts (or may even be special cases of those) for functional equations (see Kartsatos [8]). Condition (31) is a far cry from the well known nonoscillation criteria of Hille

$$\int_{t_0}^{\infty} tq(t)dt < \infty$$

for linear ordinary equations.

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ANNIHILATION OF IDEALS IN COMMUTATIVE RINGS

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Four theorem are proved concerning the annihilation of finitely generated ideals contained in the set of zero divisors of a commutative ring.

1. Introduction. An important theorem in commutative ring theory is that if I is an ideal in a Noetherian ring and if I consists entirely of zero divisors, then the annihilator of I is nonzero. This result fails for some non-Noetherian rings, even if the ideal I is finitely generated. We say that a commutative ring R has *Property* (A) if every finitely generated ideal of R consisting entirely of zero divisors has nonzero annihilator. Property (A) was originally studied by Y. Quentel in [7]. (Our Property (A) is Quentel's Condition (C).) Theorem 1 shows that all nontrivial graded rings have Property (A). (For our purposes a *nontrivial graded ring* is a ring R graded over the integers such that R contains an element x, not a zero divisor, of positive homogenous degree.) Theorem 2 completely characterizes those reduced rings with Property (A).

Property (A) is closely connected with two other conditions on a reduced ring. One is the *annihilator condition* (a.c.): If (a, b) is an ideal of R, then there exists $c \in R$ such that $\mathrm{Ann}(a, b) = \mathrm{Ann}(c)$. The other condition is that $\mathrm{MIN}(R)$, the space of minimal prime ideals of R, is compact. Our Theorem 3 shows that for a reduced coherent ring R Property (A), (a.c.), and the total quotient ring of R being a von Neumann regular ring are equivalent conditions; and that each (and hence all) of these conditions imply that $\mathrm{MIN}(R)$ is compact. Finally, in Theorem 4, we prove that every reduced nontrivial graded ring satisfies (a.c.).

We assume that all rings are commutative with identity. If R is such a ring, let $T(R)$ be the total quotient ring of R, let $Z(R)$ be the set of zero divisors of R, and let $Q(R)$ denote the complete ring of quotients of R as defined in [5]. Elements of R that are not zero divisors are called *regular elements*.

2. Graded rings.. Y. Quentel, [7, p. 269], proved that if R is a reduced ring, then the polynomial ring $R[X]$ satisfies Property (A). We generalize this to arbitrary nontrivial graded rings, and hence to polynomial rings that are not necessarily reduced.

THEOREM 1. *If R is nontrivial graded ring, then R satisfies Property* (A).

Proof. Let $I = (a_1, \cdots, a_p)$ be an ideal of R contained in $Z(R)$. For $i = 1, \cdots, p$, let $a_i = \sum_{k=m_i}^{n_i} b_k^{(i)}$ be the homogeneous decomposition of a_i, where $\deg b_k^{(i)} = k$. Let x be a regular homogeneous element in R of degree $t > 0$. Construct an element a as follows:

$$a = a_1 + a_2x^{s_2} + \cdots + a_px^{s_p} ,$$

where the s_i are integers such that $ts_2 + m_2 > n_1$, and $ts_i + m_i > n_{i-1} + ts_{i-1}$; $i = 3, \cdots, p$. There exists a nonzero homogeneous element c such that $ca = 0$. (The proof of this is identical to the proof of McCoy's Theorem: If f is a zero divisor in $R[X]$, then there is a nonzero $b \in R$ such that $bf = 0$.)

Since $deg[b_k^{(i)}x^{s_i}] \neq \deg[b_h^{(j)}x^{s_j}]$ unless $i = j$ and $k = h$, the homogeneous compontets of a are $\{b_k^{(i)}x^{s_i}\}_{i=1,\cdots,p}^{k=m_i,\cdots,n_i}$. Thus, by the unique representation in terms of the homogeneous components $cb_k^{(i)}x^{s_i} = 0$ for all i, k. Since $x \notin Z(R)$, $cb_k^{(i)} = 0$ for all i, k. Therefore, $c \in \mathrm{Ann}(I)$.

COROLLARY 1. *If R is any ring, then the polynomial ring $R[X]$ satisfies Property* (A).

3. Reduced rings.

In this section all rings are assumed to be reduced.

THEOREM 2. *For a reduced ring R, the following statements are equivalent*:

(1) *R has Property* (A);

(2) *$T(R)$ has property* (A);

(3) *If I is a finitely generated ideal of R contained in $Z(R)$, then I is contained in a minimal prime ideal of R;*

(4) *Every finitely generated ideal of R contained in $Z(R)$, extends to a proper ideal in $Q(R)$.*

Proof. (1) $\leftrightarrow$ (2) is clear.

(1) $\rightarrow$ (3): Assume that I is a finitely generated ideal contained in $Z(R)$, but not contained in a minimal prime ideal of R. Then $cI = 0$ implies that c is in every minimal prime ideal of R; i.e., $c=0$.

(3) $\rightarrow$ (1): Let $I = (x_1, \cdots, x_n) \subset P$, P a minimal prime ideal of R. By [2, p. 111], choose $z_i \in \mathrm{Ann}(x_i)$, $z_i \notin P$. Then $z = z_1z_2 \cdots z_n \neq 0$ and $z \in \bigcap_{i=1}^n \mathrm{Ann}(x_i) = \mathrm{Ann}(I)$.

(1) $\rightarrow$ (4): If I is a finitely generated ideal contained in $Z(R)$, then $IQ(R)$ has nonzero annihilator in $Q(R)$. Hence, $IQ(R) \subsetneqq Q(R)$.
has nonzero annihilator in $Q(R)$. Hence, $IQ(R) \subsetneqq Q(R)$.

(4) $\rightarrow$ (1): Assume that I is a finitely generated dense ideal of R such that $I \subset Z(R)$. (A subgroup H of a ring R is *dense*, if

$\operatorname{Ann} H = 0$.) Then I is dense in $Q(R)$, [5, p. 41], and whence $IQ(R)$ is dense in $Q(R)$. But $Q(R)$ is a von Neumann regular ring, [5, p. 42]; and von Neumann regular rings have Property (A), [3, p. 30]. By the equivalence of (1) and (3) of this theorem, $IQ(R)$ is not contained in any minimal prime ideal of $Q(R)$. But in $Q(R)$, minimal prime ideals are maximal. Therefore, $IQ(R) = Q(R)$, a contradiction.

The results about the compactness of $\mathrm{MIN}(R)$ that we need are summarized in Theorems A and B.

THEOREM A. *The following conditions on a reduced ring R are equivalent*:

(1) *$Q(R)$ is a flat R-module*;

(2) *$\mathrm{MIN}(R)$ is compact*;

(3) $\{M \cap R \colon M \in \operatorname{Spec} Q(R)\} = \mathrm{MIN}(R)$;

(4) *If $a \in R$ and if $U = \{M \in \operatorname{Spec} Q(R) \colon a \notin M \cap R\}$, then there exists a finitely generated ideal I such that*

$$\operatorname{Spec} Q(R) \backslash U = \{M \in \operatorname{Spec} Q(R) \colon I \not\subset M \cap R\}\ ;$$

(5) *If X is an indeterminate, then $T(R[X])$ is a von Neumann regular ring.*

Proof. A. C. Mewburn, in [6], proved the equivalence of (1) through (4). Quentel proved that (2) and (5) are equivalent, [7].

THEOREM B. *The following conditions on a reduced ring R are equivalent*:

(1) *$T(R)$ is a von Neumann regular ring*;

(2) *R satisfies Property* (A) *and* $\mathrm{MIN}(R)$ *is compact*;

(3) *R satisfies* (a.c.) and $\mathrm{MIN}(R)$ *is compact.*

Proof. In [7], Quentel proved the equivalence of (1) and (2); while M. Henriksen and M. Jerison, [2], showed that (1) and (3) are the same.

A ring R is *coherent* in case I is a finitely generated ideal of R implies there is an exact sequence $R^m \to R^n \to I \to 0$.

THEOREM 3. *For a reduced coherent ring R, the following conditions are equivalent*:

(1) *R has Property* (A);

(2) *R has* (a.c.);

(3) *$T(R)$ is a von Neumann regular ring.*

Proof. (1) $\to$ (3): In view of Theorem B(2) we must show that

MIN(R) is compact. Let $x \in R$. Since R is a coherent ring, $\mathrm{Ann}(x) = I$ is a finitely generated ideal of R, [1, p. 462]. Let $U = \{M \in \mathrm{Spec}\, Q(R)$: $x \notin M \cap R\}$. Assume that $I \subset M \cap R$ for some $M \in \mathrm{Spec}\, Q(R) \backslash U$, then the ideal $(I, x) \subset M \cap R$. It is clear that $M \cap T(R)$ is a proper ideal of $T(R)$ and that $M \cap R = M \cap T(R) \cap R$. Hence, $(I, x) \subset M \cap R \subset Z(R)$. By Property (A), $\mathrm{Ann}(I, x) \neq 0$. But this contradicts the fact that the ideal $(I, x) = xR + \mathrm{Ann}(x)$ is dense, [5, p. 42]. By Theorem A(4), MIN(R) is compact.

(2) $\rightarrow$ (3): Let $x \in R$, then $\mathrm{Ann}(x) = (z_1, \cdots, z_n)$ and $\mathrm{Ann}\{\mathrm{Ann}(x)\} = \mathrm{Ann}(z_1, \cdots, z_n) = \mathrm{Ann}(z)$. This last condition, given in [2], implies that MIN(R) is compact (even if R does not have a unit).

(3) $\rightarrow$ (1) and (3) $\rightarrow$ (2) are clear.

COROLLARY 2. *Let R be a reduced coherent ring.*

(1) *If R satisfies any (and hence all) of the conditions of Theorem 3, the* MIN(R) *is compact.*

(2) *If R is a nontrivial graded ring, then $T(R)$ is a von Neumann regular ring.*

THEOREM 4. *If R is a reduced nontrivial graded ring, then R satisfies* (a.c.).

Proof. Let (a, b) be an ideal in R. If $(a, b) \not\subset Z(R)$, then $\mathrm{Ann}(a, b) = \mathrm{Ann}(1)$. Assume that $(a, b) \subset Z(R)$, and write a and b in terms of their homogeneous components; say, $a = a_m + \cdots + a_n$ and $b = b_h + \cdots + b_k$. Let x be a homogeneous element of R, $x \notin Z(R)$, of degree $t > 0$. Choose an integer s satisfying $h + st > n$ and let $c = a_m + \cdots + a_n + b_h x^s + \cdots + b_k x^s$.

Since R in a reduced, $\mathrm{Ann}(c) = \cap P$, where P varies over the minimal prime ideals of R not containing c. By Lemma 3 of [8, p. 153], each P is a homogeneous ideal. Hence, $\cap P = \mathrm{Ann}(c)$ is also homogeneous.

Let d be a homogeneous element in $\mathrm{Ann}(c)$. Then $da_i = 0$ and $db_j x^s = 0$ for all i, j. Then, $da = 0 = db$ and we have $\mathrm{Ann}(c) \subset \mathrm{Ann}(a, b)$. The other inclusion is obvious.

Let R be a graded ring which contains a regular homogeneous element. Define $T_q = \{a/b$: a and b are homogeneous, b is regular, and $q = \text{degree } a - \text{degree } b\}$. Just as in the integral domain case, [8, p. 157], ΣT_q is a graded ring containing R as a graded subring.

COROLLARY 3. *Let R be a reduced nontrivial graded ring. The following statements are equivalent:*

(1) MIN(R) *is compact;*

(2) $\mathrm{MIN}(T_0)$ *is compact;*
(3) $T(R)$ *is a von Neumann regular ring.*

Proof. $(1) \leftrightarrow (3)$ by Theorem B.

$(1) \leftrightarrow (2)$: If S is the set of regular homogeneous elements of R, then $R_S = \Sigma T_q$ and $\mathrm{MIN}(R)$ is homeomorphic to $\mathrm{MIN}(R_S)$. By [4, Lemma 1], there is a one-to-one order preserving correspondence between the graded prime ideals of R_S and the graded prime ideals of T_0. It follows from [8, p. 153] that the minimal prime ideals of a graded ring are graded. Thus, $\mathrm{MIN}(R_S)$ is homeomorphic to $\mathrm{MIN}(T_0)$.

REMARKS. (1) $\mathrm{MIN}(R)$ compact $\nrightarrow$ Property A or (a.c.). This follows from an example in [6]. (2) Property (A) $\nrightarrow \mathrm{MIN}(R)$ compact. By [6. p. 427], there is a ring R for which $\mathrm{MIN}(R)$ is not compact. Applying Theorem B(5), $T(R[X])$ is not von Neumann regular. But $R[X]$ has Property (A), [7, p. 269]. Thus, $\mathrm{MIN}(R[X])$ cannot be compact.

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NORM ATTAINING OPERATORS ON LEBESGUE SPACES

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Let X and Y be Lebesgue spaces (AL-spaces). Then the norm attaining operators mapping X to Y are dense in the space of all linear bounded operators from X to Y.

For any two real Banach spaces X and Y by $B(X, Y)$ we denote the Banach space of all bounded linear operators from X to Y. In [7] Uhl proved that for any strictly convex Banach space Y the norm attaining operators are (norm) dense in $B(L^1[0, 1], Y)$ if and only if Y has the Radon-Nikodym property. The question of whether the norm attaining operators are dense in $B(L^1[0, 1], L^1[0, 1])$ has remained unsolved (cf. [7], p. 299). Here we answer this question in the affirmative. In fact we prove a slightly more general result.

First we introduce some notations. Let I stand for the unit interval. For any function μ defined on the product algebra in $I \times I$ by $\mu^i (i = 1, 2)$ we denote the corresponding marginal functions defined on the Borel subsets of I:

$$\mu^1(A) = \mu(A \times I) ,$$
$$\mu^2(B) = \mu(I \times B) .$$

The vector lattice of all finite signed Borel measures on $I \times I$ will be denoted by M. Given any two finite positive Borel measures m_1, m_2 on I we write $M(m_1, m_2)$ for the set of all measures μ in M such that $|\mu|^i$ is absolutely continuous with respect to $m_i (i = 1, 2)$ and

$$\frac{d|\mu|^1}{dm_1} \in L^\infty(m_1) .$$

The measures m_1 and m_2 will be fixed throughout the rest of the paper.

Let us recall that $B(L^1(m_1), L^1(m_2))$ is a Banach lattice under its canonical order (see [5], IV Theorem 1.5 (ii)).

The forthcoming theorem establishes an isomorphism between $M(m_1, m_2)$ and $B(L^1(m_1), L^1(m_2))$, and extends a corresponding result of J. R. Brown on doubly stochastic operators ([1], p. 18). As was kindly indicated by the referee, our Theorem 1 is also related to N. J. Kalton's representation of the endomorphisms from L^p to L^p for $0 < p \leqq 1$ ([3], Theorem 3.1).

By $\langle\cdot,\cdot\rangle$ we denote the canonical bilinear form on $L^\infty(m_2)^* \times L^\infty(m_2)$.

THEOREM 1. *The space* $M(m_1, m_2)$ *is a vector lattice ideal in* M *and to each* $\mu \in M(m_1, m_2)$ *there corresponds a unique operator* $T_\mu \in B(L^1(m_1), L^1(m_2))$ *such that*

$$\langle T_\mu f, h\rangle = \int f(x)h(y)d\mu(x, y)$$

for all $f \in L^1(m_1)$ *and* $h \in L^\infty(m_2)$. *Moreover, the mapping* $\mu \to T_\mu$ *is a vector lattice isomorphism of* $M(m_1, m_2)$ *onto* $B(L^1(m_1), L^1(m_2))$ *and*

$$\| T_\mu \| = \left\| \frac{d\,|\mu|^1}{dm_1} \right\|_\infty$$

for every $\mu \in M(m_1, m_2)$.

Proof. First we note that $M(m_1, m_2)$ is a vector subspace of M. Since $\nu \in M(m_1, m_2)$ whenever $0 \leqq \nu \in M$ and $\nu \leqq \mu \in M(m_1, m_2)$, we observe that $M(m_1, m_2)$ is a lattice ideal (and clearly a sublattice) in M. If $\mu \in M(m_1, m_2)$ then it is easy to see that the bilinear form

$$[f, h] = \int f(x)h(y)d\mu(x, y)$$

is well-defined and continuous on $L^1(m_1) \times L^\infty(m_2)$. Therefore there exists a unique operator $T_\mu \in B(L^1(m_1), L^\infty(m_2)^*)$ such that

$$[f, h] = \langle T_1, f, h\rangle$$

(see e.g., [5], IV § 2). Clearly the mapping $\mu \to T_\mu$ is one-to-one and $\mu \geqq 0$ if and only if T_μ is a positive operator in the Banach lattice sense. Moreover, for an arbitrary $\nu \geqq 0$ in $M(m_1, m_2)$ and for any $h \in L^\infty(m_2)$ we have $\langle T_\nu 1, h\rangle = \int h d\nu^2$, so

$$T_\nu 1 = \frac{d\nu^2}{dm_2} \in L^1(m_2) \ ,$$

whence $T_\nu f \in L^1(m_2)$ for any $f \in L^\infty(m_1)$. Consequently, $T_\nu \in B(L^1(m_1), L^1(m_2))$ by continuity. Since every $\mu \in M(m_1, m_2)$ is a difference of two positive measures in $M(m_1, m_2)$ and $\mu \to T_\mu$ is a linear map, we have $T_\mu \in B(L^1(m_1), L^1(m_2))$ for all $\mu \in M(m_1, m_2)$.

We now show that $\mu \to T_\mu$ is an "onto" mapping. Since $B(L^1(m_1), L^1(m_2))$ is a Banach lattice, it suffices to prove that every positive operator $T \in B(L^1(m_1), L^1(m_2))$ is of the form T_μ. Given any such T we define a set function

$$\mu(A \times B) = \langle T\chi_A, \chi_B \rangle$$

on all Borel rectangles in $I \times I$. Evidently μ extends uniquely to a finitely additive positive measure (denoted also by μ) on the product algebra. The marginal measures $\mu^1(A) = \int_A T^*1 dm_1$ and $\mu^2(B) = \int_B T1 dm_2$ are finite, positive, and countably additive, so they are compact by the classical result of Ulam. Since μ is a nondirect product of μ^1 and μ^2, it is countably additive by Theorem 1 (i) in [4]. The unique extension of μ to a finite positive (countably additive) Borel measure on $I \times I$ is again denoted by μ. By a standard approximation argument,

$$\int f(x)h(y)d\mu(x, y) = \langle Tf, h \rangle$$

for all $f \in L^1(m_1)$ and $h \in L^\infty(m_2)$. Therefore $T = T_\mu$. Finally, we note that for every $\mu \in M(m_1, m_2)$

$$\begin{aligned} \| T_\mu \| &= \| T_{|\mu|} \| = \sup \| T_{|\mu|} f \|_1 = \sup \langle T_{|\mu|} f, 1 \rangle \\ &= \sup \int f(x) d|\mu|^1(x) = \sup \int f(x) \frac{d|\mu|^1}{dm_1}(x) dm_1(x) \\ &= \left\| \frac{d|\mu|^1}{dm_1} \right\|_\infty , \end{aligned}$$

where the suprema are taken over all nonnegative functions $f \in L^1(m_1)$ with $\|f\|_1 \leqq 1$.

COROLLARY 1. *Let $\nu \in M(m_1, m_2)$. If there exists a function $g \in L^\infty(m_2)$ with $|g| = 1$ such that the Radon-Nikodym derivative of the marginal measure $(g(y)d\nu(x, y))^1$ with respect to m_1 equals*

$$\left\| \frac{d|\nu|^1}{dm_1} \right\|_\infty$$

on a set B of positive m_1 measure, then the operator T_ν attains its norm on the unit ball in $L^1(m_1)$.

Proof. We put $d\lambda(x, y) = g(y)d\nu(x, y)$. Then

$$\begin{aligned} \langle T_\nu(\chi_B/m_1(B)), g \rangle &= \frac{1}{m_1(B)} \int \chi_B(x) d\lambda(x, y) \\ &= \frac{1}{m_1(B)} \int_B \frac{d\lambda^1}{dm_1} dm_1 = \left\| \frac{d|\nu|^1}{dm_1} \right\|_\infty , \end{aligned}$$

implying $\|T_\nu(\chi_B/m_1(B))\|_1 = \|T_\nu\|$ by Theorem 1.

The algebra of sets generated by all dyadic-rational rectangles in $I \times I$ will be denoted by $\mathscr{A}$. The σ-algebra generated by $\mathscr{A}$ coincides with the Borel algebra in $I \times I$.

THEOREM 2. *The norm attaining operators are dense in* $B(L^1(m_1), L^1(m_2))$.

Proof. Let $T \in B(L^1(m_1), L^1(m_2))$. By Theorem 1 we have $T = T_\mu$ for some measure μ in $M(m_1, m_2)$. Without any loss of generality we may assume

$$\left\|\frac{d|\mu|^1}{dm_1}\right\|_\infty = 1 .$$

Given $0 < \varepsilon < 1$, the set

$$D = \left\{x \in I : \frac{d|\mu|^1}{dm_1}(x) > 1 - \frac{\varepsilon}{4}\right\}$$

is of positive m_1 measure, say, $m_1(D) = \delta > 0$. Now let P, $(I \times I) - P$ be the Hahn decomposition for μ with μ^+ concentrated on P (see [2], § 29 Theorem A). Since P is a Borel set, there exists $\tilde{P} \in \mathscr{A}$ such that $|\mu|(P \Delta \tilde{P}) < \delta\varepsilon/4$ ([2], § 13 Theorem D). We define a new measure $\tilde{\mu}$ by

$$d\tilde{\mu} = \chi_{\tilde{P}} d\mu^+ - \chi_{(I\times I)-\tilde{P}} d\mu^- .$$

Evidently $\tilde{P}, (I \times I) - \tilde{P}$ is the Hahn decomposition for $\tilde{\mu}$ and $d|\mu - \tilde{\mu}| = \chi_{P\Delta\tilde{P}} d|\mu|$. Since $|\mu - \tilde{\mu}|(I \times I) < \delta\varepsilon/4$, the Radon-Nikodym derivative of $|\mu - \tilde{\mu}|^1$ with respect to m_1 must be less than $\varepsilon/4$ on some set $C \subset D$ of positive m_1 measure. As $\tilde{P} \in \mathscr{A}$, there exists a natural number n such that $\tilde{P}$ is a union of finitely many squares corresponding to the dyadic partition of I into 2^n subintervals of equal length. Let I_0 be any such open subinterval intersecting C on a set $B = C \cap I_0$ of positive m_1 measure. We let

$$d\nu(x, y) = \chi_B(x)\left(\frac{d|\mu|^1}{dm_1}\right)^{-1}(x) d\tilde{\mu}(x, y) + \chi_{I-B}(x) d\mu(x, y) .$$

Note first that

$$\begin{aligned} d|\nu - \mu| = \Big|\chi_B(x)\left(\frac{d|\mu|^1}{dm_1}\right)^{-1}(x)\, |d(\tilde{\mu} - \mu)(x, y) \\ + \left(1 - \frac{d|\mu|^1}{dm_1}(x)\right) d\mu(x, y)\Big| \leqq 2\chi_C(x) d|\tilde{\mu} - \mu|(x, y) + \frac{\varepsilon}{2} d|\mu|(x, y) . \end{aligned}$$

Therefore

$$\frac{d|\nu - \mu|^1}{dm_1} < 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon ,$$

whence $\| T_\nu - T_\mu \| = \| T_{\nu-\mu} \| \leqq \varepsilon$. Moreover,

$$\frac{d|\mu|^1}{dm_1} = 1 \text{ on } B \text{ and } \leqq 1 \text{ elsewhere.}$$

The set $(I_0 \times I) \cap \widetilde{P}$ is a finite union of squares of the form $I_0 \times I_k (k = 1, \cdots, m)$, where each I_k is an element of the dyadic partition of I into 2^n subintervals of equal length. Therefore $(B \times I) \cap \widetilde{P}$ is the finite union of the Borel rectangles $B \times I_k$. We define a function $g \in L^\infty(m_2)$ as follows

$$g(y) = \begin{cases} 1 \text{ if } y \in \cup I_k , \\ -1 \text{ otherwise.} \end{cases}$$

Clearly the Radon-Nikodym derivative of the marginal measure $(g(y)d\nu(x, y))^1$ coincides with

$$\frac{d|\nu|^1}{dm_1} = 1$$

on B. Therefore, by Corollary 1, T_ν attains its norm and the proof is completed.

By the known representation theorems for Lebesgue spaces (see e.g., [5], II 8.5 Corollary and [2], § 41 Theorem C, or [6], 26.4.9 Exercise (C)), every separable Lebesgue space (i.e., separable AL-space in terms of [5]) is Banach lattice isomorphic with $L^1(m)$ for some finite positive Borel measure m on I. Therefore we obtain the following corollary to our result:

COROLLARY 2. *Let X and Y be separable Lebesgue spaces. Then the norm attaining operators are dense in $B(X, Y)$.*

After the paper was accepted for publication, the last corollary has been generalized to arbitrary (nonseparable) Lebesgue spaces as a result of the author's conversations with Professors J. Bourgain and H. P. Lotz. The proof is outlined below:

Theorem 1 remains true if we replace (I, m_i) by (J_i, m_i) with J_i compact Hausdorff and m_i a finite regular (compact) positive measure on the Borel σ-algebra $\mathscr{B}_i$, and with M being the space of all finite signed measures on the product σ-alglebra $\mathscr{B}_1 \times \mathscr{B}_2$. Indeed, the marginal measures $\int_A T^*1dm_1$, $\int_B T1dm_2$ are compact since the measures m_i are regular, and so Theorem 1 (i) of [4] is still applicable. The rest of the proof remains unchanged.

Theorem 2 is valid for the general spaces $L^1(J_i, m_i)$ with essentially the same proof as before, $\mathscr{A}$ being replaced now by the algebra of all finite unions of Borel rectangles in $J_1 \times J_2$.

Now if X_1, X_2 are arbitrary Lebesgue spaces then every $T \in B(X_1, X_2)$ can be approximated by norm attaining operators. Indeed, let (x_n) be a sequence in X_1 such that $\|x_n\| \leqq 1$ and $\lim \|Tx_n\| = \|T\|$. The Banach lattice ideal Y_1 spanned by (x_n) is a Lebesgue subspace with a weak order unit. Also the image TY_1 is contained in a Lebesgue subspace $Y_2 \subset X_2$ with a weak order unit. By the Kakutani representation theorem there exist compact spaces J_i with finite regular positive measures m_i such that $Y_i = L^1(J_i, m_i)$. By the above, the restriction T_1 of T to Y_1 can be approximated within a given $\varepsilon > 0$ by a norm attaining operator $T_0 \in B(Y_1, Y_2)$ satisfying $\|T_0\| = \|T\|$. If P denotes the canonical band projection of X_1 onto Y_1 then it is easy to see that $T_0P + T(I - P)$ has norm $\|T_0\|$, is norm attaining, and approximates T within ε.

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POINTWISE COMPACTNESS AND MEASURABILITY

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Among other results it is proved that if $(X, \mathfrak{A}, \mu)$ is a probability space, E a Hausdorff locally convex space such that $(E', \sigma(E', E))$ contains an increasing sequence of absolutely convex compact sets with dense union, and $f: X \to E$ weakly measurable with $f(X) \subset K$, a weakly compact convex subset of E, then f is weakly equivalent to $g: X \to E$ with $g(X)$ contained in a separable subset of K.

In [8] and [9] some remarkable results are obtained for the pointwise compact subsets of measurable real-valued functions and some interesting applications to strongly measurable Banach space-valued functions are established. In this paper we continue those ideas a little further. We first give a somewhat different proof of ([9], Theorem 1) and then apply it to give a generalization of classical Phillip's theorem ([5]). Also some result about equicontinuous subsets of $C(X)$, the space of all continuous real-valued functions on (X, τ_ρ) (τ_ρ is the lifting topology, [10], p. 59; in [8] this topology is denoted by T_ρ) are obtained.

All locally convex spaces are taken over reals and notations of [6] are used. For a topological space Y, $C(Y)$ (resp. $C_b(Y)$) will denote the set of all (resp. all bounded) real-valued continuous functions of Y. N will denote the set of natural numbers.

In this paper $(X, \mathfrak{A}, \mu)$ is a complete probability measure space. Let $\mathscr{L}$ be the set of all real-valued $\mathfrak{A}$-measurable functions on X, $\mathscr{L}^\infty$, the essentially bounded elements of $\mathscr{L}$, and M^∞, the bounded elements of $\mathscr{L}$. We fix a lifting, [10], $\rho: \mathscr{L}^\infty \to M^\infty$ and on X we always take the lifting topology τ_ρ ([10], p. 59). For $f \in \mathscr{L}$, $g \in \mathscr{L}$, we write $f = g$ if $f(x) = g(x)$, $\forall x \in X$, and $f \equiv g$ if $f(x) = g(x)$, a.e. $[\mu]$. For a Hausdorff locally convex space E, a function $f: X \to E$ is said to be weakly measurable if $h \circ f$ is $\mathfrak{A}$-measurable, $\forall h \in E'$, the topological dual of E. Two weakly measurable functions $f_i: X \to E$, $i = 1, 2$, are said to be weakly equivalent if $h \circ f_1 \equiv h \circ f_2$, $\forall h \in E'$. The space $\mathscr{L}_1$ and norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ have the usual meanings. We shall call a topological space, countably compact if every sequence in it has a cluster point, and sequentially compact if every sequence has a convergent subsequence.

We start with a different proof of the following result of [9].

THEOREM 1 ([9], *Theorem* 1). *Let H be a subset of $\mathscr{L}$ such that for any $h_1 \in H$, $h_2 \in H$, $h_1 \neq h_2$ implies $h_1 \not\equiv h_2$. Then, with the pointwise topology on H, the following are equivalent*:

(i) *H is sequentially compact*;

(ii) *H is compact and metrizable.*

If H is convex, then each of (i) *and* (ii) *is also equivalent to*:

(iii) *H is compact*;

(iv) *H is countably compact.*

Proof. By ([6], Theorem 11.2, p. 187) each of (i), (ii), (iii), (iv) implies that H is relatively compact in R^X, with product topology. Thus each of these conditions implies that H is pointwise bounded. Denote by φ the homeomorphism, $[0, \infty] \to [0, 1]$, $x \to x/(1+x)$. For any $\alpha \in I$, the directed net of all finite subsets of H, let $h_\alpha = \sup\{|h|: h \in \alpha\}$, and $p_\alpha = \rho(\varphi \circ h_\alpha)$. $\{p_\alpha\}$ is a monotone bounded net in $C_b(X)$, which is boundedly complete. Let $\sup p_\alpha = p \in C_b(X)$. This means there is an increasing sequence $\{\alpha(n)\} \subset I$ such that $p = \sup p_{\alpha(n)}$ (this follows from the fact that $\mu(p) = \sup \mu(p_\alpha)$). Since $p_\alpha \equiv \varphi \circ h_\alpha$, we get $p_\alpha^{-1}\{1\}$ is μ-null, $\forall \alpha$. From this it follows that $K = p^{-1}\{1\}$ is μ-null. Thus $q = (\varphi^{-1} \circ p)\chi_{X/K}$ is a measurable function such that $|h| \leqq q$ a.e. $[\mu]$, $\forall h \in H$.

(i) $\Leftrightarrow$ (ii) is simple ([8], Prop. 1, p. 197), the metric d of (ii) being defined by $d(f, g) = \|(f - g)/1 + q\|_1$. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are trivial. Now we come to the proof of (iv) $\Rightarrow$ (i). Take a sequence $\{f_n'\} \subset H$. Since $1/(1+q)H$ is relatively weakly compact in $(\mathscr{L}_1, \|\cdot\|_1)$ there exists a subsequence $\{f_n\}$ of $\{f_n'\}$ and an $f_0 \in \mathscr{L}_1$ such that $1/(1+q)f_n \to f_0$ weakly. Thus there exists a sequence $\{g_n\}$ in the convex hull of $\{f_n: 1 \leqq n < \infty\}$ (note $\{g_n\} \subset H$) such that $1/(1+q)g_n \to f_0$ a.e. $[\mu]$ (because a convergent sequence in $(\mathscr{L}_1, \|\cdot\|_1)$ has a subsequence converging a.e. $[\mu]$). Taking f to be a cluster point of $\{g_n\}$ in H, we get $1/(1+q)f \equiv f_0(\mu)$. We claim $f_n \to f$ in H. If $f_n \nrightarrow f$ there exists an $x \in X$, an $\varepsilon > 0$, and a subsequence $\{f_n''\}$ of $\{f_n\}$ such that one of the two following conditions are satisfied:

(i) $f_n''(x) > f(x) + \varepsilon$, $\forall n$;

(ii) $f_n''(x) < f(x) - \varepsilon$, $\forall n$.

Since $1/(1+q)f_n'' \to 1/(1+q)f$ weakly, proceeding as before we get a sequence $\{g_n''\}$ in the convex hull of $\{f_n'': 1 \leqq n < \infty\}$ such that $1/(1+q)g_n'' \to 1/(1+q)f$ a.e. $[\mu]$. If f'' is a cluster point of $\{g_n''\}$ in H we get $f'' \equiv f(\mu)$ but because of (i) or (ii), $f''(x) \neq f(x)$, a contradiction. This proves that H is sequentially compact.

This result is also proved in [11] by a different method.

By a classical theorem of Phillips [5], if $f: X \to E$, E being a Banach space, is weakly measurable and $f(X)$ is relatively weakly compact in E, then f is weakly equivalent to a strongly measurable function ([8], Theorem 3, p. 200). What one really needs to do is to find a weakly equivalent function g such that $g(X)$ is separable. The next theorem is a generalization of Phillips' theorem.

THEOREM 2. *Let $(E, \mathscr{T})$ be a Hausdorff locally convex space such that there exists an increasing sequence $\{A_n\}$ of absolutely convex compact subsets of $(E', \sigma(E', E))$ whose union is dense in $(E', \sigma(E', E))$. Suppose $f: X \to E$ is weakly measurable and $f(X) \subset K$, for some weakly compact convex subset of E. Then there exists a weakly measurable function $g: X \to E$, $g \equiv f(w)$ and $g(X) \subset K_0$, a separable closed convex subset of K.*

Proof. Since $(E, \sigma(E, E'))$ can be considered as a subspace of $R^{E'}$, with product topology, f can be considered as $f: X \to R^{E'}$. For each $h \in E'$, define $g(h) = \rho(h \circ f)$ and let $g: X \to R^{E'}$, $(g)_h = g(h)$, $\forall h \in E'$. g is evidently continuous. If $g(x_0) \notin K$ for some $x_0 \in X$, there exists, by separation theorem ([6], p. 65), an $h \in E'$ such that $h \circ g(x_0) >$ $\sup(K)$. This is a contradiction since $h \circ f \leqq \sup h(K)$ implies $\rho(h \circ f) \leqq \sup h(K)$. Evidently $g \equiv f(w)$. Fix $n \in N$. By Theorem 1, $B_n = \{h \circ g: h \in A_n\}$, with the topology of pointwise convergence on X, is a compact metric space. We metrize E by the seminorms p_n, $p_n(x) = \sup\{|h(x)|: h \in A_n\}$. We denote this metric topology by $\mathscr{T}_0$. For each n, $E_n = (C(B_n), ||\cdot||)$ is a separable Banach space (here $||\cdot||$ is sup norm), and so $F = \prod_{n=1}^{\infty} E_n$ is a separable Frechet space. Let X_0 be the quotient space obtained from X by the equivalent relation, $x \equiv y \Leftrightarrow g(x) = g(y)$. Each $x \in X_0$ gives rise to $x \in C(B_n)$, $x(t) = t(x)$ for each $t \in B_n$, for every n. Thus X_0 can be embedded in F, $x_0 \to$ $(x_0, x_0, \cdots) \in F$. Taking, on X_0, the topology induced by F, we easily verify that $g: X_0 \to (E, \mathscr{T}_0)$ is continuous and so $(g(X), \mathscr{T}_0)$ is separable. Let $K_0 =$ the closed convex hull, in $(E, \mathscr{T})$, of a countable dense subset of $(g(X), \mathscr{T}_0)$. If $g(X) \not\subset K_0$, by separation theorem, there exists an $h \in E'$ and $x_0 \in X$ such that $h \circ g(x_0) > \sup h(K_0)$. Since $(E, \mathscr{T}_0)' \supset \bigcup_{n=1}^{\infty} A_n$, $q \circ g(x_0) \leqq \sup q(K_0)$, $\forall q \in \bigcup_{n=1}^{\infty} A_n$. Now there exists a net $\{h_\alpha\} \subset \bigcup_{n=1}^{\infty} A_n$ such that $h_\alpha \to h$ uniformly on each compact convex subset of $(E, \sigma(E, E'))$. From this it follows $h \circ g(x_0) \leqq \sup h(K_0)$, a contradiction. This proves the result.

REMARK 3. If E is metrizable then $(E', \sigma(E', E))$ contains a sequence of compact absolutely convex sets whose union is E'. If Y is a completely regular Hausdorff space containing a σ-compact dense set and $E = C_b(Y)$ with strict topology β_0, β_1, then it is proved in ([3], Theorem 3) that $(E', \sigma(E', E))$ has an increasing sequence of absolutely convex compact sets with dense union — here E is not metrizable.

REMARK 4. The function $g: X \to (E, \sigma(E, E'))$, obtained in this theorem, is measurable in the sense of ([2], Def. 4, p. 89).

The next theorem, in some sense, is a generalization of ([9], Theorem 3).

THEOREM 5. *Let E be a Hausdorff locally convex space such that there exist, in $(E', \sigma(E', E))$, an increasing sequence $\{A_n\}$ of absolutely convex compact sets whose union is E'. Suppose $g: X \to E$ is weakly measurable such that $g \circ f \neq 0$ implies $g \circ f \not\equiv 0$, for every $f \in E'$. Then $g(X)$ is contained in a separable subspace of E.*

Proof. In the notations of Theorem 2, $B_n = \{h \circ g: h \in A_n\}$ are compact and metrizable, with the topology of pointwise convergence, and $\mathscr{T}_0$ is the metric topology, on E, of uniform convergence on A_n. Proceeding exactly as in Theorem 2, we prove that $g(X)$ is a separable subset of $(E, \mathscr{T}_0)$. Let $F = (E, \mathscr{T}_0)'$ and $E_0 =$ the closed separable subspace, in $(E, \mathscr{T})$, generated by a countable dense subset of $(g(X), \mathscr{T}_0)$. If $g(x_0) \notin E_0$ for some $x_0 \in X$ there exists, by separation theorem, an $h \in E'$ such that $h \circ g(x_0) > 0$ and $h \equiv 0$ on E_0. Since $E' = \bigcup_{n=1}^{\infty} A_n \subset F$, $h \circ g(x_0) \leqq \sup (h \circ g(X)) \leqq \sup h(E_0) = 0$, a contradiction. This proves the result.

In the next theorem we do not assume H to be uniformly bounded ([8], Theorem 4, p. 203).

THEOREM 6. *Let H be a pointwise bounded subset of $C(X)$. If H is equicontinuous then, with the topology of pointwise convergence on X, its closure in $C(X)$ is compact and metrizable. Conversely if H is sequentially compact then there is a μ-null set A such that H is equicontinuous at each point of the open set $X \backslash A$ of (X, τ_ρ).*

Proof. If H is equicontinuous then its pointwise closed convex hull H_0, in R^X, lies in $C(X)$ and is compact and convex, and so the result follows from Theorem 1.

Conversely suppose H is sequentially compact. Then, by Theorem 1, H is compact and metrizable. By the generalized Egoroff's theorem ([4], p. 198) there exists a $\mathfrak{A}$-partition of $X = \bigcup_{i=0}^{\infty} X_i$, with $\mu(X_0) = 0$ and $\mu(X_i) > 0$, $\forall i \geqq 1$ such that $H|_{X_i}$ is compact in the topology of uniform convergence on X_i, $\forall i \geqq 1$.

$Y_i = X_i \cap \rho(X_i)$, $i \geqq 1$, are nonvoid, disjoint, open subsets of (X, τ_ρ) and $\mu(A) = 0$, where $A = X \setminus \bigcup_{i=1}^{\infty} Y_i$. By the Ascoli Theorem ([1], Ch. X, §2.5), $H|_{Y_i}$ are equicontinuous for each i. The result follows now.

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COMMUTATION WITH SKEW ELEMENTS IN RINGS WITH INVOLUTION

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This paper describes the structure of additive subgroups and of subrings which are invariant under Lie commutation with higher commutators of the skew-symmetric elements in 2-torsion free rings with involution. Except for cases arising when the subring is central, or when the ring satisfies a polynomial identity of small degree, the invariant subring must contain an ideal of the ring. With the same exceptions, the invariant subgroup must contain either the derived Lie ring of the set of skew-symmetric elements in some ideal, or the Lie product of the set of skew-symmetric elements in the ideal with the set of symmetric elements in the ideal. Furthermore, the appropriate one of these Lie products is not Lie solvable.

The first general results of this kind were obtained for simple rings by Herstein [4], who characterized the Lie ideals of K, the set of skew-symmetric elements, and then by Baxter [2], who did the same for the Lie ideals of $[K, K]$, the derived ring of K. Their work has been extended in several ways. For prime rings, the Lie ideals of both K and $[K, K]$ were studied by Erickson [3], and an investigation of additive subgroups of K invariant under commutation with $[K, K]$ in semi-prime rings was made in [7]. This was followed by a description of arbitrary additive subgroups invariant under commutation with $[K, K]$ [9], and of subgroups of K invariant under commutation with higher commutators of K [10]. Returning to simple rings, Herstein [5] showed that no noncentral proper subring could be invariant under commutation with K, except in certain small dimensional cases. This work was extended to semi-prime rings and commutation with $[K, K]$ in [8]. Our purpose here is to complete this chain of results by describing the structure of additive subgroups and of subrings invariant under commutation with higher commutators of K.

Throughout the paper, R will denote a 2-torsion free ring with involution, $*$; $S(R) = S = \{r \in R \mid r^* = r\}$, the symmetric elements of R; $K(R) = K = \{r \in R \mid r^* = -r\}$, the skew-symmetric elements of R; and $Z(R) = Z$, the center of R. The Lie product $[A, B]$ of subsets A and B of R is the additive subgroup generated by all commutators $[a, b] = ab - ba$ for $a \in A$ and $b \in B$. A higher commutator of K is a Lie product of K with itself, some fixed number of times in a given association. For example, $[[K, K], K] = V$ is a higher commutator of K, as is $K^{(1)} = [K, K]$ or $[V, K^{(1)}]$. In general, write $K^{(i+1)} = [K^{(i)}, K^{(i)}]$.

The goal of the theorems mentioned above is to show that Lie invariant additive subgroups of K contain $[K(J), K]$ for J a nonzero *-ideal of R, and that invariant subrings contain a nonzero *-ideal. Even for simple rings, one encounters two exceptions; when the invariant object is central, and when R is no more than sixteen dimensional over its center. These exceptions exist for R a prime ring also, and the second must include the possibility that R is an order in such a simple ring, in which case we say that R satisfies S_8. As one would expect for semi-prime rings, one of the three possibilities should hold in each prime image. In fact, a stronger result can be proved. In [10] it is shown that an invariant subgroup of K contains $[K(J), K]$, which is "very" noncommutative or R decomposes as a direct product of the two kinds of exceptions. To make these notions precise, we recall two definitions from [10].

DEFINITION. Let R be a 2-torsion free semi-prime ring and set $X = \{P \mid P \text{ is a *-prime ideal of } R \text{ with } 2R \not\subset P\}$. Let

$$Q_1 = \cap \{P \in X \mid R/P \text{ does not satisfy } S_8\}$$

and $Q_2 = \cap \{P \in X \mid R/P \text{ satisfies } S_8\}$. If for some subset $T \subset R$, $T + Q_1 \subset Z(R/Q_1)$, then (Q_1, Q_2) is called a *splitting of R for T*.

When A is an additive subgroup of K invariant under commutation with some higher commutator of K, then to say that there is a splitting of R for A is clearly the same as being able to "construct" R from the two kinds of exceptions discussed above. If no such splitting exists, one associates to A a *-ideal of R with the property described in our next definition.

DEFINITION. Let R be a 2-torsion free semi-prime ring, A a subset of R, and J a *-ideal of R. Then J is called a *controlling ideal for A* if for each $P \in X$ satisfying $K^{(i)}(J) \subset P$, either R/P satisfies S_8 or $A + P \subset Z(R/P)$.

The existence of a controlling ideal for A gives information about A with respect to every $P \in X$. For example, if there were no splitting of R for A, but $A \supset K^{(1)}(T)$ for some *-ideal T of R with $K^{(i)}(T) \neq 0$, one might have $T \subset P$ for some $P \in X$ with neither $A + P \subset Z(R/P)$ nor R/P satisfying S_8. Even if $K^{(1)}(I + P) \subset A + P$ for a *-ideal $I + P$ of R/P, there is no obvious way to lift this ideal back to some I in R with $K^{(1)}(I) \subset A$, or to do this simultaneously for many primes. However, an ideal J controlling A with $A \supset K^{(1)}(J)$, uniformly satisfies $A + P \supset K^{(1)}(J + P)$ for every $P \in X$ and $K^{(i)}(J + P) \neq 0$ unless R/P is one of the two exceptional cases.

Next we make two easy observations to which we shall refer several times. Henceforth, we assume that for $P \in X$, the involution on R/P is given by $(r + P)^* = r^* + P$.

LEMMA 1. *Let R be a semi-prime ring and A an additive subgroup of R satisfying $[A, K^{(i)}] \subset A$. Then for each $P \in X$, $K^{(1)}(R/P) \subset K(R) + P$, and so, $[A + P, K^{(i+1)}(R/P)] \subset A + P$.*

Proof. Clearly, it suffices to show that $K^{(1)}(R/P) \subset K(R) + P$. But if $x + P, y + P \in K(R/P)$, then $(xy - yx) + P = (xy - y^*x^*) + P \in K(R) + P$.

LEMMA 2. *Let R be a semi-prime ring and J a *-ideal of R. If for some $P \in X$, $K^{(i)}(J) \subset P$, then either $J \subset P$ or R/P satisfies S_8.*

Proof. If $J \not\subset P$, $J + P$ is a nonzero *-ideal of R/P with $K^{(i)}(J + P) = 0$. It can be shown that this condition forces $J + P$ to satisfy S_4, although one can get directly that $J + P$ satisfies S_8 by using Lemma 1, applying [10; Lemma 3], and then applying [7; Lemma 2, p. 735]. It follows that R/P must satisfy S_8 since it has an ideal which does.

Before our first main result, which extends [9; Theorem 1, p. 77] to higher commutators, note that if V is any higher commutator of K, then $V \subset K$ and $[V, K] \subset V$. An essential ingredient in our arguments is [10; Theorem 1] applied to higher commutators of K, which we state as

THEOREM A. *Let R be a semi-prime ring and V a higher commutator of K. There exists an ideal $I^* = I$ of R which is a controlling ideal for V, and which satisfies $V \supset [K(I), K]$ and $\bar{V} \supset I$, where $\bar{V}$ is the subring generated by V.*

With the preliminaries done, we can now prove our first main result, about invariant additive subgroups of S.

THEOREM 1. *Let R be a semi-prime ring, A an additive subgroup of S, and V a higher commutator of K so that $[A, V] \subset A$. Then either there is a splitting of R for A, or there exists a *-ideal I of R controlling A with $A \supset [K(I), S(I)] = Y$ and $Y^{(i)} \neq 0$ for any i.*

Proof. By Theorem A, $V \supset [K(J), K]$ for $J^* = J$, an ideal of R controlling V. Let $B = J \cap A$, and observe that $[B, K^{(1)}(J)] \subset B$, and that J is a semi-prime ring. Using [9; Theorem 1, p. 77] we may conclude that either there is a splitting of J for B, or that there exists an ideal $T^* = T$ of J with $B \supset [S(T), K(T)]$. The last paragraph of the proof of [9; Theorem 1, p. 81] shows that for any $P \in X(J)$ with $T \subset P$, either J/P satisfies S_8 or $B + P \subset Z(J/P)$. This together with Lemma 2, shows that T is a controlling ideal for B.

Assume first that there is a splitting of J for B. It follows that there is a splitting of R for B [10; Theorem 2]. Hence, for each $P \in X$ either R/P satisfies S_8 or $(A \cap J) + P \subset Z(R/P)$. Since $[A, K^{(1)}(J)] \subset A \cap J$, one obtains $[A, K^{(2)}(J)] \subset P$, if R/P does not satisfy S_8. Should $K^{(i)}(J) \subset P$, then because J is a controlling ideal for V, $V + P \subset Z(R/P)$. An easy induction argument shows that $K^{(j)} \subset V$ for some j, so that $K^{(j+1)} \subset P$. The fact that P is a proper ideal of R and Lemma 2 give that R/P must satisfy S_8. On the other hand, if $K^{(i)}(J) \not\subset P$, then $K^{(2)}(J) + P$ is not commutative and $[K^2(J) + P, K^{(1)}(R/P)] \subset K^{(2)}(J) + P$ by Lemma 1, so [8; Theorem 2, p. 90] may be used to conclude that the subring generated by $K^{(2)}(J) + P$ contains a nonzero *-ideal of R/P, unless R/P satisfies S_8. Thus the condition $[A, K^2(J)] \subset P$ forces either R/P to satisfy S_8 or $A + P \subset Z(R/P)$. Consequently, a splitting of J for B gives rise to a splitting of R for A.

Next, assume that $A \supset B \supset [S(T), K(T)]$, for T an ideal of J controlling B. Set $I = JTJ$, a *-ideal of R. Clearly, $A \supset [S(I), K(I)]$ and we claim that I is a controlling ideal for A. Let $P \in X$ and suppose that $K^{(i)}(I) \subset P$. By Lemma 2, either R/P satisfies S_8 or $I \subset P$. Assuming that R/P does not satisfy S_8, the *-primeness of P, together with the facts that J is a *-ideal of R, and $T^* = T \subset J$, gives $T \subset P \cap J$. If $J \not\subset P$ then $P \cap J \in X(J)$, so $T \subset P \cap J$ means that $J/P \cap J$ satisfies S_8 or $B + (P \cap J) \subset Z(J/P \cap J)$, since T is a controlling ideal for B. The first possibility is equivalent to the nonzero ideal $J + P$ of R/P satisfying S_8, which would force R/P to satisfy S_8. In the second case, $(A \cap J) + P \subset Z(R/P)$ and our argument in the last paragraph shows that $A + P \subset Z(R/P)$ if R/P does not satisfy S_8. The same argument shows that R/P must satisfy S_8 when $J \subset P$. By definition, I is a controlling ideal for A.

Finally, assume that $Y^{(i)} = 0$ for $Y = [S(I), K(I)]$. We claim that this gives a splitting of R for A. Let $P \in X$ and note that $Y^{(i)} + P \subset P$, $Y^{(1)} + P \subset K(R/P)$, and by Lemma 2 $[Y^{(1)} + P, K^{(1)}(R/P)] \subset Y^{(1)} + P$. From [10; Lemma 3] we have either $Y^{(1)} + P \subset Z(R/P)$ or that R/P satisfies S_8. In the first case, a result of Amitsur [1; Theorem 1, p. 63] shows that $(I + P)/P$ satisfies a polynomial identity, and so, R/P satisfies the same identity. Of course, if $I \subset P$ we would be finished by our earlier arguments. Consequently, localizing R/P at its central symmetric elements gives a semi-simple finite dimensional algebra Q [6].

Since in this localization, $I + P$ becomes Q, $S(I) + P$ localizes to $S(Q)$, and $K(I) + P$ localizes to $K(Q)$, it follows that in Q, $[[S, K], [S, K]] \subset Z(Q)$. A consideration of the possible cases shows that Q is at most four dimensional over its center. Very briefly, if Q is not simple, or has an involution of the second kind, then $Q^{(3)} = 0$, and otherwise one can split Q to obtain matrices over a field, where straightforward computations give the result. Consequently, R/P must satisfy S_8 (in fact, S_4) so $Y^{(i)} = 0$ forces a splitting of R for A, completing the proof of the theorem.

Combining Theorem 1 with [10; Theorem 4] gives the version of [9; Theorem 2, p. 82] for higher commutators of K.

THEOREM 2. *Let R be a semi-prime ring, V a higher commutator of K, and A an additive subgroup of R satisfying $[A, V] \subset A$. Then one of the following holds:*

(i) *$A \supset [K(I), K] = L$ for $I^* = I$ an ideal of R controlling $A \cap K$, and $L^{(i)} \neq 0$;*

(ii) *$A \supset [K(I), S(I)] = Y$ for $I^* = I$ an ideal of R controlling $A \cap S$, and $Y^{(i)} \neq 0$;*

(iii) *there is a splitting of R for $A \cap S + A \cap K$.*

If in addition, $A^* = A$, then (iii) *can be replaced by:* (iii)′ *there is a splitting of R for A.*

In trying to improve Theorem 2 (iii) to (iii)′, the same counter-example and considerations as in [9] show that some additional assumption is required. Before discussing the nature of the involution on R, we point out that if in Theorem 2 (iii), for each $P \in X$ with R/P not satisfying S_8, P is not a prime ideal of R, then in fact $A + P \subset Z(R/P)$. To prove this, note first that if P is not a prime ideal of R, then $P = Q \cap Q^*$ for a Q prime ideal of R. Now $Q + Q^*$ is a nonzero ideal of R/Q^* and $q + Q^* = (q - q^*) + Q^*$, so $Q + Q^* \subset K + Q^*$. If the higher commutator V in Theorem 2 contains $K^{(i)}$, then $[A, Q^{(i)}] + Q^* \subset [A, K^{(i)}] + Q^* \subset A + Q^*$, so $[A, Q^{(i)}] + Q^* \subset (A \cap Q) + Q^* \subset (A \cap K) + Q^* \subset Z(R/Q^*)$. Since $Q^{(i)} + Q^*$ is a Lie ideal in R/Q^*, it follows that either $A + Q^* \subset Z(R/Q^*)$, or $Q^{(i)} + Q^* \subset Z(R/Q^*)$, unless R/Q^* satisfies S_4 [11; Lemma 8, p. 120]. The possibility $Q^{(i)} + Q^* \subset Z(R/Q^*)$ and repeated use of [11; Lemma 7, p. 120] force $Q + Q^* \subset Z(R/Q^*)$, which in turn means that R/Q^* is commutative. Repeating the whole argument with Q and Q^* interchanged shows that $A + P \subset Z(R/P)$ unless R/P satisfies S_4. We isolate one special case of Theorem 2 to which our observation applies.

COROLLARY. *If in Theorem 2, R is a *-prime ring which is not prime, then $A \cap S + A \cap K \subset Z$ forces $A \subset Z$ unless R satisfies S_8.*

As in [9], the obstruction to showing that a splitting of R for $A \cap S + A \cap K$ forces a splitting of R for A occurs in prime rings whose extended centroid has an induced involution of the second kind [13; Theorem 4.1, p. 511]. When this involution is of the first kind, we can prove the result corresponding to [9; Theorem 7, p. 93] for higher commutators.

THEOREM 3. *Let R be a prime ring with extended centroid C, and assume that the involution induced on C is the identity map. If V is a higher commutator of K and A is an additive subgroup of R satisfying $[A, V] \subset A$, then $A \cap S + A \cap K \subset Z$ implies that either $A \subset Z$ or R satisfies S_8.*

Proof. Let I be the controlling ideal for V given by Theorem A. Then I is a prime ring and $K^{(1)}(I) \subset V$ implies that $[A \cap I, K^{(1)}(I)] \subset A \cap I$. We wish to apply [9; Theorem 7, p. 93] to I and $A \cap I$, but first we must verify that the involution on C_I, the extended centroid of I, is the identity map. This follows from work of Martindale since the extended centroid is the center of a certain quotient ring and these quotient rings coincide for R and for I [12; Theorem 1, p. 440]. A proof of this result, using the definitions in [13] follows easily from the fact that any ideal T of I contains the ideal ITI of R and an ideal N of R contains the ideal NI of I. This observation and [13; proof of Theorem 4.1, p. 511-512] show that C and C_I have the same kind of involution. Applying [9; Theorem 7, p. 93] gives either $A \cap I \subset Z(I)$ or I satisfies S_8. Since I satisfying S_8 forces R to satisfy S_8, assume that $A \cap I \subset Z(I)$. Thus $A \cap I \subset Z(R)$, and in particular $[A, K^{(1)}(I)] \subset Z$, forcing $[A, K^{(2)}(I)] = 0$. As in the first part of the proof of Theorem 1, we must have $A \subset Z$ unless R satisfies S_8, completing the proof of the theorem.

Using the same ideas as above, we can obtain the higher commutator version of [8; Theorem 3, p. 92] for invariant subrings. Note that for subrings, the nature of the involution is immaterial.

THEOREM 4. *Let R be a semi-prime ring, V a higher commutator of K, and A a subring of R satisfying $[A, V] \subset A$. Then either $A \supset M^* = M$, a noncommutative ideal of R controlling A, or there is a splitting of R for A.*

Proof. By Theorem A, $V \supset [K(I), K]$ for $I^* = I$ an ideal of R controlling V. Clearly, $B = A \cap I$ satisfies $[B, K^{(1)}(I)] \subset B$, so [8; Theorem 3, p. 92] applies to the subring B of I to yield a splitting of I for B, or that $B \supset T^* = T$, a noncommutative ideal of I. We observe that the proof of [8; Theorem 3, p. 92] actually shows that T is a controlling ideal for B, since T can be chosen to be a controlling ideal for $B \cap K$ by [10; Theorem 1], which is all that is necessary. If $B \supset T$ holds, then $A \supset B \supset ITI = M$. The fact that M is a controlling ideal for A follows exactly as in the proof of Theorem 1. Should M be commutative, the semi-primeness of R would force $M \subset Z(R)$. In particular, $T^3 \subset Z(T)$ and $Z(T)^2T \subset Z(T)$. Thus $0 = [Z(T)^2T, T] = Z(T)^2[T, T]$, so $Z(T)[1, T] = 0$ from the fact that T is a semi-prime ring. But now $T^3[T, T] = 0$ so that $(T[T, T])^2 = 0$, forcing $T[T, T] = 0$. Hence $[T, T] \subset T \cap \mathrm{Ann}(T) = 0$, contradicting the assumption that T is not commutative. To complete the proof of the theorem, it suffices to treat the case when there is splitting of I for B. As in the proof of Theorem 1, such a splitting gives a splitting of R for B [10; Theorem 2], and the fact that $B \supset [A, K^{(1)}(I)]$ for I a controlling ideal of R for V yields a splitting of R for A.

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A RADON-NIKODYM THEOREM FOR FINITELY ADDITIVE BOUNDED MEASURES

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An exact Radon-Nikodym theorem is obtained for finitely additive bounded scalar measures defined on a field, the additional condition being a local condition on the dominant average range. The traditional technique of transferring the problem to the Stone space, which results in approximate Radon-Nikodym derivatives, is circumvented by isolating an Exhaustion principal for finitely additive measures which is then utilized to obtain the necessary decompositions.

Examples are given to illustrate the basic difficulties which arise in differentiating with respect to signed finitely additive measures and it is demonstrated that one difficulty arises from a lack of a suitable Hahn decomposition of the differentiating measures. The concept of an exhaustive Hahn decomposition is defined for finitely additive measures and is compared to the related concepts of an approximate Hahn decomposition as well as the standard Hahn decomposition. It is shown that μ having an exhaustive Hahn decomposition is equivalent to $|\mu|$ having a Radon-Nikodym derivative with respect to μ and this result is then applied, in this situation, to obtain a simplified Radon-Nikodym theorem.

The question of characterizing indefinite integrals of finitely additive measures has been under consideration for a number of years. There have been two basic approaches to this problem, both seemingly arising from a desire to characterize the absolutely continuous bounded measures. The first was to enlarge the class of integrable functions to include objects other than point functions and to then obtain an equivalence between absolute continuity and integral representation. Rickart [10] obtained such an equivalence by including the multi-valued contractive set functions, while Tucker and Wayment [12], in the setting of finitely additive operator-valued measures, obtained a similar equivalence between an enlarged class of integrable objects and a generalized definition of absolute continuity. The second approach is that of the Radon-Nikodym Bochner theorem [3, p. 315, Theorem 14] which utilized the Stone space to characterize the absolutely continuous, bounded variation measures as those which can be approximated arbitrarily close in variation by integrals of integrable simple functions. There does not seem to be any characterization of indefinite integrals of point functions with respect to a finitely additive bounded scalar measure prior to this paper.

The method of proof is interesting in that it is shown that if m is representable as an integral with respect to μ, then there exists certain "nice" decompositions of X such that both μ and m satisfy a restricted form of countable additivity with respect to these decompositions. This is sufficient to allow arguments similar to those used in the Bochner integral case [Maynard, 8, Theorem 2.1]. In fact the lack of various decompositions seems to be the key to the difficulties which arise in the finitely additive situation.

2. An exhaustion principle.

The notation and definitions employed in this paper will be the same as those of Dunford and Schwartz [3, Cnapter III] which is an equivalent development, in our setting, to that of Gould [7]. Let X be a set, Σ a field of subsets of X, and $\mu: \Sigma \to \boldsymbol{R}$ a finitely additive bounded measure ($\equiv$ set function). As usual $|\mu|$ will denote the total variation of μ and is a positive finitely additive measure and Σ^+ will denote the subset of Σ consisting of sets with positive μ-variation. In addition we will use the notation $\delta(A)$ to denote the diameter of a set $A \subset \boldsymbol{R}$.

DEFINITION 2.1. A set property P is said to be *locally exhausting* in (X, Σ, μ) if there exists an α, $0 < \alpha \leqq 1$, such that for each $E \in \Sigma^+$ there exists $F \subset E$, $F \in \Sigma^+$, such that $|\mu|(F) \geqq \alpha|\mu|(E)$ and F has property P.

DEFINITION 2.2. A countable (possibly finite) disjoint collection $\{X_i\}_{i \in I} \subset \Sigma^+$ is said to be *exhausting in X* if, given any $\varepsilon > 0$, there exists $N > 0$ such that

$$|\mu|\Big(X \sim \bigcup_{1=i}^{N} X_i\Big) < \varepsilon .$$

LEMMA 2.3 (*Exhaustion principle*). *If P is a locally exhausting set property in (X, Σ, μ), then there exists a countable (possibly finite) set of disjoint subsets, $\{X_i\}_{i \in I} \subset \Sigma^+$, such that each X_i has property P and $\{X_i\}_{i \in I}$ is exhausting in X.*

Proof. Since P is locally exhausting, there exists $X_1 \subset X$, $X_1 \in \Sigma^+$, such that X_1 has P and $|\mu|(X_1) \geqq \alpha|\mu|(X)$. Proceed by induction. If $|\mu|(X \sim \bigcup_{i=1}^n X_i) = 0$, then the process terminates and $\{X_i\}_{i=1}^n$ satisfies the conclusions of the lemma. If $|\mu|(X \sim \bigcup_{i=1}^n X_i) > 0$, choose $X_{n+1} \subset X \sim \bigcup_{i=1}^n X_i$, $X_{n+1} \in \Sigma^+$, such that X_{n+1} has property P and $|\mu|(X_{n+1}) \geqq \alpha|\mu|(X \sim \bigcup_{i=1}^n X_i)$. If the process never terminates we obtain a disjoint sequence $\{X_i\}_{i=1}^\infty \subset \Sigma^+$ such that each X_i has property P.

If $\lim_{n\to\infty} |\mu|(X \sim \bigcup_{i=1}^{n} X_i) \neq 0$, then there exists a $\beta > 0$ such that $|\mu|(X \sim \bigcup_{i=1}^{n} X_i) > \beta$, for $1 \leqq n < \infty$. Thus

$$|\mu|(X_n) \geqq \alpha |\mu|\Big(X \sim \bigcup_{i=1}^{n-1} X_i\Big) > \alpha\beta > 0$$

for every n, and since $\{X_i\}_{i=1}^{\infty}$ is disjoint, this violates the boundedness of μ.

DEFINITION 2.4. A set property P is said to be a *null difference property* if whenever $E \in \Sigma^+$ has property P and $F \in \Sigma^+$ such that $|\mu|(E \Delta F) = 0$, then F has property P.

LEMMA 2.5. *P is a locally exhausting null difference property in a complete bounded finitely additive measure space (X, Σ, μ), then there exists a countable (possibly finite) set of disjoint subsets, $\{X_i\}_{i\in I} \subset \Sigma^+$, such that $X = \bigcup_{i \in I} X_i$, each X_i has property P, and $\{X_i\}_{i\in I}$ is exhausting in X.*

Proof. By the Exhaustion principle there exists a set $\{X_i\}_{i\in I}$ satisfying all conclusions except that X need not equal $\bigcup_{i\in I} X_i$. But since $\{X_i\}_{i\in I}$ is exhausting in X we have that $X \sim \bigcup_{i\in I} X_i$ is a μ-null set and hence is measurable by completeness of (X, Σ, μ). Thus since P is a null difference property, $X \sim \bigcup_{i\in I} X_1$ may be adjoined to X_1 without altering any of the desired properties.

3. **A Radon-Nikodym theorem.** The approach to be used in obtaining a Radon-Nikodym theorem for bounded finitely additive measure is similar to the locally small average range approach for the Bochner integral. The major difficulty in this approach lies in a possible instability of the average range due to locally large values $|\mu|(E)/|\mu(E)|$ of the integrating measure. This is due to the lack of a Hahn decomposition for bounded finitely additive measures. A secondary problem is that while a local property will yield a countable maximal decomposition of the space, the measures need not be countably additive with respect to this decomposition. It is easy to construct examples on the field of finite and cofinite subsets of the integers with locally small average range but without locally exhausting small average range.

We consider first the various types of average ranges which are useful in Radon-Nikodym theorems for the Bochner integral, operator-valued measures, and finitely additive measures. Suppose $m: \Sigma \to R$ is another finitely additive measure. The standard average range which occurs in the Radon-Nikodym theorem for the Bochner integral [Rieffel [11], Maynard [8]] has the following definition.

DEFINITION 3.1. For each $E \in \Sigma^+$, the average range of m with respect to μ over E is: $A_m(E) = \{m(F)/\mu(F): F \subset E, \mu(F) \neq 0\}$.

However without a Hahn decomposition the local structure of $A_m(E)$ may always be poorly behaved when the ratios, $|\mu|(F)/|\mu(F)|$, are large and hence to avoid this problem we consider, with finitely additive measures, the dominant average range.

DEFINITION 3.2. For each $E \in \Sigma^+$, the dominant average range of m with respect to μ over E is

$$A_m^*(E) = \left\{m(F)/\mu(F): F \subset E, F \in \Sigma^+ , \quad \text{and} \quad |\mu(F)| > \frac{1}{2}|\mu|(F)\right\} .$$

The third average range we will consider is the ε-approximate average range which is useful for operator-valued measures, Maynard [7], but is primarily used here for convienence and to illustrate the connections between the various average ranges.

DEFINITION 3.3. For each $E \in \Sigma^+$, the ε-approximate average range of m with respect to μ over E is

$$A(E, \varepsilon) = \{x \in \boldsymbol{R}: |m(F) - x\mu(F)| \leqq \varepsilon|\mu|(F), \forall F \subset E, F \in \Sigma\} .$$

The following two properties are the key properties involved in the Radon-Nikodym theorem for finitely additive measures.

DEFINITION 3.4. m is said to have *locally exhausting small dominant average range* iff for each $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that for $E \in \Sigma^+$ there exists $F \subset E$, $F \in \Sigma^+$, with $|\mu|(F) > \alpha(\varepsilon)|\mu|(E)$ and $\delta(A_m^*(F)) < \varepsilon$.

DEFINITION 3.5. m is said to have *locally exhausting approximate average range* iff for each $\varepsilon > 0$ there exists $\alpha(\varepsilon) > 0$ such that for $E \in \Sigma^+$ there exists $F \subset E$, $F \in \Sigma^+$, with $|\mu|(F) > \alpha(\varepsilon)|\mu|(E)$ and $A(F, \varepsilon) \neq \varnothing$.

DEFINITION 3.6. If $m, \mu: \Sigma \to \boldsymbol{R}$ are finitely additive measures, then m is μ-continuous iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mu|(E) < \delta$ implies that $|m|(E) < \varepsilon$.

It should be emphasized that the definitions of μ-continuity in [5] and [8], requiring only that $|m(E)| < \varepsilon$, are too restrictive as noted in [4] and should be the above definition.

LEMMA 3.7. *Let (X, Σ, μ) be a bounded finitely additive measure* *space and* $m\colon \Sigma \to \boldsymbol{R}$ *be a* μ*-continuous finitely additive measure. Then* m *has locally exhausting small dominant average range iff* m *has locally exhausting approximate average range.*

Proof. Suppose m has locally exhausting approximate average range, and let $\varepsilon > 0$ be given and $\alpha(\varepsilon)$ the guaranteed constant corresponding to $\varepsilon/4$. Then if $E \in \Sigma^+$, there exists $F \subset E$, $F \in \Sigma^+$, $|\mu|(F) > \alpha(\varepsilon)|\mu|(E)$ such that $A(F, \varepsilon/4) \neq \varnothing$.

Choose $x \in A(F, \varepsilon/4)$. Then if $F_1 \subset F$, $F_1 \in \Sigma^+$, such that $|\mu(F_1)| > 1/2|\mu|(F_1)$ we have

$$\left|\frac{m(F_1) - x}{\mu(F_1)}\right| = |m(F_1) - x\mu(F_1)| \cdot \frac{1}{|\mu(F_1)|} \leqq \frac{\varepsilon}{4}\frac{|\mu|(F_1)}{|\mu(F_1)|} < \frac{\varepsilon}{2}\,.$$

Thus $\delta(A_m^*(F)) < \varepsilon$ and m has locally exhausting small dominant average range.

Suppose that m has locally exhausting small dominant average range. Let $\varepsilon > 0$ be given and $\alpha(\varepsilon)$ the constant corresponding to ε. Then given $E \in \Sigma^+$, there exists $F \subset E$, $F \in \Sigma^+$, such that $\delta(A_m^*(F)) < \varepsilon$. Choose $F_1 \subset F$ such that $|\mu(F_1)| > 1/2|\mu|(F_1)$. Then it suffices to show that $m(F_1)/\mu(F_1) \in A(F, \varepsilon)$.

Let $C \subset F$, $C \in \Sigma^+$. If $|\mu|(C) = 0$ then by μ-continuity, $m(C) = 0$ and we have the desired inequality. If $|\mu|(C) \neq 0$, then let $\delta = \min(u^+(C), \mu^-(C))$ where $\mu^+(C) = \sup_{D\subset C}\mu(D)$ and $\mu^-(C) = -\inf_{D\subset C}\mu(D)$. If $\delta = 0$ the argument is trivial so suppose $\delta > 0$. Then by Darst [5] there exist disjoint sets A, B such that $C = A \cup B$ with property that $\mu^+(B) < \delta/4 < |\mu|(B)/4$ and $\mu^-(A) < \delta/4 < |\mu|(A)/4$. Then

$$|\mu(A)| = |\mu^+(A) - \mu^-(A)| > |\mu|(A) - 2\mu^-(A) > \frac{|\mu|(A)}{2}$$

and similarily $|\mu(B)| > |\mu|(B)/2$. Thus

$$\begin{aligned}\left|m(C) - \frac{m(F_1)}{\mu(F_1)}\mu(C)\right| &= \left|m(A\cup B) - \frac{m(F_1)}{\mu(F_1)}\mu(A\cup B)\right| \\ &\leqq \left|m(A) - \frac{m(F_1)}{\mu(F_1)}\mu(A)\right| + \left|m(B) - \frac{m(F_1)}{\mu(E_1)}\mu(B)\right| \\ &\leqq \left|\frac{m(A)}{\mu(A)} - \frac{m(F_1)}{\mu(F_1)}\right||\mu(A)| \\ &\quad + \left|\frac{m(B)}{\mu(B)} - \frac{m(F_1)}{\mu(F_1)}\right||\mu(B)| < \varepsilon|\mu|(A) + \varepsilon|\mu|(B) \\ &= \varepsilon|\mu|(C)\,.\end{aligned}$$

Thus $m(F_1)/\mu(F_1) \in A(F, \varepsilon) \neq \varnothing$ and hence m has locally exhausting approximate average range. As the third example in §4 demonstrates, it is not true that either of these two conditions imply that m has locally exhausting small average range, even if m is an indefinite integral. We are now prepared to prove our main theorem after we point out a restricted form of countable additivity which will enable us to mimic proofs in the countably additive case.

LEMMA 3.8. *Let m and μ be two $\boldsymbol{R}$-valued measures in (X, Σ), Σ a field, such that m is μ-continuous. Then μ is uniformly countably additive with respect to a disjoint sequence $\{E_i\}_{i=1}^{\infty} \subset \Sigma$*(i.e., $\forall F \in \Sigma^+$, $\mu(F) = \sum_{i=1}^{\infty} \mu(F \cap E_i)$ *where convergence is uniform in F*) *iff $\{E_i\}_{i=1}^{\infty}$ is exhausting in X. In addition if $\{E_i\}_{i \in I}$ is exhausting in X with respect to μ, then m is also uniformly countably additive with respect to $\{E_i\}_{i=1}^{\infty}$.*

The following bound on the ε-approximate average range can easily be calculated.

LEMMA 3.9. *Let m and μ be two $\boldsymbol{R}$-valued measures in (X, Σ). Then for $\varepsilon > 0$, $\delta(A(E, \varepsilon)) \leqq 2\varepsilon$, $E \in \Sigma^+$.*

THEOREM 3.10 [*Radon-Nikodym theorem*]. *Let (X, Σ, μ) be a bounded finitely additive measure space, Σ a field of subsets of X and μ a signed measure. If m is a finitely additive $\boldsymbol{R}$-valued measure, then there exists a μ-integrable function $f: X \to \boldsymbol{R}$ such that $m(E) = \int_E f d\mu$, $\forall E \in \Sigma$ iff*

(a) *m is bounded, μ-continuous and*

(b) *for all $\delta > 0$ there exists $F_\delta \subset X$, $F_\delta \in \Sigma$ such that*

 (i) *$\mu(X \sim F_\delta) < \delta$*

 (ii) *$A_m^*(F_\delta)$ is bounded and*

 (iii) *m has locally exhausting small dominated average range in F_δ.*

Proof. We may assume throughout the proof that (X, Σ, μ) is complete since a function integrable with respect to the completion is integrable with respect to (X, Σ, μ) and has the same integral values.

($\Rightarrow$) Suppose $m(E) = \int_E f d\mu$. Then (a) is well known [Dunford and Schwartz, 3, III 2.18 and 20]. Let $\delta > 0$ be given. Then there exists a simple function f_n such that $\mu^*\{x: |f(x) - f_n(x)| > 1\} < \delta$. Choose $A \in \Sigma$ such that $A \supset \{x: |f(x) - f_n(x)| > 1\}$ and $\mu(A) < \delta$ and let $F_\delta = X \sim A$. Hence F_δ satisfies (i). Let $N = \sup\{|f_n(x)|: x \in F_\delta\} + 1$. Thus $|f(x)| \leqq N$ for all $x \in F_\delta$. Now if $E \subset F_\delta$, $|\mu(E)| > 1/2|\mu|(E)$, then $|m(E)| = \left|\int_E f d\mu\right| \leqq 2N|\mu|(E) \leqq 4N|\mu(E)|$ and hence $A_m^*(F_\delta)$ is bounded.

Let $\varepsilon > 0$ be given and let $\alpha(\varepsilon) = \min\{1/16, \varepsilon/8N\}$ and suppose $E \in \Sigma^+$, $E \subset F_\delta$. Since f is totally measurable on F_δ, there exists a measurable partition $\{X_i\}_{i=0}^n$ of E such that $|\mu|(X_0) < |\mu|(E)/4$ and $\delta(f(X_i)) < \varepsilon/2$, $1 \leqq i \leqq n$. Now by Lemma 3.7 it suffices to show the equivalence with locally exhausting approximate average range.

Claim 1. $f(X_i) \subset A(X_i, \varepsilon/2)$, $1 \leqq i \leqq n$.

Proof. Let $r \in f(X_i)$ and let $F \subset X_i$, $F \in \Sigma^+$. Then

$$|m(F) - r\mu(F)| = \left|\int_F f - r d\mu\right| \leqq \frac{\varepsilon}{2}|\mu|(F)$$

since $|f(x) - r| \leqq \varepsilon/2$ for all $x \in X_i$. Thus $f(X_i) \subset A(X_i, \varepsilon/2)$.

We now cover the interval $[-N, N]$ with the disjoint intervals $E_k \equiv [-N + k\varepsilon/2, -N + (k+1)\varepsilon/2)$, $0 \leqq k \leqq [\![4N/\varepsilon]\!] \equiv Q$ where $[\![\cdot]\!]$ is the greatest integer function.

For each k, $0 \leqq k \leqq Q$, we define the following set of indices:

$$I_k = \{i : A(X_i, \varepsilon/2) \cap E_k \neq \varnothing\} .$$

Now $A(X_1, \varepsilon/2)$ must intersect at least one E_k since $f(X_i) \subset [-N, N]$ and can intersect no more than two since $\delta(A(X_i, \varepsilon/2)) \leqq \varepsilon$.

Claim 2. There exists $k \geqq 0$ such that

$$|\mu|\Big(\bigcup_{i \in I_k} X_i\Big) > \alpha(\varepsilon)|\mu|(E) .$$

Proof. Suppose not. We already know that

$$|\mu|\Big(\bigcup_{i=1}^n X_i\Big) \geqq |\mu|(E) - |\mu|(X_0) \geqq \frac{3|\mu|(E)}{4} ,$$

but on the other hand

$$\begin{aligned} |\mu|\Big(\bigcup_{i=1}^n X_i\Big) &= |\mu|\Big(\bigcup_{k=0}^Q \Big\{\bigcup_{i \in I_k} X_i\Big\}\Big) \leqq \sum_{k=0}^Q |\mu|\Big(\bigcup_{i \in I_k} X_i\Big) \\ &\leqq (Q+1)\alpha(\varepsilon)|\mu|(E) \leqq \Big(\frac{4N}{\varepsilon} \cdot \frac{\varepsilon}{8N} + \frac{1}{16}\Big)|\mu|(E) \\ &\leqq \Big[\frac{1}{2} + \frac{1}{16}\Big]|\mu|(E) < \frac{3}{4}|\mu|(E). \quad \Rightarrow \Leftarrow . \end{aligned}$$

Thus there exists I_k such that $|\mu|(\bigcup_{i \in I_k} X_i) > \alpha(\varepsilon)|\mu|(E)$. Let $F = \bigcup_{i \in I_k} X_i$.

Claim 3. $A(F, \varepsilon) \neq \varnothing$.

Proof. Let $r = -M + ((k+1)/2)\varepsilon$ and suppose $F' \subset F$, $F' \in \Sigma^+$. Now for each X_i, $i \in I_k$, choose $r_i \in A(X_i, \varepsilon/2) \cap E_k$. Then $|r - r_i| \leqq \varepsilon/2$ since $r, r_i \in \bar{E}_k$. Now

$$\begin{aligned} |m(F') - r\mu(F')| &\leqq \sum_{i \in I_k} |m(F' \cap X_i) - r_i\mu(F' \cap X_i)| \\ &\quad + \sum_{i \in I_k} |r_i - r|\,|\mu|(F' \cap X_i) \\ &\leqq \sum_{i \in I_k} \frac{\varepsilon}{2}|\mu|(F' \cap X_i) + \sum_{i \in I_k} \frac{\varepsilon}{2}|\mu|(F' \cap X_i) = \varepsilon|\mu|(F') . \end{aligned}$$

Thus $r \in A(F, \varepsilon) \neq \varnothing$.

Hence since $|\mu|(F) > \alpha(\varepsilon)|\mu|(E)$ we have finished demonstrating the necessity of our conditions.

($\Leftarrow$) Suppose m satisfies (a) and (b) and hence has locally exhausting approximate range.

We will use the following notation. If $z = (z_1, \cdots, z_n) \in N^n$, then $p(z) = (z_1, \cdots, z_{n-1})$, $q(z) = z_n$, and $(z, i) = (z_1, \cdots, z_n, i) \in A^{n+1}$, where the dependence on n is suppressed in an effort for notational simplicity.

Now there exists a disjoint sequence of sets $\{F_N\} \subset \Sigma^+$, which of exhausting in X, guaranteed by conditions (a) and (b). We will obtain a density for m on each F_N and then sum to obtain the entire density. Fix N.

Now the set property, $A(F, 1/2) \neq \varnothing$, is a locally exhausting null difference property in F_N and hence there exists a disjoint countable set $\{Y_z^1\}_{z \in A_1} \subset \Sigma^+$, $A_1 \subset N$, with $\{Y_z^1\}$ exhausting in X, $F_N = \bigcup_{z \in A_1} Y_z^1$, and $A(Y_z^1, 1/2) \neq \varnothing$.

Since $A(F, 1/2^2) \neq \varnothing$ is a locally exhausting null difference property in each Y_z^1 we may decompose each in an exhausting manner, $Y_z^1 = \bigcup_{i \in A_z^2} Y_{(z,i)}^2$, where $A(Y_{(z,i)}^2, 1/2^2) \neq \varnothing$.

Let $A_2 = \{z \in N^2 : p(z) \in A_1,\ q(z) \in A_{p(z)}^2\}$. Thus $F_N = \bigcup_{z \in A_2} Y_z^2$ and this decomposition is exhausting.

In general if $\{Y_z^n\}_{z \in A_n}$ is exhausting in F_N, $A_n \subset N^n$, $F_N = \bigcup_{z \in A_n} Y_z^n$, we may decompose each Y_z^n in an exhausting manner and obtain the decomposition $\{Y_z^{n+1}\}_{z \in A_{n+1}}$ where

$$Y_z^n = \bigcup_{i \in A_z^{n+1}} Y_{(z,i)}^{n+1},\ A_z^{n+1} \subset N,\ A(Y_{(z,i)}^{n+1}, 1/2^{n+1}) \neq \varnothing$$

$$F_N = \bigcup_{z \in A_{n+1}} Y_z^{n+1},\ A_{n+1} = \{z \in N^{n-1} : p(z) \in A_n,\ q(z) \in A_{p(z)}^{n+1}\} .$$

We now define a sequence of functions, $f_n : F_N \to \boldsymbol{R}$, in the following manner. For each n and each $z \in A_n$ choose $x_z^n \in A(Y_z^n, 1/2^n)$ and let $f_n = \sum_{z \in A_n} x_z^n \chi_{Y_z^n}$.

Claim 1. f_n is totally measurable, bounded, and hence integrable and $\int_E f_n d\mu = \sum_{z \in A_n} x_z^n \, \mu(E \cap Y_z^n)$.

Proof. Since $\{Y_z^n\}_{z \in A_n}$ is exhausting in F_N, the finite sums converge in measure to f_n and hence f_n is totally measurable. f_n is bounded since the dominated average ranges are bounded and hence the 1-approximate average ranges are bounded in F_N.

Claim 2. $\{f_n(t)\}_{n=1}^{\infty}$ is uniformly Cauchy for $t \in F_N$.

Proof. Let $\varepsilon > 0$ be given and choose M such that $1/(2M) < \varepsilon$.

If $t \in F_N$, there exists a sequence $\{z_n\}$, $z_n \in A_n$, such that $t \in Y_{z_n}^n$. Thus if $n, m > M$ with $m > n$ we have that

$$f_n(t) = x_{z_n}^n \in A(Y_{z_n}^n, 1/2^n) \subset A(Y_{z_m}^m, 1/2^n) \quad \text{and}$$
$$f_m(t) = x_{z_m}^m \in A(Y_{z_m}^m, 1/2^m) \subset A(Y_{z_m}^m, 1/2^n) \ .$$

But $\delta(A(Y_{zm}^m, 1/2^n)) \leqq 1/2^{n-1}$ and hence $|f_n(t) - f_m(t)| \leqq 1/2^{n-1} \leqq 1/2^m < \varepsilon$ for any $t \in F_N$.

We thus can define $g_N(t) = \lim_{n \to \infty} f_n(t)$: $F_N \to \boldsymbol{R}$.

Claim 3. g_N is totally measurable, bounded and hence integrable.

Proof. $f_n \to g_N$ uniformly and hence in measure which implies that g_N is totally measurable. g_N is bounded since the functions $\{f_n\}$ are uniformly bounded.

Claim 4. $\int_E g_N d\mu = \lim_{n \to \infty} \int_E f_n d\mu$, $\forall E \in \Sigma$, $E \subset F_N$.

Proof. The functions $\{f_n\}_{n=1}^{\infty}$ are uniformly bounded and converge uniformly, and hence in measure, to g_N on F_N. Thus by the Dominated Convergence theorem we obtain that g_N is integrable and $\int_E g_N d\mu = \lim_{n \to \infty} \int_E f_n d\mu$, $\forall E \in \Sigma$.

Claim 5. $\int_E g_N d\mu = m(E)$, $\forall E \in \Sigma$, $E \subset F_N$.

Proof. Let $\varepsilon > 0$ be given. Then there exists n such that $\left| \int_E g_N d\mu - \int_E f_n d\mu \right| < \varepsilon/2$ and such that $1/2^n < \varepsilon/8|\mu|(E)$. Now choose $K > 0$ such that

$$\text{(i)} \quad \left| \int_E f_n d\mu - \sum_{\substack{z \leqq (K \cdots K) \\ z \in A_n}} x_z^n \mu(E \cap Y_z^n) \right| < \frac{\varepsilon}{4}$$

and

$$\text{(ii)} \quad |m|\Big(E \sim \bigcup_{\substack{(z \leqq (K \cdots K)) \\ z \in A_n}} Y_z^n\Big) < \frac{\varepsilon}{8}\,.$$

Then

$$\Big|m(E) - \sum_{\substack{z \leqq (K \cdots K) \\ z \in A_n}} x_z^n \mu(E \cap Y_z^n)\Big|$$

$$\leqq \Big|m\Big(E \sim \bigcup_{\substack{z \leqq (K \cdots K) \\ z \in A_n}} (E \cap Y_z^n)\Big)\Big| + \sum_{\substack{z \leqq (K \cdots K) \\ z \in A_n}} |m(E \cap Y_z^n) - x_z^n \mu(E \cap Y_z^n)|$$

$$\leqq \frac{\varepsilon}{8} + \sum_{z \leqq (K, \cdots, K)} \frac{1}{2^n} |\mu|(E \cap Y_z^n) \quad \text{since} \quad x_z^n \in A(Y_z^n, 1/2^n)$$

$$\leqq \frac{\varepsilon}{8} + \frac{1}{2^n} |\mu|(E)\,, \quad \text{since} \quad \{Y_z^n\} \quad \text{is exhausting}\,,$$

$$\leqq \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}\,.$$

Thus

$$\begin{aligned} \Big|\int_E g_N d\mu - m(E)\Big| &\leqq \Big|\int_E g_N d\mu - \int_E f_n d\mu\Big| \\ &+ \Big|\int_E f_n d\mu - \sum_{\substack{z \leqq (K \cdots K) \\ z \in A_n}} x_z^n \mu(E \cap Y_z^n)\Big| \\ &+ \Big|\sum_{z \leqq (K, \cdots, K)} x_z^n \mu(E \cap Y_z^n) - m(E)\Big| \\ &\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon\,. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\int_E g_N d\mu = m(E)$.

If we extend each g_N to be zero off F_N and let $h_k = \sum_{N=1}^k g_N$ and $f = \lim_{k \to \infty} h_k = \sum_{N=1}^\infty g_N$, it suffices [Dunford and Schwartz, III, 3.6] to show the following three conditions are satisfied.

(i) $h_k \to f$ in measure,

(ii) for each $\varepsilon > 0$ there is a $E_\varepsilon \in \Sigma$ such that

$$\int_{X \sim E_\varepsilon} |h_k(s)|\, d|\mu| < \varepsilon,\ k = 1, 2, \cdots, \quad \text{and}$$

(iii) $\lim_{|\mu|(E) \to 0} \int_E |h_k|\, d|\mu| = 0$, uniformly in k.

The first two conditions follow easily from the exhaustive nature of $\{F_N\}$. If $\varepsilon > 0$ is given, choose $\delta > 0$ such that $|\mu|(E) < \delta$ implies $|m|(E) < \varepsilon$.

Then for any k and any $E \in \Sigma$, $|\mu|(E) < \delta$, we have

$$\int_E |h_k|\, d|\mu| = \int_{E \cap \left(\bigcup_{N=1}^k F_N\right)} |h_k|\, d|\mu| = |m|\Big(E \cap \Big(\bigcup_{N=1}^k F_N\Big)\Big) < \varepsilon\,.$$

Thus for $E \in \Sigma$,

$$\int_E f d\mu = \sum_{N=1}^{\infty} \int_E g_N d\mu = \sum_{N=1}^{\infty} m(E \cap F_N) = m(E)$$

since $\{F_N\}$ is exhausting in X.

COROLLARY 3.11. *Let (X, Σ, μ) be a positive bounded finitely additive measure space. If m is a finitely additive $\boldsymbol{R}$-valued measure, then there exists a μ-integrable function $f: X \to \boldsymbol{R}$ such that $m(E) = \int_E f d\mu$, $\forall E \in \Sigma$, iff*

(a) *m is bounded, μ-continuous, and*

(b) *for all $\delta > 0$ there exists $F_\delta \subset X$, $F_\delta \in \Sigma$ such that*

(i) *$\mu(X \sim F_\delta) < \delta$,*

(ii) *$A_m(F_\delta)$ is bounded and*

(iii) *m has locally exhausting small average range in F_δ.*

Proof. If μ is positive then $\mu = |\mu|$ and hence $A_m(E) = A_m^*(E)$.

4. Examples. The failure of absolute continuity and boundedness to imply the existence of a density arises, it appears, from the lack of appropriate decompositions of the space which are obtainable in the countably additive case on a σ-algebra.

When the domain is a σ-algebra, it is impossible to suitably separate the support of countably additive measures and finitely additive measures which yields the failure. If m is Lebesgue measures on $[0, 1]$ and Σ the Lebesgue measurable subsets of $[0, 1]$, we have, for any nonzero $\mu \in [L^\infty(m)]^* = ba(\Sigma, m)$ such that $\mu \geqq 0$ and μ is purely finitely additive, that m is $(m + \mu)$-continuous. However there exists no density f such that $m(E) = \int_E f d(m + \mu) = \int_E f dm + \int_E f d\mu$ since $\int_E f d\mu$ must be identically zero, (otherwise it is purely finitely additive) and hence $f = 1$ a.e. Thus $\mu \equiv 0$ on $\sum$ which yields the desired contradiction.

If the doman is a field, not a σ-field, then we can illustrate the failure utilizing countably additive measures since we do not have a Hahn decomposition. Let $X = [0, 1)$, Σ the field generated by the half open intervals, $[a, b)$. Let m represent Lebesgue measure on $[0, 1)$ and choose a Lebesgue measurable set $A \subset [0, 1)$ which intersects every interval in a set with positive Lebesgue measure. Define $m(E) = \mu(E \cap A) - \mu(E \cap A^c)$, $E \in \Sigma$. Of course $A \notin \Sigma$. Then m is μ-continuous and m is bounded, in fact $|m| = \mu$. Now m cannot be an indefinite integral with respect to $|m|$ since for $E \in \Sigma^+$, $\delta(A_m(E)) = 2$ and hence m does not even have locally small average range.

A similar example can be used to show that while indefinite integrals need have locally small dominated average range they need not have even locally bounded average range. Let X, Σ, A, and m be as above and $v(E) = \int_E x dm$.

Then if $E \in \Sigma^+$, there exists a subset $F \in \Sigma^+$, $F \subset E$, such that $m(F) = 0$ and yet $v(F) \neq 0$. Then by m-continuity of v there are sets, $\{B\}$, $B \subset F$ such that the values $\{m(B)\}$ are arbitrarily small and yet $\{v(B)\}$ are uniformly bounded away from zero and hence the average range is never bounded.

The above examples depend upon a lack of suitable decompositions of the underlying space. The effect of appropriate Hahn decompositions is to eliminate many of the difficulties.

DEFINITION 4.1. Let $\mu: \Sigma \to \boldsymbol{R}$ be a bounded finitely additive measure. Then μ has a *Hahn decomposition* iff there exist disjoint sets A, $B \in \Sigma$, $X = A \cup B$, such that $\mu^+(B) = \mu^-(A) = 0$.

μ has an *approximate Hahn decomposition* iff for each $\varepsilon > 0$ there exists disjoint sets A_ε, $B_\varepsilon \in \Sigma$, $X = A_\varepsilon \cup B_\varepsilon$, such that $\mu^+(B_\varepsilon) < \varepsilon$ and $\mu^-(A_\varepsilon) < \varepsilon$.

μ has an *exhaustive Hahn decomposition* iff there exist two increasing sequences $\{A_n\}$, $\{B_n\} \subset \Sigma$ such that $\mu^+(B_n) = \mu^-(A_n) = 0$ and $|\mu|(X \sim (A_n \cup B_n)) \to 0$ as $n \to \infty$.

An exhaustive Hahn decomposition is equivalent to the countably additive extension on the Stone space having a Hahn decomposition where each set is, within a null set, a countable union of images from Σ^+. The second example in this section shows that finitely additive bounded measures need not have exhaustive Hahn decompositions. Darst [3, Lemma 2.1] has shown, however, that every finitely additive measure has an approximate Hahn decomposition and, of course, every countably additive measure on a σ-field has a Hahn decomposition.

The Radon-Nikodym theorem simplifies when the integrating measure has an exhausting Hahn decomposition as the following simple lemmas demonstrate.

LEMMA 4.2. *If μ is a bounded finitely additive measure on (X, Σ), Σ a field, then there exists a μ-integrable f such that $|\mu|(E) = \int_E f d\mu$, iff μ has an exhaustive Hahn decomposition. If Σ is a σ-field then $|\mu|(E) = \int_E f d\mu$ iff μ has a Hahn decomposition.*

LEMMA 4.3. *If μ is a bounded finitely additive measure with an exhaustive Hahn decomposition, then any bounded finitely additive* *measure has locally exhaustive small dominated average range with respect to μ iff it has locally exhaustive small average range.*

These lemmas yield the following theorem.

THEOREM 4.4. *Let (X, Σ, μ) be a bounded finitely additive measure space with an exhaustive Hahn decomposition. If m is a finitely additive $\boldsymbol{R}$-valued measure, then there exists a μ-integrable function $f: X \to \boldsymbol{R}$ such that $m(E) = \int_E f d\mu$, $\forall E \in \Sigma$ iff*

(a) *m is bounded, μ-continuous, and*

(b) *for all $\delta > 0$ there exists $F_\delta \subset X$, $F_\delta \in \Sigma$, such that*

(i) *$\mu(X \sim F_\delta) < \delta$*

(ii) *$A_m(F_\delta)$ is bounded and*

(iii) *m has locally exhausting small average range in F_δ.*

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PEIRCE IDEALS IN JORDAN TRIPLE SYSTEMS

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We show that an ideal in a Peirce space $J_i (i = 1, 1/2, 0)$ of a Jordan triple system J is the Peirce i-component of a global ideal precisely when it is invariant under the multiplications $L(J_{1/2}, J_{1/2}), P(J_{1/2})P(J_{1/2})$ (for $i=1$); under $L(J_{1/2}, J_{1/2})$, $P(J_{1/2})P(J_{1/2})$, $P(J_{1/2})P(e)P(J_{1/2})$, $L(J_{1/2}, e)P(J_0, J_{1/2})$ (for $i = 0$); under $L(J_1)$, $L(J_0)$, $L(J_{1/2}, e)L(e, J_{1/2})$, $L(J_{1/2}, e)P(e, J_{1/2})$ (for $i = 1/2$). We use this to show that the sub triple systems J_1 and J_0 are simple when J is. The method of proof closely follows that for Jordan algebras, but requires a detailed development of Peirce relations in Jordan triple systems.

Throughout we consider Jordan triple systems (henceforth abbreviated JTS) with basic product $P(x)y$ linear in y and quadratic in x, with derived trilinear product $\{xyz\} = P(x, z)y = L(x, y)z$, over an arbitrary ring Φ of scalars. Because we are already overburdened with subscripts and indices, we prefer not to treat the general case of Jordan pairs directly, but rather derive it via hermitian JTS. For basic facts about JTS and Jordan pairs we refer to [1], [3], [6]. Our analysis of Peirce ideals will closely follow that for Jordan algebras; although the basic lines of our treatment are the same as in [4], the triple system case requires such horrible computations that we do not carry out so fine an analysis, but concentrate just on the main simplicity theorem.

1. Peirce relations in Jordan triple systems.

Any Jordan triple system satisfies the general identities

$$\text{(JT1)} \quad L(x, y)P(x) = P(x)L(y, x)$$

$$\text{(JT2)} \quad L(x, P(y)x) = L(P(x)y, y)$$

$$\text{(JT3)} \quad P(P(x)y) = P(x)P(y)P(x)$$

and the linearization

$$\text{(JT3')} \quad P(\{xyz\}) + P(P(x)y, P(z)y) = P(x)P(y)P(z) + P(z)P(y)P(x) + P(x, z)P(y)P(x, z) .$$

A more useful version of this is the identity

$$\text{(JT4)} \quad P(\{xyz\}) = P(x)P(y)P(z) + P(z)P(y)P(x) + L(x, y)P(z)L(y, x) - P(P(x)P(y)z, z) .$$

Other basic identities we require are

$$\begin{aligned}
&\text{(JT5)} \quad L(x, y)P(z) + P(z)L(y, x) = P(L(x, y)z, z)\\
&\text{(JT6)} \quad P(x)P(y, z) = L(x, y)L(x, z) - L(P(x)y, z)\\
&\text{(JT7)} \quad P(y, z)P(x) = L(z, x)L(y, x) - L(z, P(x)y)\\
&\text{(JT8)} \quad 2P(x)P(y) = L(x, y)^2 - L(P(x)y, y)\\
&\text{(JT9)} \quad [L(x, y), L(z, w)] = L(L(x, y)z, w) - L(z, L(y, x)w) .
\end{aligned}$$

(See for example JP1-3, 20, 21, 12-13, 9 in [1, pp. 13, 14, 19, 20].)

PEIRCE DECOMPOSITIONS. Now let e be a *tripotent*, $P(e)e = e$. Then J decomposes into a direct sum of Peirce spaces

$$J = J_1 \oplus J_{1/2} \oplus J_0$$

relative to e, where the *Peirce projections* are

$$\text{(1.1)} \qquad \begin{aligned} &E_1 = P(e)P(e) , \quad E_{1/2} = L(e, e) - 2P(e)P(e) ,\\ &E_0 = B(e, e) = I - L(e, e) + P(e)P(e) . \end{aligned}$$

We have

$$\text{(1.2)} \qquad L(e, e) = 2iI \quad \text{on} \quad J_i , \qquad P(e) = 0 \quad \text{on} \quad J_{1/2} + J_0 .$$

Note that $P(e)$ is not the identity on J_1, though $J_1 = P(e)J$: it induces a map of period 2 which is an involution of the triple structure and is denoted by $x \to x^*(x \in J_1)$. For reasons of symmetry we introduce a trivial involution $x \to x$ on J_0, so $*$ is defined on $J_1 + J_0$:

$$\text{(1.3)} \qquad x_1^* = P(e)x_1 , \quad x_0^* = x_0 .$$

Note that if J is a Jordan algebra and e is actually an idempotent, then $x_1^* = x_1$ too.

The Peirce relations describe how the Peirce spaces multiply. Let i be either 1 or 0, and $j = 1 - i$ its complement. Then just as in Jordan algebras we have

$$\text{(1.4)} \qquad \begin{aligned}
&\text{(PD1)} \quad P(J_i)J_i \subset J_i , \quad P(J_i)J_j = P(J_i)J_{1/2} = 0\\
&\text{(PD2)} \quad P(J_{1/2})J_{1/2} \subset J_{1/2} , \quad P(J_{1/2})J_i \subset J_j\\
&\text{(PD3)} \quad \{J_iJ_iJ_{1/2}\} \subset J_{1/2} , \quad \{J_{1/2}J_{1/2}J_i\} \subset J_i\\
&\text{(PD4)} \quad \{J_iJ_{1/2}J_j\} \subset J_{1/2}\\
&\text{(PD5)} \quad \{J_iJ_jJ\} = 0 .
\end{aligned}$$

(For all this see [6] and [1, p. 44].) These show that the Peirce spaces are invariant under the multiplications mentioned in the introduction.

PEIRCE IDENTITIES. For a finer description of multiplication between Peirce spaces it is useful to reduce Jordan triple products to bilinear products whenever possible. We introduce a dot operation $x \cdot y$ (corresponding to $x \circ y$ in Jordan algebras) for elements a_k in Peirce spaces J_k, and a component product $E_i(x_{1/2}, y_{1/2})$ (corresponding to the J_i-component of $x_{1/2} \circ y_{1/2}$) as follows:

$$(1.5)\quad \begin{array}{lll} \text{(B1)} & x_1 \cdot y_{1/2} = y_{1/2} \cdot x_1 = \{x_1 e y_{1/2}\} \quad L(x_1) = L(x_1, e)\colon J_{1/2} \longrightarrow J_{1/2} \\ \text{(B2)} & x_0 \cdot y_{1/2} = y_{1/2} \cdot x_0 = \{x_0 y_{1/2} e\} \quad L(x_0) = P(x_0, e)\colon J_{1/2} \longrightarrow J_{1/2} \\ \text{(B3)} & x_1^2 = P(x_1)e,\ x_1 \cdot y_1 = \{x_1 e y_1\} \quad L(x_1) = L(x_1, e)\colon J_1 \longrightarrow J_1 \\ \text{(B4)} & E_1(x_{1/2}, y_{1/2}) = \{x_{1/2} y_{1/2} e\} \qquad J_{1/2} \times J_{1/2} \longrightarrow J_1 \\ \text{(B5)} & E_0(x_{1/2}, y_{1/2}) = \{x_{1/2} e y_{1/2}\},\ E_0(x_{1/2}) = P(x_{1/2})e\colon J_{1/2} \times J_{1/2} \longrightarrow J_0 \\ \text{(B6)} & L_1(x_{1/2}) = L(x_{1/2}, e),\ L_0(x_{1/2}) = L(e, x_{1/2}) \text{ so that} \\ & L_i(x_{1/2}) a_i = a_i \cdot x_{1/2},\ L_i(x_{1/2}) a_j = 0,\ L_i(x_{1/2}) y_{1/2} = E_j(y_{1/2}, x_{1/2})\ . \end{array}$$

It turns out that the only Jordan products x^2 or $x \circ y$ which are not expressible in triple terms are

$$x_0^2,\ x_0 \circ y_0,\ E_1(x_{1/2})\ .$$

The need to avoid these products causes many complications when passing from Jordan algebra results to triple system results.

For example, let e be an ordinary symmetric idempotent in an associative algebra A with involution, made into a triple system $J = JT(A, {}^*)$ via $P(x)y = xy^*x$. Then the Peirce spaces are the usual ones, $J_1 = A_{11}$, $J_{1/2} = A_{10} + A_{01}$, $J_0 = A_{00}$. The bilinear products we have introduced take the form

$$\begin{aligned} x_1 \cdot y_{1/2} &= x_1 y_{1/2} + y_{1/2} x_1 \\ x_0 \cdot y_{1/2} &= x_0 y_{1/2}^* + y_{1/2}^* x_1 \\ E_1(x_{1/2}, y_{1/2}) &= E_1(x_{1/2} y_{1/2}^* + y_{1/2}^* x_{1/2}) \\ E_0(x_{1/2}, y_{1/2}) &= E_0(x_{1/2} y_{1/2} + y_{1/2} x_{1/2})\ . \end{aligned}$$

This suggests that because of the * the products $x_0 \cdot y_{1/2}$ and $E_1(x_{1/2}, y_{1/2})$ are going to behave anomalously.

1.6. PROPOSITION. *The triple products of Peirce elements are expressed in terms of bilinear products by*

$$\begin{array}{ll} \text{(P1)} & P(x_{1/2}) y_{1/2} = x_{1/2} \cdot E_1(x_{1/2}, y_{1/2}) - y_{1/2} \cdot E_0(x_{1/2}) \\ \text{(P2)} & \{x_{1/2} y_{1/2} z_{1/2}\} = x_{1/2} \cdot E_1(z_{1/2}, y_{1/2}) + z_{1/2} \cdot E_1(x_{1/2}, y_{1/2}) \\ & \qquad - y_{1/2} \cdot E_0(x_{1/2}, z_{1/2}) \\ \text{(P3)} & \{x_{1/2} a_i y_{1/2}\} = E_j(x_{1/2}, a_i^* \cdot y_{1/2}) = E_j(y_{1/2}, a_i^* \cdot x_{1/2}) \\ \text{(P4)} & \{x_{1/2} y_{1/2} a_i\} = E_i(x_{1/2}, a_i^* \cdot y_{1/2}) \\ \text{(P5)} & \{a_i b_i z_{1/2}\} = a_i \cdot (b_i^* \cdot z_{1/2}) \end{array}$$

(P6) $\{a_i z_{1/2} b_j\} = a_i \cdot (z_{1/2} \cdot b_j^*) = (a_i^* \cdot z_{1/2}) \cdot b_j$

(P7) $e \cdot z_{1/2} = z_{1/2}$

(P8) $E_i(x_{1/2}, y_{1/2})^* = E_i(y_{1/2}, x_{1/2})$

and we can write

(P9) $L(x_{1/2}, a_i) = L_i(x_{1/2} \cdot a_i^*),\ L(a_i, x_{1/2}) = L_j(a_i^* \cdot x_{1/2})$.

The triple product of elements $x = x_1 + x_{1/2} + x_0$, $y = y_1 + y_{1/2} + y_0$ *may be written as*

$$\begin{aligned} P(x)y &= P(x_1)y_1 + P(x_0)y_0 + P(x_{1/2})y_{1/2} + P(x_{1/2})(y_1 + y_0) + \{x_1 y_{1/2} x_0\} \\ &\quad + \{x_1 y_1 x_{1/2}\} + \{x_0 y_0 x_{1/2}\} + \{x_1 y_{1/2} x_{1/2}\} + \{x_0 y_{1/2} x_{1/2}\} \\ &= P(x_1)y_1 + P(x_0)y_0 + \{x_{1/2} \cdot E_1(x_{1/2}, y_{1/2}) - y_{1/2} \cdot E_0(x_{1/2})\} \\ &\quad + P(x_{1/2})(y_1 + y_0) + x_1 \cdot (x_0 \cdot y_{1/2}) + x_1 \cdot (y_1^* \cdot x_{1/2}) + x_0 \cdot (y_0 \cdot x_{1/2}) \\ &\quad + E_1(x_{1/2}, x_1^* \cdot y_{1/2}) + E_0(x_{1/2}, x_0 \cdot y_{1/2}) . \end{aligned} \tag{1.7}$$

Proof. Most of these product rules can be established either by using JT5 to move $L(x, y)$ inside a triple product $P(z)w$, or by using the linearization of JT2 to interchange x and z in a product $\{x(P(y)z)w\}$. Thus (P1) is $P(x)y = P(x)\{yee\}$ (by 1.2)) $= \{\{eyx\}ex\} - \{ey(P(x)e)\}$ (by JT5) $= E_1(x, y) \cdot x - y \cdot E_0(x)$, and (P2) is its linearization. (P7) follows from PD2, $\{eez_{1/2}\} = z_{1/2}$, and (P8) is vacuous for $i = 0$ by triviality of * and symmetry of E_0, while for $i = 1$ $P(e)\{xye\} = P(e)L(e, y)x = -L(y, e)P(e)x + P(\{yee\}, e)x = -0 + \{yxe\}$ by JT5. For (P3)-(P6) we will need (P9),

$$\begin{aligned} L(x_{1/2}, a_1) &= L(x_{1/2} \cdot a_1^*, e) \qquad & L(a_1, x_{1/2}) &= L(e, a_1^* \cdot x_{1/2}) \\ L(x_{1/2}, a_0) &= L(e, x_{1/2} \cdot a_0) \qquad & L(a_0, x_{1/2}) &= L(a_0 \cdot x_{1/2}, e) . \end{aligned}$$

To establish this for a_1 we note $L(x_{1/2}, a_1) = L(x_{1/2}, P(e)a_1^*) = -L(a_1^*, P(e)x_{12}) + L(\{x_{1/2} e a_1^*\}, e)$ (linearized JT2) $= L(x_{1/2} \cdot a_1^*, e)$ and dually for $L(a_1, x_{1/2})$; for a_0 we have $L(x_{1/2}, a_0) = L(\{x_{1/2}ee\}, a_0) = -L(\{x_{1/2}a_0 e\}, e) + L(x_{1/2}, \{eea_0\}) + L(e, \{ex_{1/2}a_0\}) = -0 + 0 + L(e, x_{1/2} \cdot a_0)$ and dually for $L(a_0, x_{1/2})$. By B6 we can write these in the uniform manner (P9). Applying these to $x_{1/2}$ yields (P3) and (P4) respectively, and applying them to a_i, b_j respectively yields (P5) and (P6). □

Even in a Jordan algebra the products $P(x_i)y_i$ and $P(x_{1/2})y_i$ cannot be reduced to bilinear products if there is no scalar $1/2 \in \Phi$ (though $2P(x_{1/2})y_i$, and more generally $P(x_{1/2}, y_{1/2})a_i$, can be reduced by (P3)).

It will be convenient to introduce the abbreviation

$$\begin{aligned} &P^*(x_{1/2}) = {}^* \circ P(x_{1/2}) \circ {}^* \quad (\text{i.e., } P^*(x_{1/2})a_1 = P(x_{1/2})a_1^* , \\ &P^*(x_{1/2})a_0 = (P(x_{1/2})a_0)^*, \text{ so } P(P^*(x_{1/2})a_i) = P^*(x_{1/2})P(a_i)P^*(x_{1/2})) . \end{aligned} \tag{1.8}$$

We now list the basic Peirce identities. Many of these have appeared in [6], or in [1], [2] disguised as alternative triple identities.

1.9. PEIRCE IDENTITIES. *The following identities hold for elements* $a_i, b_i, c_i \in J_i (i = 1, 0, j = 1 - i)$ *and* $x, y, z \in J_{1/2}$:

(PI1) *we have a Peirce specialization* $a_i \to L(a_i)$ *of* J_i *in* $\mathrm{End}(J_{1/2})$:

(i) $P(a_i)b_i \cdot z = a_i \cdot (b_i^* \cdot (a_i \cdot z))$ $\quad L(P(a_i)b_i^*) = L(a_i)L(b_i)L(a_i)$

(ii) $e \cdot z = z$ $\quad L(e) = Id$

(iii) $a_1^2 \cdot z = a_1 \cdot (a_1 \cdot z)$ $\quad L(a_1^2) = L(a_1)^2$

(iv) $(a_1 \cdot b_1) \cdot z = a_1 \cdot (b_1 \cdot z) + b_1 \cdot (a_1 \cdot z)$

$$L(a_1 \cdot b_1) = L(a_1)L(b_1) + L(b_1)L(a_1)$$

(PI2) $P(a_i)E_i(x, y)^* = E_i(a_i \cdot x, a_i^* \cdot y)$

(PI3) $L(a_i, b_i)E_i(x, y) = E_i(a_i \cdot (b_i^* \cdot x), y) + E_i(x, a_i^* \cdot (b_i \cdot y))$

(PI4) $a_1 \cdot E_1(x, y) = E_1(a_1 \cdot x, y) + E_1(x, a_1^* \cdot y)$

(PI5) $P(z)E_i(x, y) = E_j(z, E_j(y, z) \cdot x) - E_j(P(z)x, y)$

(PI6) $P(E_i(x, y))a_i = P(x)P^*(y)a_i + P^*(y)P(x)a_i + E_i(x, P(y)(a_i^* \cdot x))$

(PI7) $\{P(x)a_i\} \cdot y + P(x)(a_i \cdot y) = E_i(x, y) \cdot (a_i^* \cdot x)$

(PI8) $\{P^*(x)a_i\} \cdot y + a_i \cdot P(x)y = E_i(a_i \cdot x, y) \cdot x$

(PI9) $P(x)\{a_1 x b_0\} = P(x)a_1 \cdot (b_0 \cdot x) = P(x)b_0 \cdot (a_1^* \cdot x)$

(PI10) $P(a_i \cdot x)b_j = P(a_i)P^*(x)b_j,\ P(a_i \cdot x)b_i = P^*(x)P(a_i)b_i$

(PI11) $P(a_i)P(x)b_j = P^*(a_i^* \cdot x)b_j,\ P(x)P(a_i)b_i = P^*(a_i^* \cdot x)b_i$

(PI12) $L(a_i, b_i)P(x)c_j = P(a_i \cdot (b_i^* \cdot x), x)c_j = E_i(a_i \cdot (b_i^* \cdot x), c_j^* \cdot x)$

(PI13) $L(a_i, b_i)P^*(x)c_j = P^*(a_i^* \cdot (b_i \cdot x), x)c_j = E_i(c_j \cdot x, a_i \cdot (b_i^* \cdot x))$

(PI14) $P(x)\{a_i b_i c_i\} = P(x, b_i \cdot (a_i^* \cdot x))c_i = E_j(x, c_i^* \cdot (b_i \cdot (a_i^* \cdot x)))$

(PI15) $E_0(a_0 \cdot x) = P(a_0)E_0(x),\ E_0(a_1 \cdot x) = P^*(x)a_1^2$

(PI16) $P(a_i \cdot x)y = a_i \cdot P(x)(a_i^* \cdot y)$

(PI17) $P(a_1 \cdot x, x)y = a_1 \cdot P(x)y + P(x)(a_1^* \cdot y)$.

Proof. The Peirce specialization relation PI1(i) follows from JT5, using B6: $P(a_i)b_i \cdot z = L_i(z)P(a_i)b_i = \{-P(a_i)L_j(z) + P(L_i(z)a_i, a_i)\}b_i = -0 + \{(z \cdot a_i)b_i a_i\}$(by PD1) $= a_i \cdot (b_i^* \cdot (a_i \cdot z))$ by P5. We have already noted $e \cdot z_{1/2} = z_{1/2}$, whence (ii). Setting $b_1 = e$ in (i) yields (iii), and linearization yields (iv).

The identities involving the E_i follow from JT5 and JT4. For PI2 and PI5 we have B6 $P(u)E_i(x, y) = P(u)L_j(y)x = -L_i(y)P(u)x + \{(L_i(y)u)xu\}$ (by JT5); when $u = a_i$ we get $-0 + \{(a_i \cdot y)xa_i\} = E_i(a_i \cdot y, a_i^* \cdot x)$(by P4) as in PI2, and when $u = z$ we get $-E_j(P(z)x, y) + E_j(z, x \cdot E_j(z, y)^*)$ (by P4) $= E_j(z, E_j(y, z) \cdot x) - E_j(P(z)x, y)$ (by P8) as in PI5. For PI3, $L(a_i, b_i)E_i(x, y) = L(a_i, b_i)L_j(y)x = L_j(y)L(a_i, b_i)x -$

$[L_j(y), L(a_i, b_i)]x = E_i(L(a_i, b_i)x, y) - L(L_j(y)a_i, b_i)x + L(a_i, L_i(y)b_i)x$ (by JT9) $= E_i(a_i \cdot (b_i^* \cdot x), y) - 0 + \{a_i(b_i \cdot y)x\} = E_i(a_i \cdot (b_i^* \cdot x), y) + E_i(x, a_i^* \cdot (b_i \cdot y))$ (by P4). PI4 is the special case $b_1 = e$ of PI3. For PI6 we use JT3′ for $i = 1$: $P(\{xye\})a_1 = \{P(x)P(y)P(e) + P(e)P(y)P(x) - P(P(x)y, P(e)y) + P(e, x)P(y)P(e, x)\}a_1 = P(x)P(y)a_1^* + (P(y)P(x)a_1)^* - 0 + E_1(x, P(y)(a_1^* \cdot x))$, while for $i = 0$ we use JT4: $P(\{xey\})a_0 = \{P(x)P(e)P(y) + P(y)P(e)P(x) + L(x, e)P(y)L(e, x) - P(P(x)P(e)y, y)\}a_0 = P(x)(P(y)a_0)^* + P(y)(P(x)a_0)^* + E_0(x, P(y)(a_0 \cdot x)) - 0$.

The identities involving $P(x)a_i$ are established in the same ways. For (PI7), $P(x)a_i \cdot y + P(x)(a_i \cdot y) = \{L_j(y)P(x) + P(x)L_i(y)\}a_i = P(L_j(y)x, x)a_i = P(E_i(x, y), x)a_i$ (by JT5) $= E_i(x, y) \cdot (a_i^* \cdot x)$ (by P5). For (PI8) we use linearized JT1: for $i = 1$, $\{(P(x)a_1^*)ye\} + \{(P(x)y)a_1^*e\} = \{x\{a_1^*xy\}e\}$, for $i = 0$ $\{(yP(x)a_0)e\} + \{a_0(P(x)y)e\} = \{\{a_0xy\}xe\}$, and we use P8. For (PI9), $P(x)\{a_ixa_j\} = P(x)L(a_i, x)a_j = L(x, a_i)P(x)a_j$ (by JT1) $= \{xa_iP(x)a_j\} = P(x)a_j \cdot (a_i^* \cdot x)$. For (PI10) with $i = 1$ we have by JT4 that $P(\{a_1ex\})b_k = \{P(a_1)P(e)P(x) + P(x)P(e)P(a_1) - P(P(a_1)P(e)x, x) + L(a_1, e)P(x)L(e, a_1)\}b_k = \{P(a_1)P(e)P(x) + P(x)P(e)P(a_1)\}b_k$. If $k = 0$ this becomes $P(a_1)P(e)P(x)b_0 = P(a_1)(P(x)b_0)^* = P(a_1)P^*(x)b_0$, while for $k = 1$ becomes $P(x)P(e)P(a_1)b_1 = P(x)(P(a_1)b_1)^* = P^*(x)P(a_1)b_1$ by (1.8). Similarly if $i = 0$ we have $P(\{a_0xe\})b_k = \{P(a_0)P(x)P(e) + P(e)P(x)P(a_0) - P(P(a_0)P(x)e, e) + L(a_0, x)P(e)L(x, a_0)\}b_k = \{P(a_0)P(x)P(e) + P(e)P(x)P(a_0)\}b_k$, reducing if $k = 0$ to $P(e)P(x)P(a_0)b_0 = P^*(x)P(a_0)b_0$ and if $k = 1$ to $P(a_0)P(x)P(e)b_1 = P(a_0)P^*(x)b_1$. Since * is an involution on J_i, J_j, (PI11) follows by applying * to (PI10) (with a_i, b_k replaced by a_i^*, b_k^*). Similarly (PI13) follows by applying * to (PI12) (with a_i, b_i replaced by a_i^*, b_i^*), where (PI12) follows from JT5: $L(a_i, b_i)P(x)c_j = \{-P(x)L(b_i, a_i) + P(\{a_ib_ix\}, x)\}c_j = P(a_i \cdot (b_i^* \cdot x), x)c_j$ (by P5) $= E_i(a_i \cdot (b_i^* \cdot x), c_j^* \cdot x)$ (by P3). For (PI14), $P(x)\{a_ib_ic_i\} = -L(b_i, a_i)P(x)c_i + P(\{b_ia_ix\}, x)c_i$ (by JT5) $= -0 + \{(b_i \cdot (a_i^* \cdot x))c_ix\} = E_j(x, c_i^* \cdot (b_i \cdot (a_i^* \cdot x)))$ (by P3). (PI15) is just the particular case $b = e$ of (PI10). For (PI16) with $i = 0$, $P(a_0 \cdot x)y = E_1(a_0 \cdot x \cdot y) \cdot (a_0 \cdot x) - E_0(a_0 \cdot x) \cdot y = a_0 \cdot \{E_1(a_0 \cdot x, y)^* \cdot x\} - P(a_0)E_0(x) \cdot y$ (by PI15) $= a_0 \cdot \{E_1(y, a_0 \cdot x) \cdot x\} - a_0 \cdot \{E_0(x) \cdot (a_0 \cdot y)\}$ (by PI1i) $= a_0 \cdot \{E_1(x, a_0 \cdot y) \cdot x - E_0(x) \cdot (a_0 \cdot y)\}$ (by symmetry of P3) $= a_0 \cdot \{P(x)(a_0 \cdot y)\}$. For $i = 1$, $P(a_1 \cdot x)y = E_1(a_1 \cdot x, y) \cdot (a_1 \cdot x) - E_0(a_1 \cdot x) \cdot y = \{-a_1 \cdot (E_1(a_1 \cdot x, y) \cdot x)\} + \{E_1(a_1^2 \cdot x, y) + E_1(a_1 \cdot x, a_1^* \cdot y)\} \cdot x - P^*(x)a_1^2 \cdot y$ (by (PI1iv), (PI4), (PI15)) $= -a_1 \cdot (E_1(a_1 \cdot x, y) \cdot x) + P(a_1)E_1(x, y)^* \cdot x + E_1(a_1^2 \cdot x, y) \cdot x + \{a_1^2 \cdot P(x)y - E_1(a_1^2 \cdot x, y) \cdot x\}$ (by (PI2), (PI8)) $= a_1 \cdot \{-E_1(a_1 \cdot x, y) \cdot x + E_1(x, y) \cdot (a_1 \cdot x) + a_1 \cdot [E_1(x, y) \cdot x - E_0(x) \cdot y]\}$ (by PI1i, iii) $= a_1 \cdot \{E_1(x, a_1^* \cdot y) - E_0(x) \cdot (a_1^* \cdot y)\}$ (by (PI4), (P6)) $= a_1 \cdot \mathrm{P}(x)(a_1^* \cdot y)$. (PI17) is just the linearization $a_1 \to a_1, e$ of PI16, or it follows from JT5. □

Observe that the proof of PI16 depended only on PI1, 2, 4, 8, 15. Note also that there is no analogue of PI1iv for J_0, so we cannot commute an $L(a_0)$ past an $L(b_0)$ at the expense of an $L(a_0 \cdot b_0)$, which means that if K_0 is an ideal in J_0 we do not have $L(J_0)L(K_0) \subset L(K_0)N(J_0)$ as we do for an ideal K_1 in J_1. Similarly there is no analogue of PI4 or PI17 for $i = 0$.

THE BRACKET PRODUCT ON $J_{1/2}$. Even more basic than the inherited triple product $P(x)y$ on $J_{1/2}$ are the *bracket products*

$$\langle xyz\rangle_i = E_i(x, y)\cdot z,\ \langle x; z\rangle_0 = E_0(x)\cdot z\ . \tag{1.10}$$

This gives two trilinear compositions on $J_{1/2}$, the one for $i = 0$ being symmetric in the first two variables

$$\langle xyz\rangle_0 = \langle yxz\rangle_0\ .$$

Formulas P1, P2 show

$$\begin{aligned} P(x)y &= \langle xyx\rangle_1 - \langle x; y\rangle_0 \\ \{xyz\} &= \langle xyz\rangle_1 + \langle zyx\rangle_1 - \langle xzy\rangle_0\ . \end{aligned} \tag{1.11}$$

In the special case of a maximal idempotent where $J_0 = 0$ we see $P(x)y = \langle xyx\rangle_1$, so the bracket product coincides with the triple product; Loos [1, 2] has abstractly characterized such products $\langle , , \rangle$ on such $J_{1/2}$ as *alternative triple systems*. We will show that in general even if $J_0 \neq 0$ the product $\langle xyz\rangle_1$ still behaves somewhat like an alternative triple product.

The interaction of the bracket with multiplications from the diagonal Peirce spaces is given by

$$\begin{aligned} L(a_i, b_i)\langle xyz\rangle_i = \langle L(a_i, b_i)x, y, z\rangle_i &+ \langle x, L(a_i^*, b_i^*)y, z\rangle_i \\ &- \langle x, y, L(b_i^*, a_i^*)\rangle_i \end{aligned} \tag{1.12}$$

$$a_1\cdot\langle xyz\rangle_1 = \langle a_1\cdot x, y, z\rangle_1 + \langle x, a_1^*\cdot y, z\rangle_1 - \langle x, y, a_1\cdot z\rangle_1 \tag{1.13}$$

$$L(a_i, b_i)\langle xyz\rangle_j = \langle x, y, L(a_i^*, b_i^*)z\rangle_j \tag{1.14}$$

$$L(a_i)\langle xyz\rangle_j = \langle y, x, L(a_i^*)z\rangle_j \tag{1.15}$$

$$a_1\cdot\langle xyx\rangle_1 - \langle a_1\cdot x, y, x\rangle_1 = E_0(x)\cdot(a_1^*\cdot y) - P(x)a_1^*\cdot y\ . \tag{1.16}$$

Unfortunately (1.13) with 1 replaced by 0 is false (even in triple systems $JT(A, {}^*)$ derived from associative algebras), and there does not seem to be any analogous identity for the interaction of $\langle\ ,\ ,\ \rangle_0$ with J_0.

To verify these identities, note for (1.12) $L(a_i, b_i)E_i(x, y)\cdot z = a_i\cdot(b_i^*\cdot(E_i(x, y)\cdot z))$ (by P5) $= \{a_i b_i E_i(x, y)\}\cdot z - E_i(x, y)\cdot(b_i^*\cdot(a_i\cdot z))$ (by linearized PI1i) $= \{E_i(a_i\cdot(b_i^*\cdot x), y) + E_i(x, a_i^*\cdot(b_i\cdot y))\}\cdot z - E_i(x, y)\cdot\{b_i^* a_i^* z\}$ (by PI3, P5) $= \langle L(a_i, b_i)x, y, z\rangle_i + \langle x, L(a_i^*, b_i^*)y, z\rangle_i - \langle x, y, L(b_i^*, a_i^*)z\rangle_i$ (by P5). We obtain (1.13) by setting $b_1 = e$ in (1.12). For (1.14),

$L(a_i, b_i)E_j(x, y)\cdot z = L(a_i)L(b_i^*)L(E_j(x, y))z = L(E_j(x, y))L(a_i^*)L(b_i)z$ (using P6 twice) $= \langle x, y, L(a_i^*, b_i^*)z\rangle_j$ (using P8). When $i = 1$ (1.15) follows from (1.14) by setting $b_i = e$; in general we argue as before $L(a_i)L(E_j(x, y))z = L(E_j(x, y)^*)L(a_i^*)z = \langle y, x, a_i^*\cdot z\rangle_j$. For (1.16), $a_1\cdot\langle xyx\rangle = a_1\cdot\{P(x)y + E_0(x)\cdot y\}$ (by (1.10), P1) $= \{-P^*(x)a_1\cdot y + E_1(a_1\cdot x, y)\cdot x\} + E_0(x)\cdot(a_1^*\cdot y)$ (by PI8, P6) $= E_0(x)\cdot(a_1^*\cdot y) - P(x)a_1^*\cdot y + \langle a_1\cdot x, y, x\rangle_1$.

Next we have some intrinsic bracket relations for the more important bracket $\langle x, y, z\rangle = \langle x, y, z\rangle_1$:

$$\langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle \tag{1.17}$$

$$\begin{aligned}\langle uv\langle xyx\rangle\rangle - \langle\langle uvx\rangle yx\rangle &= \langle x\langle vuy\rangle x\rangle - \langle xy\langle uvx\rangle\rangle \\ &= E_0(x)\cdot\langle vuy\rangle - E_0(E_0(x)\cdot v, u)\cdot y \\ &\quad + E_0(x, [E_1(x, v)\cdot u - E_0(x, u)\cdot v])\cdot y\end{aligned} \tag{1.18}$$

$$\langle\langle xyx\rangle yw\rangle - \langle xy\langle xyw\rangle\rangle = \{P(e)P(y)P(x) - P(x)P(y)\}e\cdot w \tag{1.19}$$

$$\langle x\langle yxy\rangle w\rangle - \langle xy\langle xyw\rangle\rangle = \{P(x)P(y) - P(e)P(y)P(x)\}e\cdot w \tag{1.20}$$

$$\langle\langle xyx\rangle vw\rangle - \langle x\langle vxy\rangle w\rangle = \{P(e)P(y, v)P(x) - P(x)P(y, v)\}e\cdot w \tag{1.21}$$

$$\langle\langle xyz\rangle yw\rangle - \langle x\langle yzy\rangle w\rangle = \{P(e)P(y)P(x, z) - P(x, z)P(y)\}e\cdot w \tag{1.22}$$

$$\langle\langle uvx\rangle yw\rangle + \langle x\langle vuy\rangle w\rangle = \langle\langle xyu\rangle vw\rangle + \langle u\langle yxv\rangle w\rangle\ . \tag{1.23}$$

Here (1.17) is just (1.13) for $a_1 = E_1(u, v)$, $a_1^* = E_1(v, u)$, while (1.23) is a consequence of the symmetry in uv, xy on the left side of (1.17). Setting $a_1 = E_1(u, v)$ in (1.16) yields $\langle uv\langle xyx\rangle\rangle - \langle\langle uvx\rangle yx\rangle (= \langle x\langle vuy\rangle x\rangle - \langle xy\langle uvx\rangle\rangle$ by (1.17)) $= E_0(x)\cdot(E_1(v, u)\cdot y) - P(x)E_1(v, u)\cdot y = E_0(x)\cdot(E_1(v, u)\cdot y) - E_0(x, E_0(u, x)\cdot v)\cdot y + E_0(P(x)v, u)\cdot y$ (by PI5) $= E_0(x)\cdot(E_1(v, u)\cdot y) - E_0(x, E_0(u, x)\cdot v)\cdot y + E_0(E_1(x, v)\cdot x, u)\cdot y - E_0(E_0(x)\cdot v, u)\cdot y$ (by P1) $= E_0(x)\cdot(E_1(v, u)\cdot y) - E_0(E_0(x)\cdot v, u)\cdot y + E_0(x, [E_1(x, v)\cdot u - E_0(x, u)\cdot v])\cdot y$ (by P3 and symmetry of E_0), which is (1.18). The formulas (1.19), (1.20), (1.21), (1.22) are respectively

$$E_1(\langle xyx\rangle, y) - E_1(x, y)^2 = \{P(e)P(y)P(x) - P(x)P(y)\}e \tag{1.19$'$}$$

$$E_1(x, \langle yxy\rangle) - E_1(x, y)^2 = \{P(x)P(y) - P(e)P(y)P(x)\}e \tag{1.20$'$}$$

$$E_1(\langle xyx\rangle, v) - E_1(x, \langle vxy\rangle) = \{P(e)P(y, v)P(x) - P(x)P(y, v)\}e \tag{1.21$'$}$$

$$E_1(\langle xyz\rangle, y) - E_1(x, \langle yzy\rangle) = \{P(e)P(y)P(x, z) - P(x, z)P(y)\}e\ . \tag{1.22$'$}$$

Here (1.19′) will follow by setting $v = y$ in (1.21′) (or $z = x$ in (1.22′)) and using (1.20′). For (1.20′) note $E_1(x, y)^2 = P(E_1(x, y))e = P(x)P^*(y)e + P^*(y)P(x)e + E_1(x, P(y)(x\cdot e))$ (by PI6) $= P(x)P(y)e + (P(y)P(x)e)^* + E_1(x, P(y)x) = E_1(x, \langle yxy\rangle - P(y)e\cdot x) + P(x)P(y)e + P(e)P(y)P(x)e =$

$E_1(x, \langle yxy\rangle) - \{x(P(y)e)x\} + P(x)P(y)e + P(e)P(y)P(x)e = E_1(x, \langle yxy\rangle) + P(e)P(y)P(x)e - P(x)P(y)e$. For (1.21′) note that $E_1(P(x)y + E_0(x)\cdot y, v) - E_1(x, E_1(v, x)\cdot y) = \{(P(x)y)ve\} + \{yE_0(x)v\}^* - \{xyE_1(v, x)^*\}$ (by P1, P3, P4)$-$ $\{L(P(x)y, v) + P(e)P(y, v)P(x) - L(x, y)L(x, v)\}e = \{P(e)P(y, v)P(x) - P(x)P(y, v)\}e$ by JT6. Finally, for (1.22′) we have $E_1(y, E_1(x, y)\cdot z)^* - E_1(x, E_1(y, z)\cdot y) = \{yzE_1(x, y)^*\}^* - \{xyE_1(y, z)^*\} = P(e)L(y, z)L(y, x)e - L(x,y)L(z,y)e = P(e)\{L(P(y)z,x) + P(y)P(x,z)\}e - \{L(x,P(y)z) + P(x,z)P(y)\}e$ (by JT6, JT7) $= E_1(P(y)z, x)^* - E_1(x, P(y)z) + \{P(e)P(y)P(x, z) - P(x, z)P(y)\}e = \{P(e)P(y)P(x, z) - P(x, z)P(y)\}e$ (by P8).

In the special case that $J_0 = 0$ we obtain the easy half of Loos′ characterization [1, p. 76] of alternative triple systems.

1.24. PROPOSITION. *If* $K_{1/2} \subset J_{1/2}$ *is a bracket subalgebra* ($\langle K_{1/2}K_{1/2}K_{1/2}\rangle \subset K_{1/2}$) *with* $E_0(K_{1/2}) = P(K_{1/2})e = 0$ *(for example,* $K_{1/2} = J_{1/2}$ *if* $J_0 = 0$, *or* $K_{1/2} = P(x)J_{1/2}$ *or* $K_{1/2} = P(x)J_{1/2} + \Phi x$ *principal inner ideals determined by an* $x \in J_{1/2}$ *with* $P(x)e = 0$), *then* $K_{1/2}$ *becomes an alternative triple system under the bracket*

$$\langle xyz\rangle = E_1(x, y)\cdot z = \{\{xye\}ez\} \qquad (x, y, z \in K_{1/2}) .$$

The Jordan triple product on $K_{1/2}$ *is then* $P(x)y = \langle xyx\rangle$.

Proof. The axioms for an alternative triple system are

$$\text{(AT1)} \quad \langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle$$
$$\text{(AT2)} \quad \langle uv\langle xyx\rangle\rangle = \langle\langle uvx\rangle yx\rangle$$
$$\text{(AT3)} \quad \langle xy\langle xyz\rangle\rangle = \langle\langle xyx\rangle yz\rangle .$$

Here (AT1) follows from (1.17), and (AT2), (AT3) from (1.18), (1.19) since $E_0(K_{1/2}) = P(K_{1/2})e = 0$. By (P1) we have $P(x)y = E_1(x, y)\cdot x = \langle xyx\rangle$ in this case.

If x has $P(x)e = 0$ then the inner ideals $K_{1/2} = P(x)J_{1/2} \subset P(x)J_{1/2} + \Phi x = K'_{1/2}$ kill e, $P(K_{1/2})e = P(K'_{1/2})e = 0$. Indeed, by JT3 we have $P(K_{1/2}) = P(x)P(J_{1/2})P(x)$, and by JT1 $P(K'_{1/2}) = P(P(x)J_{1/2}) + P(P(x)J_{1/2}, x) + \Phi P(x) = \{P(x)P(J_{1/2}) + L(x, J_{1/2}) + \Phi\}P(x)$. To see next that these inner ideals are bracket-closed subalgebras, first note that since $P(K'_{1/2})J_{1/2} \subset K_{1/2} \subset K'_{1/2}$ by innerness we have $\langle xyx\rangle = P(x)y \in K_{1/2}$, hence by linearization $\langle xyz\rangle + \langle zyx\rangle \in K_{1/2}$, for any $x, z \in K'_{1/2}$ and any $y \in J_{1/2}$. Next we show $\langle K_{1/2}J_{1/2}x\rangle$ and $\langle xJ_{1/2}K_{1/2}\rangle$ are contained in $K_{1/2}$; by skewness it suffices to prove the latter, where $\langle xJ_{1/2}K_{1/2}\rangle = E_1(x, J_{1/2})\cdot P(x)J_{1/2} \subset -P(x)(E_1(x, J_{1/2})^*\cdot J_{1/2}) + P(E_1(x, J_{1/2})\cdot x, x)J_{1/2}$ (by PI17) $\subset P(x)J_{1/2} + P(\langle xJ_{1/2}x\rangle, x)J_{1/2} \subset P(K'_{1/2})J_{1/2} \subset K_{1/2}$. Finally, $\langle K_{1/2}J_{1/2}K_{1/2}\rangle = E_1(K_{1/2}, J_{1/2})\cdot K_{1/2} \subset -P(x)(E_1(K_{1/2}, J_{1/2})^*\cdot J_{1/2}) + P(E_1(K_{1/2}, J_{1/2})\cdot x, x)J_{1/2} \subset P(x)J_{1/2} + P(\langle K_{1/2}J_{1/2}x\rangle, x)J_{1/2} \subset P(K'_{1/2})J_{1/2}$ (by the previous case) $\subset K_{1/2}$. Thus in fact we have the stronger closure $\langle K'_{1/2} J_{1/2} K'_{1/2}\rangle \subset K_{1/2}$. $\square$

In any alternative triple system we obtain an ordinary bilinear alternative multiplication by fixing the middle factor: the homotopes $A^{(u)}$ with products $x\cdot_u y = \langle xuy\rangle$ are alternative.

1.25. PROPOSITION. *If $K_{1/2}$ is a bracket-closed subspace of $J_{1/2}$ with $P(K_{1/2})e = 0$, then for any $u \in K_{1/2}$ the homotope $K_{1/2}^{(u)}$ with product*

$$x \cdot_u y = \langle xuy\rangle$$

is an alternative algebra. If u is a tripotent with $P(u)e = 0$ then we have an involutory map $x \to P(u)x = \bar{x}$ on $K_{1/2} = J_{1/2}(e) \cap J_1(u) = P(u)J_{1/2}(e)$, and the bracket can be recovered as

$$\langle xyz\rangle = (x \cdot_u \bar{y}) \cdot_u z \,. \tag{1.26}$$

If in addition $E_1(u, u) = \{uue\} = e$ then u acts as unit for $P(u)J_{1/2}(e)$, and $x \to \bar{x}$ is an involution of the multiplicative structure.

Proof. By 1.24 we know $K_{1/2}$ is an alternative triple system under the bracket, hence the homotope $K_{1/2}^{(u)}$ is an alternative algebra [1, p. 64]. When u is tripotent $P(u)^3 = P(u)$, so $P(u)$ is involutory on $P(u)J_1$, and furthermore for $x, y, z \in P(u)J_{1/2}$ we have $(x \cdot_u y)\cdot_u z - \langle x\bar{y}z\rangle = \langle\langle xuy\rangle uz\rangle - \langle x\langle uyu\rangle z\rangle = \{P(e)P(u)P(x, y) - P(x, y)P(u)\}e\cdot z$ (by 1.22) $= 0$ since $P(K_{1/2})e = P(u)P(J_{1/2})P(u)e = 0$. Thus we recover the bracket on $P(u)J_{1/2}$ from the bilinear product and the involution.

When $\{uue\} = E_1(u, u) = e$ in addition then u is a left unit, $u \cdot_u y = E_1(u, u)\cdot y = e\cdot y = y$. If we knew $x \to \bar{x}$ reversed multiplication this would imply $\bar{u} = u$ was also a right unit; we can also argue directly, $x \cdot_u u = \langle xuu\rangle = E_1(x, u)\cdot u = \{xuu\} - E_1(u, u)\cdot x + E_0(x, u)\cdot u = L(u, u)(P(u)^2x) - e\cdot x + 0$ (since $E_0(K_{1/2}) = 0$) $= P(P(u)u, u)P(u)x - x$ (using JT1) $= 2P(u)^2x - x = x$.

To see $x \to \bar{x}$ is indeed an involution, first use the right unit to see $x \cdot_u y = (x \cdot_u y) \cdot_u u = \langle x\bar{y}u\rangle$,

$$x \cdot_u y = \langle xuy\rangle = \langle x\bar{y}u\rangle \qquad (\text{when } \{uue\} = e)\,. \tag{1.27}$$

Then

$$\begin{aligned}\overline{x \cdot_u y} &= \langle u\langle xuy\rangle u\rangle \\ &= \langle\langle uxu\rangle yu\rangle - \{P(e)P(x, y)P(u) - P(u)P(x, y)\}e\cdot u \text{ (by 1.27)} \\ &= \langle\bar{x}yu\rangle - 0 \qquad (\text{again } P(K_{1/2})e = 0) \\ &= \bar{x} \cdot_u \bar{y} \qquad (\text{above})\,.\end{aligned}$$

Thus the involution condition is precisely (1.27).

The condition $E_1(u, u)\cdot y = y$ is necessary well as sufficient for (1.27) to hold. Indeed, using (1.21), (1.18) and $P(K_{1/2})e = 0$ one can show in general that $P(u)\{\langle xuy\rangle - \langle x\bar{y}u\rangle\} = \langle u\langle xuy\rangle u\rangle - \langle u\langle x\bar{y}u\rangle u\rangle = \langle\langle uyu\rangle xu\rangle - \langle uu\langle \bar{y}xu\rangle\rangle = \{\mathrm{Id} - L(E_1(u, u))\}\langle \bar{y}xu\rangle$, which again establishes sufficiency; for necessity set $x = u$, so $\langle uuy\rangle - \langle u\bar{y}u\rangle = E_1(u, u)\cdot y - P(u)\bar{y} = E_1(u, u)\cdot y - y$. $\square$

These alternative structures on the subsystems $P(u)J_{1/2}$ are important for the study of collinear idempotents [5]. These are families of tripotents $\{e_1, \cdots, e_n\}$ with $P(e_i)e_j = 0$, $\{e_ie_ie_j\} = e_j$ for $i \neq j$, and the $P(e_j)J_{1/2}(e_i) = J_{1/2}(e_i) \cap J_1(e_j)$ carry isomorphic alternative structures. (The motivating example is the collinear matrix units $\{e_{11}, e_{12}, \cdots, e_{1n}\}$ in $M_n(\Phi)$ under xy^tx.)

2. Ideal-building. A subspace $K \subset J$ is an *ideal* if it is both an *outer ideal*

(2.1) $$P(J)K \subset K$$

(2.2) $$L(J, J)K \subset K$$

and an *inner ideal*

(2.3) $$P(K)J \subset K\,.$$

If K is already an outer ideal, the inner condition (2.3) reduces to

(2.3′) $$P(k_i)J \subset K \text{ for some spanning set } \{k_i\} \text{ for } K\,.$$

Note that the operators $L(y, z)$ cannot be derived from the $P(x)$'s.

From now on we fix a tripotent e with corresponding Peirce decomposition

$$J = J_1 \oplus J_{1/2} \oplus J_0\,.$$

Since the Peirce projections (1.1) are multiplication operators, any ideal $K \lhd J$ breaks into Peirce pieces

$$K = K_1 \oplus K_{1/2} \oplus K_0 \qquad (K_i = K \cap J_i)\,.$$

Using the expression (1.7) for the product $P(x)y$ in terms of bilinear products, we obtain a componentwise criterion for K to be an ideal (exactly like that in Jordan algebras).

2.4. IDEAL CRITERION. *A subspace $K = K_1 \oplus K_{1/2} \oplus K_0$ is an ideal in the* JTS *$J = J_1 \oplus J_{1/2} \oplus J_0$ iff for $i = 1, 0$ and $j = 1 - i$ we have*

(C1) *K_i is an ideal in J_i*
(C2) $E_i(J_{1/2}, K_{1/2}) \subset K_i$
(C3) $J_i \cdot K_{1/2} \subset K_{1/2}$
(C4) $K_i \cdot J_{1/2} \subset K_{1/2}$
(C5) $P(J_{1/2})K_i \subset K_j$
(C6) $P(k_{1/2})J_i \subset K_j$ *for some spanning set* $\{k_{1/2}\}$ *for* $K_{1/2}$.

If $1/2 \in \Phi$ *then* (C5) *and* (C6) *are superfluous.*

Proof. Clearly the conditions are necessary, since any product with a factor in K must fall back in K. Just as in the Jordan algebra case, they also suffice. Outerness (2.1) $P(J)K \subset K$ follows by (1.7) since $P(J_i)K_i \supset K_i$ (by (C1)), $P(J_{1/2})K_i \subset K_j$ (by (C5)), $J_{1/2} \cdot E_1(J_{1/2}, K_{1/2}) \subset K_{1/2}$ (by (C2), (C4)), $K_{1/2} \cdot J_0 \subset K_{1/2}$ (by (C3)), $J_1 \cdot (J_0 \cdot K_{1/2}) \subset K_{1/2}$ (by (C3)), $J_i \cdot (K_i^* \cdot J_{1/2}) \subset K_{1/2}$ (by (C4), (C3) — note that $K_i^* = K_i$ for any ideal $K_i \lhd J_i$ since the involution is given by a multiplication), and $E_i(J_{1/2}, J_i^* \cdot K_{1/2}) \subset K_i$ (by (C3), (C2)).

Outerness (2.2) $L(J, J)K = P(J, K)J \subset K$ follows by the linearization of (1.7). First note

$$\text{(C2')} \quad E_i(K_{1/2}, J_{1/i}) \subset K_i$$

since $E_i(K_{1/2}, J_{1/2}) = E_i(J_{1/2}, K_{1/2})^* \subset K_i^* \subset K_i$. We have $\{J_i J_i K_i\} \subset K_i$ (by (C1)), $\{J_{1/2} J_i K_{1/2}\} \subset E_j(J_{1/2}, J_i^* \cdot K_{1/2}) \subset K_j$ (by P3, (C3), (C2)), $K_{1/2} \cdot E_1(J_{1/2}, J_{1/2}) \subset K_{1/2}$ (by (C3)), $J_{1/2} \cdot E_1(K_{1/2}, J_{1/2}) + J_{1/2} \cdot E_1(J_{1/2}, K_{1/2}) \subset K_{1/2}$ (by (C2′), (C2), (C4)), $J_{1/2} \cdot P(J_{1/2}, K_{1/2})e = J_{1/2} \cdot E_0(J_{1/2}, K_{1/2}) \subset K_{1/2}$ (by (C2), (C4)), $J_i \cdot (K_i^* \cdot J_{12}) + K_i \cdot (J_i^* \cdot J_{1/2}) \subset K_{1/2}$ (by (C4), (C3)), $E_i(K_{1/2}, J_i^* \cdot J_{1/2}) \subset E_i(K_{1/2}, J_{1/2}) \subset K_i$ (by (C2′)), and $E_i(J_{1/2}, K_i^* \cdot J_{1/2}) = E_i(J_{1/2} \cdot K_i \cdot J_{1/2}) \subset K_i$ (by (C4), (C2)).

Once K is outer we can apply (2.3′) to obtain innerness: for the spanning elements $k_r \in K_r$ we have $P(k_i)J = P(k_i)J_i \subset K_i$ by (C1) if $i = 1, 0$, while $P(k_{1/2})J_i \subset K_j$ by (C6) and $P(k_{1/2})J_{1/2} = k_{1/2} \cdot E_1(k_{1/2}, J_{1/2}) - J_{1/2} \cdot P(k_{1/2})e \subset K_{1/2} \cdot J_1 - J_{1/2} \cdot K_0 \subset K_{1/2}$ by P1, (C5), (C3), (C4). Thus K is an ideal.

When $1/2 \in \Phi$, (C5) and (C6) follow from (C2–C4) since $P(x) = 1/2P(x, x)$ where $P(J_{1/2}, J_{1/2})K_i = E_j(J_{1/2}, K_i^* \cdot J_{1/2}) \subset K_j$ by (C4), (C2), and $P(J_{1/2}, K_{1/2})J_i \subset E_j(J_{1/2}, J_i^* \cdot K_{1/2}) + E_j(K_{1/2}, J_i^* \cdot J_{1/2}) \subset K_j$ by (C3), (C2), (C2′). □

An ideal K_i in a diagonal Peirce space J_i is *invariant* if it is both *L-invariant*

$$L(J_{1/2}, J_{1/2})K_i = E_i(J_{1/2}, K_i^* \cdot J_{1/2}) \subset K_i \tag{2.5}$$

and if $i = 0$, also

$$L(J_{1/2}, e)P(J_0, J_{1/2})K_0 = E_0(J_{1/2}, J_0\cdot(K_0\cdot J_{1/2})) \subset K_0\ , \tag{2.6}$$

and *P-invariant*

$$P(J_{1/2})P(J_{1/2})K_i \subset K_i \tag{2.7}$$

and again if $i = 0$ also

$$P^*(J_{1/2})P(J_{1/2})K_0 = P(J_{1/2})P^*(J_{1/2})K_0 = P(J_{1/2})P(e)P(J_{1/2})K_0 \subset K_0\ . \tag{2.8}$$

Note that the maps $L(J_{1/2}, J_{1/2})$ and $P(J_{1/2})P(J_{1/2})$ automatically send J_i into itself (and $L(J_{1/2}, e)P(J_0, J_{1/2})$ and $P(J_{1/2})P(e)P(J_{1/2})$ send J_0 into itself).

An ideal $K_{1/2} \lhd J_{1/2}$ in the off-diagonal Peirce space is *invariant* if

$$L(J_i)K_{1/2} = J_i\cdot K_{1/2} \subset K_{1/2} \tag{2.9}$$

$$\begin{aligned} L_1(J_{1/2})L_0(J_{1/2})K_{1/2} &= L(J_{1/2}, e)L(e, J_{1/2})K_{1/2} = \langle K_{1/2}J_{1/2}J_{1/2}\rangle \subset K_{1/2} \\ L_1(J_{1/2})L_0(K_{1/2})J_{1/2} &= L(J_{1/2}, e)P(e, J_{1/2})K_{1/2} = \langle J_{1/2}K_{1/2}J_{1/2}\rangle \subset K_{1/2}\ . \end{aligned} \tag{2.10}$$

Note that these maps do send $J_{1/2}$ back into itself.

An alternate characterization of invariance in terms of the bracket products is that $K_{1/2}$ be a subspace satisfying

$$J_i\cdot K_{1/2} \subset K_{1/2} \tag{2.9$'$}$$

$$\langle J_{1/2}J_{1/2}K_{1/2}\rangle_1 + \langle J_{1/2}K_{1/2}J_{1/2}\rangle_1 + \langle K_{1/2}J_{1/2}J_{1/2}\rangle_1 \subset K_{1/2} \tag{2.10$'$}$$

$$\langle J_{1/2}K_{1/2}J_{1/2}\rangle_0 + \langle K_{1/2};\, J_{1/2}\rangle_0 \subset K_{1/2}\ , \tag{2.10$''$}$$

i.e., that $K_{1/2}$ be an ideal of the bracket algebra $J_{1/2}$. Clearly any invariant bracket ideal (2.9′)-(2.10″) is invariant in the sense of (2.9)-(2.10) and is an ordinary ideal by (1.11). Conversely, if $K_{1/2}$ is an invariant ordinary ideal it must be a bracket ideal: $\langle K_{1/2}J_{1/2}J_{1/2}\rangle_1 + \langle J_{1/2}K_{12}J_{1/2}\rangle_1$ is contained in $K_{1/2}$ by invariance (2.10), $\langle J_{1/2}J_{1/2}K_{1/2}\rangle_1 \subset J_1\cdot K_{1/2} \subset K_{1/2}$ by invariance (2.9), similarly $\langle J_{1/2}J_{1/2}K_{1/2}\rangle_0 \subset J_0\cdot K_{1/2} \subset K_{1/2}$ by (2.9), while $\langle J_{1/2}K_{1/2}J_{1/2}\rangle_0 = \langle K_{1/2}J_{1/2}J_{1/2}\rangle_0 \subset -\{J_{1/2}J_{1/2}K_{1/2}\} + \langle J_{1/2}J_{1/2}K_{1/2}\rangle_1 + \langle K_{1/2}J_{1/2}J_{1/2}\rangle_1 \subset K_{1/2}$ by ordinary idealness and closure under $\langle\ ,\ ,\ \rangle_1$, also $\langle K_{1/2};\, J_{1/2}\rangle_0 = \langle K_{1/2}J_{1/2}K_{1/2}\rangle_1 - P(K_{1/2})J_{1/2} \subset K_{1/2}$ for the same reason, with $\langle J_{1/2};\, K_{1/2}\rangle_0 \subset J_0\cdot K_{1/2} \subset K_{1/2}$ by (2.9).

If $1/2 \in \Phi$ then L-invariance (2.5) of $K_i \lhd J_i$ implies P-invariance (2.7) in view of JT8. It is not clear whether (2.5), (2.6) imply (2.8) when $1/2 \in \Phi$.

An important tool is the ability to flip an ideal from one diagonal Peirce space to another.

2.11. FLIPPING LEMMA. *If K_1 is an ideal in J_1 then*

$$K_0 = P(J_{1/2})K_1$$

is an ideal in J_0, which is invariant if K_1 is. If K_0 is an ideal in J_0 then

$$K_1 = P(J_{1/2})K_0 + P^*(J_{1/2})K_0$$

is an ideal in J_1, which again is invariant if K_0 is.

Proof. We handle both cases at once by proving

$$K_j = P(J_{1/2})K_i + P^*(J_{1/2})K_i$$

is an ideal inheriting invariance from K_i. Note again that $K_i^* = K_i$ for any ideal $K_i \lhd J_i$.

Outerness (2.1) follows from (PI11, 10):

$$P(a_j)P(x_{1/2})k_i = P^*(a_j^* \cdot x_{1/2})k_i \in P^*(J_{1/2})K_i$$
$$P(a_j)P^*(x_{1/2})k_i = P(a_j \cdot x_{1/2})k_i \in P(J_{1/2})K_i \,.$$

Outerness (2.2) follows from (PI12, 13):

$$L(a_j, b_j)P(x_{1/2})k_i = P(a_j \cdot (b_j^* \cdot x_{1/2}), x_{1/2})k_i \in P(J_{1/2})K_i$$
$$L(a_j, b_j)P^*(x_{1/2})k_i = P(a_j^* \cdot (b_j \cdot x_{1/2}), x_{1/2})k_i \in P^*(J_{1/2})K_i \,.$$

To see that K_j is inner (2.3′), for the spanning elements $P(x_{1/2})k_i$ and $P^*(x_{1/2})k_i$ we have

$$P(P(x_{1/2})k_i)J_j = P(x_{1/2})P(k_i)P(x_{1/2})J_j \subset P(x_{1/2})P(k_i)J_i \subset P(x_{1/2})K_i$$
$$P(P^*(x_{1/2})k_i)J_j = P^*(x_{1/2})P(k_i)P^*(x_{1/2})J_j \subset P^*(x_{1/2})P(k_i)J_i \subset P^*(x_{1/2})K_i$$

using (1.8) and innerness of K_i in J_i. Thus K_j is inner as well as outer, hence is an ideal in J_j.

If K_i is L-invariant (2.5) to begin with, then K_j will be L-invariant too:

$$\begin{aligned} L(x_{1/2}, y_{1/2})P(z_{1/2})k_i &= \{P(\{x_{1/2}y_{1/2}z_{1/2}\}, z_{1/2}) - P(z_{1/2})L(y_{1/2}, x_{1/2})\}k_i \quad \text{(by JT5)} \\ &\in P(J_{1/2})K_i + P(J_{1/2})L(J_{1/2}, J_{1/2})K_i \subset P(J_{1/2})K_i \\ &\qquad \text{(by } L\text{-invariance)} \end{aligned}$$

$$\begin{aligned} L(x_{1/2}, y_{1/2})P^*(z_{1/2})k_0 &= L(x_{1/2}, y_{1/2})P(e)P(z_{1/2})k_0 \\ &= \{P(\{x_{1/2}y_{1/2}e\}, e) - P(e)L(y_{1/2}, x_{1/2})\}P(z_{1/2})k_0 \quad \text{(by JT5)} \\ &\in P(J_1)P(J_{1/2})K_0 - (L(J_{1/2}, J_{1/2})P(J_{1/2})K_0)^* \\ &\subset P^*(J_{1/2})K_0 \quad \text{(by PI11, above, and } L\text{-invariance)}\,. \end{aligned}$$

L-invariance (2.6) only applies when $i = 1$. In this case it follows from L-invariance (2.5) of K_1: we have $E_0(J_{1/2}, K_1 \cdot J_{1/2}) = \{J_{1/2}K_1J_{1/2}\} \subset K_0$ by definition, and $J_0 \cdot (K_0 \cdot J_{1/2}) \subset K_1 \cdot J_{1/2}$ because $\{J_0(P(J_{1/2})K_1)J_{1/2}\} = -\{J_0(P(J_{1/2})J_{1/2})K_1\} + \{J_0J_{1/2}\{K_1J_{1/2}J_{1/2}\}\}$ (by JT2) $\subset \{J_0J_{1/2}K_1\}$ (by L-invariance of $K_1) = K_1\cdot(J_0\cdot J_{1/2})\subset K_1\cdot J_{1/2}$.

If in addition K_i is P-invariant (2.7) the same is true of K_j:

$$
\begin{aligned}
&P(x_{1/2})P(y_{1/2})(P(z_{1/2})k_i) = P(x_{1/2})(P(y_{1/2})P(z_{1/2})k_i)\in P(J_{1/2})K_i\\
&P(x_{1/2})P(y_{1/2})(P^*(z_{1/2})k_0) = P(x_{1/2})P(y_{1/2})P(e)P(z_{1/2})k_0\\
&\quad = \{P(\{x_{1/2}y_{1/2}e\}) + P(P(x_{1/2})P(y_{1/2})e,\ e) - P(e)P(y_{1/2})P(x_{1/2})\\
&\qquad - L(x_{1/2},\ y_{1/2})P(e)L(y_{1/2},\ x_{1/2})\}P(z_{1/2})k_0 \quad \text{(by JT4)}\\
&\qquad \subset\{P(J_1) - P(e)P(J_{1/2})P(J_{1/2}) - L(J_{1/2},\ J_{1/2})P(e)L(J_{1/2},\ J_{1/2})\}P(J_{1/2})K_0\\
&\qquad \subset P^*(J_{1/2})K_0 - L(J_{1/2},\ J_{1/2})P^*(J_{1/2})K_0 \quad \text{(by } P, L\text{-invariance of } K_0)\\
&\qquad \subset P^*(J_{1/2})K_0 \quad \text{(by above } L\text{-invariance of } K_1)\ .
\end{aligned}
$$

P-invariance (2.8) applies only when $i=1$. In this case it follows from P-invariance (2.7) for K_1: $P^*(J_{1/2})P(J_{1/2})K_0 = P(J_{1/2})P(e)P(J_{1/2})\{P(J_{1/2})K_1\}\subset P(J_{1/2})P(e)K_1$ (by P-invariance of $K_1) = P(J_{1/2})K_1 = K_0$.

It is not clear whether $P(J_{1/2})K_0$ inherits P-invariance when K_0 is merely P-invariant (not also L-invariant). □

We can now obtain the main result on Peirce ideals. Notice how much messier the formulation becomes for triple systems.

2.12. PROPOSITION THEOREM. *An ideal K_i in a Peirce subsystem J_i is the projection of a global ideal K in J iff K_i is invariant. In this case the ideal generated by K_i takes the form*

$$
\begin{aligned}
&(i=1)\quad K = K_1\oplus K_1\cdot J_{1/2}\oplus P(J_{1/2})K_1\\
&(i=0)\quad K = K_0\oplus\{K_0\cdot J_{1/2} + J_0\cdot(K_0\cdot J_{1/2}) + P(J_{1/2})K_0\cdot J_{1/2}\}\\
&\qquad\qquad \oplus\{P(J_{1/2})K_0 + P^*(J_{1/2})K_0\}\\
&\left(i=\frac{1}{2}\right) K = \{E_0(J_{1/2},\ K_{1/2}) + P(K_{1/2})J_1 + P(J_{1/2})P(K_{1/2})J_0 + P^*(J_{1/2})P(K_{1/2})J_0\}\\
&\qquad\qquad \oplus K_{1/2}\oplus\{E_1(J_{1/2},\ K_{1/2}) + E_1(K_{1/2},\ J_{1/2}) + P(K_{1/2})J_0 + P^*(K_{1/2})J_0\\
&\qquad\qquad + P(J_{1/2})P(K_{1/2})J_1 + P^*(J_{1/2})P(K_{1/2})J_1\}\ .
\end{aligned}
$$

If $1/2\in\Phi$ we have $P(J_{1/2})K_i = E_j(J_{1/2},\ K_i\cdot J_{1/2})$, $P(K_{1/2})J_j + P^*(K_{1/2})J_j\subset E_i(K_{1/2}, K_{1/2})$, $P(J_{1/2})P(K_{1/2})J_i + P^*(J_{1/2})P(K_{1/2})J_i\subset E_i(J_{1/2}, K_{1/2}) + E_i(J_{1/2}, K_{1/2})^*$ so the expressions for K reduce to

$$
\begin{aligned}
&(i=1)\quad K = K_1\oplus K_1\cdot J_{1/2}\oplus E_0(J_{1/2},\ K_1\cdot J_{1/2})\\
&(i=0)\quad K = K_0\oplus\{K_0\cdot J_{1/2} + J_0\cdot(K_0\cdot J_{1/2} + E_1(J_{1/2},\ K_0\cdot J_{1/2})\cdot J_{1/2})\}\\
&\qquad\qquad \oplus\{E_1(J_{1/2},\ K_0\cdot J_{1/2}) + E_1(K_0\cdot J_{1/2},\ J_{1/2})\}\\
&\left(i=\frac{1}{2}\right)\quad K = E_0(J_{1/2},\ K_{1/2})\oplus K_{1/2}\oplus\{E_1(J_{1/2},\ K_{1/2}) + E_1(K_{1/2},\ J_{1/2})\}\ .
\end{aligned}
$$

Proof. We have already noted that a Peirce component K_i must be invariant under global multiplications sending J_i into itself. Certainly the ideal generated by K_i contains all the above products; it remains only to show in each case K forms an ideal.

We begin with the easier diagonal cases $i = 1, 0$, where $K = K_i \oplus K_{1/2} \oplus K_j = K_i \oplus \{K_i \cdot J_{1/2} + J_i \cdot (K_i \cdot J_{1/2}) + P(J_{1/2})K_i \cdot J_{1/2}\} \oplus \{P(J_{1/2})K_i + P^*(J_{1/2})K_i\}$ (note for $i = 1$ that some of these products simplify: $J_1 \cdot (K_1 \cdot J_{1/2}) \subset (J_1 \cdot K_1) \cdot J_{1/2} - K_1 \cdot (J_1 \cdot J_{1/2}) \subset K_1 \cdot J_{1/2}$ by PIiv, $P^*(J_{1/2})K_1 = P(J_{1/2})K_1$ since $K_1^* = K_1$, and $P(J_{1/2})K_1 \cdot J_{1/2} \subset J_{1/2} \cdot L(J_{1/2}, J_{1/2})K_1 - K_1^* \cdot P(J_{1/2})J_{1/2} \subset J_{1/2} \cdot K_1$ by JT2).

We verify that the K_r satisfy the conditions (C1)-(C6) of (1.4). For (C1), K_i is an invariant ideal in J_i by hypothesis and $K_j = P(J_{1/2})K_i + P^*(J_{1/2})K_i$ is an invariant ideal in J_j by the Flipping Lemma 2.11. For (C5) we have $P(J_{1/2})K_i \subset K_j$ by construction, and $P(J_{1/2})K_j = P(J_{1/2})P(J_{1/2})K_i + P(J_{1/2})P^*(J_{1/2})K_i \subset K_i$ by P-invariance (2.7), (2.8). For (C2) we have $E_i(J_{1/2}, K_{1/2})$ the sum of $E_i(J_{1/2}, K_i \cdot J_{1/2})$ and $E_i(J_{1/2}, J_i \cdot (K_i \cdot J_{1/2}))$ and $E_i(J_{1/2}, P(J_{1/2})K_i \cdot J_{1/2})$ (the latter two only when $i = 0$). The first of these has $E_i(J_{1/2}, K_i \cdot J_{1/2}) = L(J_{1/2}, J_{1/2})K_i^* \subset K_i$ by (P4) and the L-invariance (2.5) of $K_i = K_i^*$. For $i = 0$ the second term $E_0(J_{1/2}, J_0 \cdot (K_0 \cdot J_{1/2}))$ falls in K_0 by the hypothesis of L-invariance (2.6). For $i = 0$ the third term becomes $E_0(J_{1/2}, P(J_{1/2})K_0 \cdot J_{1/2}) = \{J_{1/2}(P(J_{1/2})K_0)^* J_{1/2}\}$ (by P3)$\subset P(J_{1/2})P^*(J_{1/2})K_0$, which falls in K_0 by the hypothesis of P-invariance (2.8). Continuing with (C2), we examine $E_j(J_{1/2}, K_{1/2})$. By (P3) $E_j(J_{1/2}, K_i \cdot J_{1/2}) = \{J_{1/2}K_i^* J_{1/2}\} \subset P(J_{1/2})K_i \subset K_j$ by (C5). When $i = 0$ we must examine two other terms: $E_1(J_{1/2}, J_0 \cdot (K_0 \cdot J_{1/2})) = E_1(K_0 \cdot J_{1/2}, J_0 \cdot J_{1/2}) \subset E_1(K_0 \cdot J_{1/2}, J_{1/2}) = E_1(J_{1/2}, K_0 \cdot J_{1/2})^* \subset K_1^* = K_1$ as above, and $E_1(J_{1/2}, P(J_{1/2})K_0 \cdot J_{1/2}) = L(J_{1/2}, J_{1/2})(P(J_{1/2})K_0)^* = L(J_{1/2}, J_{1/2})P(e)P(J_{1/2})K_0$ where $L(x, y)P(e)P(z)k_0 = P(e)P(z)L(x, y)k_0 + P(\{xye\}, e)P(z)k_0 - P(e)P(\{yxz\}, z)k_0 \in P(e)P(J_{1/2})L(J_{1/2}, J_{1/2})K_0 + P(J_1)P(J_{1/2})K_0 - P(e)P(J_{1/2})K_0 \subset P(e)P(J_{1/2})K_0 + P^*(J_{1/2})K_0$ (by PI11 and L-invariance (2.5)) $\subset K_1$. This completes the verification of (C2). We have (C4) because $K_i \cdot J_{1/2} \subset K_{1/2}$ by construction and $K_j \cdot J_{1/2} = (P(J_{1/2})K_i) \cdot J_{1/2} + (P(J_{1/2})K_i)^* \cdot J_{1/2}$ (the two differing only when $i = 0$) where the latter is by PI8 contained in $E_i(J_{1/2}, K_i^* \cdot J_{1/2})^* \cdot J_{1/2} - K_i^* \cdot P(J_{1/2})J_{1/2} \subset K_i^* \cdot J_{1/2} - K_i^* \cdot J_{1/2}$ (by L-invariance (2.5)) $\subset K_i \cdot J_{1/2} \subset K_{1/2}$ and when $i = 0$ the former $(P(J_{1/2})K_0) \cdot J_{1/2}$ is contained in $K_{1/2}$ by construction. (There does not seem to be any way to show it falls into $K_0 \cdot J_{1/2} + J_0 \cdot (K_0 \cdot J_{1/2})$.) For (C3) note that $J_i \cdot (K_i \cdot J_{1/2}) \subset K_{1/2}$ by construction, $J_j \cdot (K_i \cdot J_{1/2}) = K_i^* \cdot (J_j^* \cdot J_{1/2}) \subset K_{1/2}$ by P6, and for $i = 0$ $J_1 \cdot [J_0 \cdot (K_0 \cdot J_{1/2})] \subset J_0 \cdot (K_0 \cdot (J_1 \cdot J_{1/2})) \subset K_{1/2}$ using P6 twice, and $J_0 \cdot [J_0 \cdot (K_0 \cdot J_{1/2})] \subset \{J_0 J_0 K_0\} \cdot J_{1/2} - K_0 \cdot (J_0 \cdot (J_0 \cdot J_{1/2}))$ (by PI1i) $\subset K_0 \cdot J_{1/2} \subset K_{1/2}$, and finally $J_r \cdot (P(J_{1/2})K_0 \cdot J_{1/2}) \subset J_r \cdot (K_1 \cdot J_{1/2}) \subset K_{1/2}$ by the above. For the last criterion (C6) we consider the spanning elements $k_i \cdot x_{1/2}$ (and, when $i = 0$, $a_0 \cdot (k_0 \cdot x_{1/2})$ and $P(x_{1/2})k_0 \cdot y_{1/2}$ as well). We observe by PI10, (C5), (C1) that $P(k_i \cdot x_{1/2})(J_i + J_j) = P^*(x_{1/2})P(k_i)J_i + P(k_i)P^*(x_{1/2})J_j \subset$

$P^*(J_{1/2})K_i + P(K_i)J_i \subset K_j + K_i$, also $P(a_0\cdot(k_0\cdot x_{1/2}))(J_1 + J_0) = P(a_0)P^*(k_0\cdot x_{1/2})J_1 + P^*(k_0\cdot x_{1/2})P(a_0)J_0 = P(a_0)P(k_0)P(x_{1/2})J_1 + P(x_{1/2})P(k_0)P(a_0)J_0 \subset P(J_0)K_0 + P(J_{1/2})K_0 \subset K_0 + K_1$, and also $P(P(x_{1/2})k_0\cdot y_{1/2})(J_1 + J_0) = P^*(y_{1/2})P(P(x_{1/2})k_0)J_1 + P(P(x_{1/2})k_0)P^*(y_{1/2})J_0 = P^*(y_{1/2})P(x_{1/2})P(k_0)P(x_{1/2})J_1 + P(x_{1/2})P(k_0)P(x_{1/2})P^*(y_{1/2})J_0 \subset P^*(J_{1/2})K_1 + P(J_{1/2})K_0 \subset K_0 + K_1$. Thus (C1)-(C6) hold, and K is an ideal.

The case $i = 1/2$ is even more tiresome. We must again verify (C1)-(C6). (C3) follows from invariance (2.9), and (C2) and (C6) follow by our construction of K_1, K_0. For the sake of symmetry we write the diagonal Peirce pieces as

$$K_i = E_i(J_{1/2}, K_{1/2}) + E_i(J_{1/2}, K_{1/2})^* + P(K_{1/2})J_j + P^*(K_{1/2})J_j + P(J_{1/2})P(K_{1/2})J_i + P^*(J_{1/2})P(K_{12})J_i\ .$$

As we remarked after (2.10), an invariant ideal is closed under all brackets:

$$(*)\qquad \{E_i(K_{1/2}, J_{1/2}) + E_i(J_{1/2}, K_{1/2})\}\cdot J_{1/2} \subset K_{1/2}\ .$$

We can now establish the rest of (C4), $K_i\cdot J_{1/2}\subset K_{1/2}$. Since $E_i(J_{1/2}, K_{1/2})^* = E_i(K_{1/2}, J_{1/2})$ by P8, we have so far that $\{E_i + E_i^*\}\cdot J_{1/2}\subset K_{1/2}$. Next, we observe $\{P(K_{1/2})J_j + P^*(K_{1/2})J_j\}\cdot J_{1/2} \subset E_j(K_{1/2}, J_{1/2})\cdot(J_j^*\cdot K_{1/2}) - P(K_{1/2})(J_j\cdot J_{1/2}) + E_j(J_{1/2}, J_j^*\cdot K_{1/2})^*\cdot K_{1/2} - J_j^*\cdot P(K_{1/2})J_{1/2}$ (by PI7, 8) $\subset J_j\cdot(J_j\cdot K_{1/2}) - P(K_{1/2})J_{1/2} + J_j^*\cdot K_{1/2} - J_j\cdot P(K_{1/2})J_{1/2} \subset K_{1/2}$ by invariance (2.9) and inner idealness $P(K_{1/2})J_{1/2}\subset J_{1/2}$. Finally, $\{P(J_{1/2})P(K_{1/2})J_i + P^*(J_{1/2})P(K_{1/2})J_i\}\cdot J_{1/2} \subset E_j(J_{1/2}, J_{1/2})\cdot[(P(K_{1/2})J_i)^*\cdot J_{1/2}] - P(J_{1/2})[P(K_{1/2})J_i\cdot J_{1/2}] + E_j(P(K_{1/2})J_i\cdot J_{1/2}, J_{1/2})\cdot J_{1/2} - P(K_{1/2})J_i\cdot P(J_{1/2})J_{1/2}$ (by PI7, 8 again) $\subset J_j\cdot K_{1/2} - P(J_{1/2})K_{1/2} + E_j(K_{1/2}, J_{1/2})\cdot J_{1/2} - K_{1/2}$ (by the previous case) $\subset K_{1/2}$ by invariance, outer idealness, and (*). Thus all 6 pieces of K_i send $J_{1/2}$ into $K_{1/2}$, completing (C4).

Next we check (C5), $P(J_{1/2})K_i \subset K_j$. We have $P(J_{1/2})\{E_i(J_{1/2}, K_{1/2}) + E_i(J_{1/2}, K_{1/2})^*\} = P(J_{1/2})\{E_i(J_{1/2}, K_{1/2}) + E_i(K_{1/2}, J_{1/2})\} \subset E_j(J_{1/2}, \langle K_{1/2}, J_{1/2}, J_{1/2}\rangle_j) - E_j(P(J_{1/2})J_{1/2}, K_{1/2}) + E_j(J_{1/2}, \langle J_{1/2}, J_{1/2}, K_{1/2}\rangle_j) - E_j(P(J_{1/2})K_{1/2}, J_{1/2})$ (by PI5) $\subset E_j(J_{1/2}, K_{1/2}) + E_j(K_{1/2}, J_{1/2}) \subset K_j$ by invariance and outer idealness. We have $P(J_{1/2})[P(K_{1/2})J_1] \subset K_1$ and $P(J_{1/2})[P(K_{1/2})J_0 + (P(K_{1/2})J_0)^*] \subset P(J_{1/2})P(K_{1/2})J_0 + P^*(J_{1/2})P(K_{1/2})J_0 \subset K_0$ by construction. For $P(J_{1/2})[P(J_{1/2})(P(K_{1/2})J_i) + P^*(J_{1/2})P(K_{1/2})J_i]$ we first have $P(J_{1/2})P(J_{1/2})P(K_{1/2})J_i = \{P(\{J_{1/2}J_{1/2}K_{1/2}\}) - P(K_{1/2})P(J_{1/2})P(J_{1/2}) + P(P(J_{1/2})P(J_{1/2})K_{1/2}, K_{1/2}) - L(J_{1/2}, J_{1/2})P(K_{1/2})L(J_{1/2}, J_{1/2})\}J_i$ (by JT4) $\subset P(K_{1/2})J_i - L(J_{1/2}, J_{1/2})P(K_{1/2})J_i \subset P(K_{1/2})J_i + \{P(K_{1/2})L(J_{1/2}, J_{1/2}) - P(\{J_{1/2}J_{1/2}K_{1/2}\}, K_{1/2})\}J_i$ (by JT5) $\subset P(K_{1/2})J_i \subset K_j$. With the *'s we consider the cases $i = 1$, $i = 0$ separately. For $i = 1$, $P(J_{1/2})P^*(J_{1/2})P(K_{1/2})J_1 = P(J_{1/2})P(e)P(J_{1/2})P(K_{1/2})J_1 \subset P(J_{1/2})\{P(\{eJ_{1/2}K_{1/2}\}) - P(K_{1/2})P(J_{1/2})P(e) + P(P(e)P(J_{1/2})K_{1/2}, K_{1/2}) - L(e, J_{1/2})P(K_{1/2})L(J_{1/2}, e)\}J_1 \subset P(J_{1/2})P(E_1(K_{1/2}, J_{1/2})J_1 + P(J_{1/2})P(K_{1/2})J_0 + 0 - P(J_{1/2})L(e, J_{1/2})P(K_{1/2})J_{1/2} \subset P^*(J_{1/2}\cdot E_1(K_{1/2}, J_{1/2})^*)J_1 + P(J_{1/2})P(K_{1/2})J_0 -$

$P(J_{1/2})E_1(K_{1/2}, J_{1/2})$ (by PI11, since $K_{1/2} \lhd J_{1/2}) \subset P^*(K_{1/2})J_1 + P(J_{1/2})P(K_{1/2})J_0 - P(J_{1/2})E_1(K_{1/2}, J_{1/2})$ (by invariance (2.10)) $\subset K_0$ (using the above relation $P(J_{1/2})E_i \subset E_j$). For $i=0$ we have $P(J_{1/2})P^*(J_{1/2})P(K_{1/2})J_0 = P(J_{1/2})P(J_{1/2})P(e)P(K_{1/2})J_0 \subset \{P(\{J_{1/2}J_{1/2}e\}) - P(e)P(J_{1/2})P(J_{1/2}) + P(P(e)P(J_{1/2})J_{1/2}, J_{1/2}) - L(e, J_{1/2})P(J_{1/2})L(J_{1/2}, e)\}P(K_{1/2})J_0$ (by JT4) $\subset P(J_1)P(K_{1/2})J_0 - P(e)[P(J_{1/2})P(J_{1/2})P(K_{1/2})J_0] + 0 - L(e, J_{1/2})P(J_{1/2})(J_{1/2} \cdot P(K_{1/2})J_0) \subset P^*(J_1^* \cdot K_{1/2})J_0 - P(e)K_1 - L(e, J_{1/2})P(J_{1/2})K_{1/2}$ (by PI11, the above, and (C4)) $\subset P^*(K_{1/2})J_0 - K_1^* - L(e, J_{1/2})K_{1/2} \subset K_1^* - E_1(K_{1/2}, J_{1/2}) \subset K_1$. Finally, we check (C1): $K_i \lhd J_i$. By PI2, 3 and invariance (2.9) we have $E_i(J_{1/2}, K_{1/2}) + E_i(K_{1/2}, J_{1/2})$ is an outer ideal in J_i. $P(K_{1/2})J_j + P^*(K_{1/2})J_j$ is also an outer ideal by invariance and PI10, 11, 12, 13. In the same way $P(J_{1/2})P(K_{1/2})J_i + P^*(J_{1/2})P(K_{1/2})J_i$ is outer, since

$$P(J_i)[P(J_{1/2})P(K_{1/2})J_i] \subset P^*(J_i^* \cdot J_{1/2})P(K_{1/2})J_i \text{ (by PI11)} \subset P^*(J_{1/2})P(K_{1/2})J_i$$

and $P(J_i)P^*(J_{1/2})P(K_{1/2})J_i \subset P(J_i \cdot J_{1/2})P(K_{1/2})J_i$ (by PI10) $\subset P(J_{1/2})P(K_{1/2})J_i$, establishing P-outerness (2.1), while L-outerness (2.2) follows from $L(J_i, J_i)[P(J_{1/2})P(K_{1/2})J_i] \subset P(J_i \cdot (J_i^* \cdot J_{1/2}), J_{1/2})P(K_{1/2})J_i$ (by PI12) $\subset P(J_{1/2})P(K_{1/2})J_i$, and $L(J_i, J_i)[P^*(J_{1/2})P(K_{1/2})J_i] = P^*(J_i^* \cdot (J_i \cdot J_{1/2}), J_{1/2})P(K_{1/2})J_i$ (by PI13) $\subset P^*(J_{1/2})P(K_{1/2})J_i$. Thus K_i is an outer ideal in J_i. For innerness (2.3′) we need only check the generators $E_i(x_{1/2}, k_{1/2})$, $E_i(x_{1/2}, k_{1/2})^*$, $P(k_{1/2})a_j$, $P^*(k_{1/2})a_j$, $P(x_{1/2})P(k_{1/2})a_i$ and $P^*(x_{1/2})P(k_{1/2})a_i$. Using (1.8) we have $P(P(k_{1/2})a_j)J_i = P(k_{1/2})P(a_j)P(k_{1/2})J_i \subset P(K_{1/2})J_j$, $P(P^*(k_{1/2})a_j)J_i = P^*(k_{1/2})P(a_j)P^*(k_{1/2})J_i \subset P^*(K_{1/2})J_j$, $P(P(x_{1/2})P(k_{1/2})a_i)J_i = P(x_{1/2})P(k_{1/2})P(a_i)P(k_{1/2})P(x_{1/2})J_i \subset P(J_{1/2})P(K_{1/2})J_i$, $P(P^*(x_{1/2})P(k_{1/2})a_i)J_i = P^*(x_{1/2})P(k_{1/2})P(a_i)P(k_{1/2})P^*(x_{1/2}) \subset P^*(J_{1/2})P(K_{1/2})J_i$, while by PI6, $P(E_i(K_{1/2}, J_{1/2}))J_i \subset P(K_{1/2})P^*(J_{1/2})J_i + P^*(J_{1/2})P(K_{1/2})J_i + E_i(K_{1/2}, K_{1/2}) \subset K_i$ and therefore $P(E_i(K_{1/2}, J_{1/2})^*)J_i^* = \{P(E_i(K_{1/2}, J_{1/2}))J_i\}^* \subset K_i^* = K_i$ as well. Thus $K_i \lhd J_i$, all conditions (C1)-(C6) are met, and $K \lhd J$.

If $1/2 \in \Phi$ the cases $i = 1, 0$ are simplified since $P(J_{1/2})K_i = 2P(J_{1/2})K_i = P(J_{1/2}, J_{1/2})K_i = E_j(J_{1/2}, K_i \cdot J_{1/2})$ (by P3 since $K_i^* = K_i$). The case $i = 1/2$ is simplified by $P(K_{1/2})J_j = P(K_{1/2}, K_{1/2})J_j = E_i(K_{1/2}, J_j^* \cdot K_{1/2}) \subset E_i(K_{1/2}, K_{1/2})$ by invariance, hence by P8 $(P(K_{1/2})J_j)^* \subset E_i({}_{1/2}, K_{1/2})$ too, and so $P(J_{1/2})(P(K_{1/2})J_i) + P^*(J_{1/2})P(K_{1/2})J_i \subset P(J_{1/2})E_j(K_{1/2}, K_{1/2}) + (P(J_{1/2})E_j(K_{1/2}, K_{1/2}))^* \subset E_i(J_{1/2}, J_j \cdot K_{1/2}) - E_i(P(J_{1/2})K_{1/2}, K_{1/2}) + \{E_i(J_{1/2}, J_j \cdot K_{1/2}) - E_i(P(J_{1/2})K_{1/2}, K_{1/2})\}^*$ (by PI5) $\subset E_i(J_{1/2}, K_{1/2}) + E_i(J_{1/2}, K_{1/2})^*$. □

We can easily describe the global ideal generated by a Peirce space.

2.13. COROLLARY. *The ideal in J generated by a Peirce $J_i(e)$ is*

$$(i = 1) \quad I(J_1) = J_1 \oplus J_{1/2} \oplus P(J_{1/2})J_1$$

$$(i = 0) \quad I(J_0) = J_0 \oplus \{J_0 \cdot J_{1/2} + P(J_{1/2})J_0 \cdot J_{1/2}\} \oplus \{P(J_{1/2})J_0 + P^*(J_{1/2})J_0\}$$

$$\left(i = \frac{1}{2}\right) \quad I(J_{1/2}) = P(J_{1/2})J_1 \oplus J_{1/2} \oplus \{E_1(J_{1/2}, J_{1/2}) + P(J_{1/2})J_0 + P^*(J_{1/2})J_0\}.$$

Proof. In each case $K_i = J_i$ is trivially invariant, so we have the explicit expressions for K given by the Projection Theorem. In case $i = 1$ the $J_{1/2}$-component simplifies by $K_1 \cdot J_{1/2} = e \cdot J_{1/2} = J_{1/2}$. In case $i = 0$ we have $J_0 \cdot (J_0 \cdot J_{1/2}) \subset J_0 \cdot J_{1/2}$ for the $J_{1/2}$-component. In case $i = 1/2$ we have for the J_0-component $E_0(J_{1/2}, J_{1/2}) = P(J_{1/2}, J_{1/2})e \subset P(J_{1/2})J_1$, $P(J_{1/2})[P(J_{1/2})J_0 + P^*(J_{1/2})J_0] \subset P(J_{1/2})J_1$ and for the J_1-component $P(J_{1/2})P(J_{1/2})J_1 + P^*(J_{1/2})P(J_{1/2})J_1 \subset P(J_{1/2})J_0 + P^*(J_{1/2})J_0$. □

When J is simple and $J_i \neq 0$ the ideal $I(J_i)$ must be all of J, leading to

2.14. PROPOSITION. *If J is simple and e a proper tripotent (nonzero and noninvertible) then*

(i) $P(J_{1/2})J_1 = J_0$,

(ii) $P(J_{1/2})J_0 + P^*(J_{1/2})J_0 + E_1(J_{1/2}, J_{1/2}) = J_1$.

If $J_0 \neq 0$ then

(iii) $P(J_{1/2})J_0 + P^*(J_{1/2})J_0 = J_1$, (iv) $J_0 \cdot J_{1/2} + P(J_{1/2})J_0 \cdot J_{1/2} = J_{1/2}$.

In characteristic $\neq 2$ we have

(v) $J_1 = E_1(J_{1/2}, J_{1/2})$, $J_0 = E_0(J_{1/2}, J_{1/2})$.

Proof. $e \neq 0$ implies $J_1 \neq 0$, so $I(J_1) = J$, yielding (i). If $J_{1/2} = 0$ then $J = J_1 \boxplus J_0$ forces either $J = J_1$ (e invertible) or $J = J_0(e = 0)$ by primeness, so we must have $J_{1/2} \neq 0$, and $I(J_{1/2}) = J$ yields (ii). We may well have $J_0 = 0$ with $J_1, J_{1/2} \neq 0$, but if $J_0 \neq 0$ then $I(J_0) = J$ yields (iii), (iv). For characteristic $\neq 2$, note $2P(J_{1/2})J_j = P(J_{1/2}, J_{1/2})J_j = E_i(J_{1/2}, J_j \cdot J_{1/2}) \subset E_i(J_{1/2}, J_{1/2}) = E_i(J_{1/2}, J_{1/2})^*$. □

In case $J_0 = 0$ we can also recover some ideal-building lemmas of Loos.

2.15. COROLLARY [1, pp. 131–132]. *Let e be a tripotent in a Jordan triple system with $J_0(e) = 0$.* (i) *If $K_{1/2}$ is an invariant bracket ideal of $J_{1/2}$ such that*

$$J_1 \cdot K_{1/2} \subset K_{1/2} \quad \langle K_{1/2} J_{1/2} J_{1/2} \rangle_1 + \langle J_{1/2} K_{1/2} J_{1/2} \rangle_1 \subset K_{1/2}$$

then the ideal in J generated by $K_{1/2}$ is $K = K_{1/2} \oplus \{E_1(K_{1/2}, J_{1/2}) + E_1(J_{1/2}, K_{1/2})\}$.

(ii) *If K_1 is an ideal of J_1 such that $L(J_{1/2}, J_{1/2})K_1 \subset K_1$ then the ideal in J generated by K_1 is $K_1 \oplus K_1 \cdot J_{1/2}$.*

Proof. (i) Note that $K_{1/2}$ is an ideal in $J_{1/2}$: Since $P(x_{1/2})y_{1/2} = E_1(x_{1/2}, y_{1/2}) \cdot x_{1/2} = \langle x_{1/2} y_{1/2} x_{1/2} \rangle$ by P1 when $J_0 = 0$, the above conditions guarantees a bracket (hence a product $P(x_{1/2})y_{1/2}$ or $P(x_{1/2}, z_{1/2})y_{1/2}$) falls in $K_{1/2}$ as soon as one factor does. This $K_{1/2}$ is invariant in the sense of (2.9), (2.10) by hypothesis, so by the Projection Theorem $K = K_1 + K_{1/2}$ where $P(K_{1/2})J_0 = P^*(J_{1/2})J_0 = P(J_{1/2})P(J_{1/2})J_1 = P^*(J_{1/2})P(J_{1/2})J_1 = 0$ when $J_0 = 0$, so K_1 reduces to $E_1(J_{1/2}, K_{1/2}) + E_1(K_{1/2}, J_{1/2})$.

(ii) K_1 is invariant since $P(J_{1/2})P(J_{1/2})K_1 = 0$, so by the Projection Theorem $K = K_1 \oplus K_1 \cdot J_{1/2}$. □

Since invariant Peirce ideals correspond to global ideals and simple JTS contain no proper global ideals, the Peirce subsystems contain no proper invariant ideals.

2.16. PROPOSITION. *If e is a tripotent in a simple Jordan triple system J, then then Peirce subsystems J_1, $J_{1/2}$, J_0 contain no proper invariant ideals.* □

We can also recover a result of Loos [1] on alternative triple systems.

2.17. COROLLARY. *If e is an idempotent in a simple Jordan triple system J with $J_0(e) = 0$, then $J_{1/2}(e)$ is simple as an alternative triple system under the bracket.*

Proof. By (2.15) $J_{1/2}$ contains no proper invariant ideals $K_{1/2}$, where the invariant ideal conditions (2.9′–2.10″) reduce to

$$J_1 \cdot K_{1/2} \subset K_{1/2} \quad \langle J_{1/2}J_{1/2}K_{1/2}\rangle_1 + \langle J_{1/2}K_{1/2}J_{1/2}\rangle_1 + \langle K_{1/2}J_{1/2}J_{1/2}\rangle_1 \subset K_{1/2} .$$

We may as well assume $J_{1/2} \neq 0$, so by (2.14) $J_1 = E_1(J_{1/2}, J_{1/2})$. Thus $J_1 \cdot K_{1/2} = E_1(J_{1/2}, J_{1/2}) \cdot K_{1/2} = \langle J_{1/2}J_{1/2}K_{1/2}\rangle_1$, and invariance under J_1 is a consequence of bracket-invariance. Therefore the nonexistence of proper invariant ideals means nonexistence of proper bracket ideals, that is, simplicity as an alternative triple system (note $J_{1/2}$ is not trivial under brackets since $0 \neq J_{1/2} = e \cdot J_{1/2} \subset E_1(J_{1/2}, J_{1/2}) \cdot J_{1/2} = \langle J_{1/2}J_{1/2}J_{1/2}\rangle_1$). □

3. Simplicity theorem.

As in the Jordan algebra case, we will quickly find J_1 inherits simplicity from J, then will use a flipping argument to establish simplicity of J_0. Before flipping we need to consider the case when the flipping process annihilates an ideal $K_0 \lhd J_0$.

3.1. KERNEL LEMMA. *The maximal ideal of J_0 annihilated by $P(J_{1/2})$ is* $\operatorname{Ker} P(J_{1/2}) = \{z_0 \in J_0 \mid P(J_{1/2})z_0 = P(J_{1/2})P(z_0)J_0 = 0\}$. *It is an invariant ideal.*

Proof. Clearly any ideal K_0 annihilated by $P(J_{1/2})$ lies in Ker $P(J_{1/2})$ since $P(K_0)J_0 \subset K_0$. It remains to show $K_0 = \operatorname{Ker} P(J_{1/2})$ is actually an invariant ideal.

K_0 is a linear subspace: it is clearly closed under scalars, and for sums $z_0 + w_0$ note

$$\begin{aligned} P(J_{1/2})P(z_0 + w_0)J_0 &= P(J_{1/2})P(z_0, w_0)J_0 = P(J_{1/2})L(w_0, J_0)z_0 \\ &= \{-L(J_0, w_0)P(J_{1/2}) + P(\{J_0 w_0 J_{1/2}\}, J_{1/2})\}z_0 \text{ (by JT5)} \\ &\subset -L(J_0, J_0)P(J_{1/2})z_0 + P(J_{1/2})z_0 = 0 \,. \end{aligned}$$

K_0 is P-outer, $P(J_0)K_0 \subset K_0$, since $P(J_{1/2})[P(a_0)z_0] = P^*(J_{1/2}\cdot a_0)z_0$ (by PI11) $\subset P^*(J_{1/2})z_0 = 0$ and $P(J_{1/2})[P(P(a_0)z_0)J_0] = P(J_{1/2})P(a_0)P(z_0)P(a_0)J_0 \subset P^*(J_{1/2}\cdot a_0)P(z_0)J_0 \subset P(e)P(J_{1/2})P(z_0)J_0 = 0$. It is L-outer, $L(J_0, J_0)K_0 \subset K_0$, since $P(J_{1/2})[L(a_0, b_0)z_0] \subset P(J_{1/2})z_0 = 0$ by PI14 and $P(J_{1/2})[P(L(a_0, b_0)z_0)J_0] \subset P(J_{1/2})\{P(a_0)P(b_0)P(z_0) + P(z_0)P(b_0)P(a_0) + L(a_0, b_0)P(z_0)L(b_0, a_0) - P(P(a_0)P(b_0)z_0, z_0)\}J_0$ (by JT4) $\subset P^*(J_{1/2}\cdot a_0)P(b_0)P(z_0)J_0 + P(J_{1/2})P(z_0)J_0 + P(J_{1/2})L(a_0, b_0)P(z_0)J_0 - P(J_{1/2})L(J_0, J_0)z_0$ (by PI11) $\subset P((J_{1/2}\cdot a_0)\cdot b_0)P(z_0)J_0 + 0 + P(J_{1/2}, J_{1/2})P(z_0)J_0 - P(J_{1/2}, J_{1/2})z_0$ (by PI10 and PI14) $\subset P(J_{1/2})P(z_0)J_0 + 0 - 0 = 0$.

K_0 is inner, $P(K_0)J_0 \subset K_0$, since $P(J_{1/2})[P(z_0)a_0] = 0$ by hypothesis and $P(J_{1/2})[P(P(z_0)a_0)J_0] = P(J_{1/2})P(z_0)P(a_0)P(z_0)J_0 \subset P(J_{1/2})P(z_0)J_0 = 0$.

K_0 is trivially P-invariant (2.7) and (2.8), $P(J_{1/2})P(J_{1/2})K_0 = P(J_{1/2})P(e)P(J_{1/2})K_0 = 0$. It is L-invariant (2.5), $L(J_{1/2}, J_{1/2})K_0 \subset K_0$, since $P(J_{1/2})[L(x_{1/2}, y_{1/2})z_0] = \{P(\{y_{1/2}x_{1/2}J_{1/2}\}, J_{1/2}) - L(y_{1/2}, x_{1/2})P(J_{1/2})\}z_0$ (by JT5) $= 0$ and

$$\begin{aligned} P(J_{1/2})[P(\{x_{1/2}y_{1/2}z_0\})J_0] &\subset P(J_{1/2})\{P(x_{1/2})P(y_{1/2})P(z_0) + P(z_0)P(y_{1/2})P(x_{1/2}) \\ &\quad + L(x_{1/2}, y_{1/2})P(z_0)L(y_{1/2}, x_{1/2}) - P(P(x_{1/2})P(y_{1/2})z_0, z_0)\}J_0 \text{ (by JT4)} \\ &\subset P(J_{1/2})P(J_{1/2})(P(y_{1/2})P(z_0)J_0) + P(J_{1/2})P(z_0)J_0 \\ &\quad + P(J_{1/2})L(J_{1/2}, J_{1/2})P(z_0)J_0 - P(J_{1/2})L(J_0, J_0)z_0 = 0 \end{aligned}$$

as above. The trickiest part is L-invariance (2.6), $E_0(J_{1/2}, J_0\cdot(K_0\cdot J_{1/2})) \subset K_0$. We first show this is killed by $P(J_{1/2})$. We have

$$\begin{aligned} &P(J_{1/2})[E_0(J_{1/2}, J_0\cdot(K_0\cdot J_{1/2}))] \\ &\quad = P(J_{1/2})\{J_{1/2}(K_0\cdot J_{1/2})J_0\} \text{ (by P4)} = P(J_{1/2})L(J_0, K_0\cdot J_{1/2})J_{1/2} \\ &\quad \subset \{-L(K_0\cdot J_{1/2}, J_0)P(J_{1/2}) + P(\{(K_0\cdot J_{1/2})J_0 J_{1/2}\}, J_{1/2})\}J_{1/2} \text{ (by JT5)} \\ &\quad \subset \{(K_0\cdot J_{1/2})J_0 J_{1/2}\} + L(J_{1/2}, J_{1/2})\{(K_0\cdot J_{1/2})J_0 J_{1/2}\} \end{aligned}$$

where $\{(K_0\cdot J_{1/2})J_0 J_{1/2}\} = E_1(K_0\cdot J_{1/2}, J_0\cdot J_{1/2})$ (by P3) $\subset E_1(K_0\cdot J_{1/2}, J_{1/2}) = E_1(J_{1/2}, K_0\cdot J_{1/2})^*$ (by P8) $= \{J_{1/2}K_0 J_{1/2}\}^*$ (by P3) $\subset (P(J_{1/2})K_0)^* = 0$.

To see $P(J_{1/2})$ also kills $P(E_0)J_0$ we use PI6 to write $P(E_0(x_{1/2}, a_0\cdot(z_0\cdot y_{1/2})))J_0 \subset P(x_{1/2})P^*(a_0\cdot(z_0\cdot y_{1/2}))J_0 + P^*(a_0\cdot(z_0\cdot y_{1/2}))P(x_{1/2})J_0 + E_0(x_{1/2}, P(a_0\cdot(z_0\cdot y_{1/2}))(J_0\cdot x_{1/2}))$. Here $P^*(a_0\cdot(z_0\cdot y_{1/2}))J_0 = P(z_0\cdot y_{1/2})P(a_0)J_0$ (by

PI11) $= P^*(y_{1/2})P(z_0)P(a_0)J_0 \subset P^*(J_{1/2})P(z_0)J_0 = 0$ by PI10, and $P^*(a_0 \cdot (z_0 \cdot y_{1/2}))J_1 = P(a_0)P(z_0 \cdot y_{1/2})J_1 = P(a_0)P(z_0)P^*(y_{1/2})J_1$ (by PI10, 11) $\subset P(a_0)P(z_0)J_0 \subset K_0$ since $K_0 \lhd J_0$, also $P(a_0 \cdot (z_0 \cdot y_{1/2}))(J_0 \cdot x_{1/2}) = a_0 \cdot \{z_0 \cdot P(y_{1/2})(z_0 \cdot (a_0 \cdot J_{1/2}))\}$ (using PI16 twice) $\subset J_0 \cdot (z_0 \cdot J_{1/2})$ so that $E_0(x_{1/2}, P) \subset E_0(J_{1/2}, J_0 \cdot (z_0 \cdot J_{1/2}))$ is killed by $P(J_{1/2})$ by the above. Thus $P(J_{1/2})$ does kill all three pieces of $P(E_0)J_0$, E_0 is contained in K_0, and K_0 is an invariant ideal. $\square$

Next we establish that $L(J_{1/2}, J_{1/2})$ and $P(J_{1/2})P(J_{1/2})$ and $P^*(J_{1/2})P(J_{1/2})$ send an ideal into its "square root" or "fourth root".

3.2. LEMMA. *For any ideal $K_i \lhd J_i (i = 1, 0)$ we have*

$$L(J_{1/2}, J_{1/2})P(K_i)J_i \subset K_i \tag{3.3}$$

$$P(J_{1/2})P(J_{1/2})P(P(K_i)J_i)J_i \subset K_i \tag{3.4}$$

$$\text{if } i = 0,\ P^*(J_{1/2})P(J_{1/2})P(J_0)P(P(K_0)J_0)J_0 \subset K_0\ . \tag{3.5}$$

Proof. (3.3) $L(x_{1/2}, y_{1/2})P(z_i)a_i = -P(z_i)L(y_{1/2}, x_{1/2})a_i + P(\{x_{1/2}y_{1/2}z_i\}, z_i)a_i$ (by JT5) $\in -P(K_i)J_i + P(J_i, K_i)J_i \subset K_i$ since K_i is an ideal.

(3.4) For $w_i \in P(K_i)J_i$ we have $P(x_{1/2})P(y_{1/2})P(w_i)J_i = \{P(\{x_{1/2}y_{1/2}w_i\}) - P(w_i)P(y_{1/2})P(x_{1/2}) - L(x_{1/2}, y_{1/2})P(w_i)L(y_{1/2}, x_{1/2}) + P(P(x_{1/2})P(y_{1/2})w_i, w_i)\}J_i$ (by JT4) $\subset P(K_i)J_i - P(K_i)J_i - L(J_{1/2}, J_{1/2})P(K_i)J_i + P(J_i, K_i)J_i$ (using (3.3) for w_i) $\subset K_i$.

(3.5) $P(x_{1/2})P(e)P(y_{1/2})P(a_0)L_0 \subset P(x_{1/2})[P(\{ey_{1/2}a_0\}) - P(a_0)P(y_{1/2})P(e) - L(e, y_{1/2})P(a_0)L(y_{1/2}, e) + P(P(e)P(y_{1/2})a_0, a_0)]L_0$ (by JT4) $\subset P(J_{1/2})P(J_{1/2})L_0 - 0 - L(e, y_{1/2})P(a_0)\{J_{1/2}eL_0\} + \{J_1L_0J_0\} = P(J_{1/2})P(J_{1/2})L_0$, so if $L_0 = P(P(K_0)J_0)J_0$ we have $P(J_{1/2})P(J_{1/2})L_0 \subset K_0$ by (3.4). $\square$

It is not clear whether (3.5) can be improved to assert $P^*(J_{1/2})P(J_{1/2})P(P(K_0)J_0)J_0 \subset K_0$.

Now we can describe a class of ideals which is guaranteed to be invariant.

3.6 PROPOSITION. *Any strongly semiprime ideal $K_1 \lhd J_1$ is invariant.*

Proof. We first prove that K_1 is L-invariant, i.e., $w_1 = L(x_{1/2}, y_{1/2})z_1 \in K_1$ for all $z_1 \in K_1$. By strong semiprimeness we will have $w_1 \in K_1$ if we can show $P(w_1)J_1 \subset K_1$. But

$$\begin{aligned} P(w_1)J_1 = \{&P(x_{1/2})P(y_{1/2})P(z_1) + P(z_1)P(y_{1/2})P(x_{1/2}) \\ &+ L(x_{1/2}, y_{1/2})P(z_1)L(y_{1/2}, x_{1/2}) - P(P(x_{1/2})P(y_{1/2})z_1, z_1)\}J_1 \text{ (by JT4)} \end{aligned}$$

$$\subset P(x_{1/2})P(y_{1/2})P(z_1)J_1 + P(K_1)J_1 + L(x_{1/2}, y_{1/2})P(K_1)J_1 - \{J_1J_1K_1\}$$
$$\subset P(x_{1/2})P(y_{1/2})P(z_1)J_1 + K_1 \text{ (using (3.3))} ,$$

so it suffices if all $u_1 = P(x_{1/2})P(y_{1/2})P(z_1)a_1$ fall in K_1. Here again it suffices if $P(u_1)J_1 \subset K_1$, and for this

$$P(u_1)J_1 = P(x_{1/2})P(y_{1/2})P(P(z_1)a_1)P(y_{1/2})P(x_{1/2})J_1$$
$$\subset P(J_{1/2})P(J_{1/2})P(P(K_1)J_1)J_1 \subset K_1 \text{ by (3.4)} .$$

Next we prove K_1 is P-invariant. Let $w_1 = P(x_{1/2})P(y_{1/2})z_1$; to show w_1 falls in K_1 it again suffices by strong semiprimeness if it pushes J_1 into K_1, i.e., if $P(w_1)J_1 = P(x_{1/2})P(y_{1/2})P(z_1)P(y_{1/2})P(x_{1/2})J_1 \subset P(x_{1/2})P(y_{1/2})P(z_1)J_1$ falls into K_1. But again this is in K_1 since it pushes J_1 into K_1, $P(P(x_{1/2})P(y_{1/2})P(z_1)a_1)J_1 \subset P(x_{1/2})P(y_{1/2})P(P(z_1)a_1)J_1 \subset K_1$ by (3.4). $\square$

Because it is such a nuisance to verify the extra invariance needed when $i = 0$, and since we will not need the result, we do not establish the analogous result for $K_0 \lhd J_0$.

3.7. COROLLARY. *Any maxinal ideal $M_1 \lhd J_1$ is invariant.*

Proof. If M_1 is maximal then $\bar{J}_1 = J_1/M_1$ is simple with invertible element $\bar{e}$, hence the Jacobson and small radicals are zero and $\bar{J}_1$ is strongly semiprime (see [1, p. 38]), so M_1 is strongly semiprime in J_1. $\square$

We now have the tools to establish our main result.

3.8. SIMPLICITY THEOREM. *If e is a tripotent in a simple Jordan triple system J, then the Peirce subsystems $J_1(e)$ and $J_0(e)$ are simple.*

Proof. We may as well assume e is proper, else the result is trivial. Then J_1 contains a nonzero tripotent and consequently is not trivial, and it has no proper ideals since any such could be enlarged to a maximal proper ideal $0 < M_1 < J_1$ (Zornifying and avoiding e), which would be invariant by 3.7, whereas by 2.15 J_i contains no proper invariant ideals.

Thus J_1 is simple. We may easily have $J_0 = 0$; we will show that if J_0 is nonzero then it must be simple. First, it is strongly semiprime: any element trivial in J_0 would be trivial in J ($P(z_0)J_0 = 0$ implies $P(z_0)J = 0$), whereas by simplicity and non-quasi-invertibility (thanks to $e \neq 0$) the system J is strongly semiprime (see [1, p. 38] again). In particular, J_0 is not trivial, and we need only show it contains no proper ideals $0 < K_0 < J_0$. Suppose on the contrary that such a K_0 exists. By (ordinary) semiprimeness we have successively $K_0' = P(K_0)K_0 \neq 0$, $K_0'' = P(K_0')K_0' \neq 0$, $K_0''' = P(K_0'')K_0'' \neq 0$. By the Flipping Lemma 2.11 $K_1''' = P(J_{1/2})K_0''' + P^*(J_{1/2})K_0'''$ is an ideal in J_1, so by simplicity of J_1 we have either $K_1''' = 0$ or $K_1''' = J_1$. In the first case K_0''' is an ideal annihilated by $P(J_{1/2})$, hence is contained in the invariant ideal Ker $P(J_{1/2})$ by 3.1; by (2.15) we know J_0 contains no proper invariant ideals, so Ker $P(J_{1/2}) \supset K_0''' > 0$ forces Ker $P(J_{1/2}) = J_0$, hence $P(J_{1/2})J_0 = 0$, contrary to (2.14iii) (assuming $J_0 \neq 0$). Thus the first case $K_1''' = 0$ is impossible.

On the other hand, consider the case $K_1''' = J_1$. Here (by (2.14i)) $J_0 = P(J_{1/2})J_1 = P(J_{1/2})K_1''' = P(J_{1/2})P(J_{1/2})K_0''' + P^*(J_{1/2})P(J_{1/2})K_0'''$ is contained in K_0 by (3.4) and (3.5) (noting $K_0'' = P(P(K_0)K_0)K_0' \subset P(P(K_0)J_0)J_0$ and $K_0''' = P(K_0'')K_0'' \subset P(J_0)(P(K_0')K_0') \subset P(J_0)P(P(K_0)J_0)J_0$ as required by (3.4) and (3.5)). But $J_0 = K_0$ contradicts propriety of K_0.

In either case the existence of a proper K_0 leads to a contradiction so no K_0 exists and J_0 too is simple. $\square$

This settles a question raised by Loos [1, p. 133] whether J_1 is simple in case J is simple and $J_0 = 0$. The result was known when J had d.c.c. on principal inner ideals. Of course, for the case $J_0 = 0$ we would not need the elaborate machinery of Peirce decompositions, since the Peirce relations and invariance are vastly simplified (for example $P(J_{1/2})P(J_{1/2})J_1 = 0$, so P-invariance is automatic).

The analogous simplicity result fails for $J_{1/2}$: $J_{1/2}$ need not inherit simplicity from J, since when $J = M_{p,q}(D)$ is the space of pxq matrices over D relative to $P(x)y = xy^*x$ $(y^* = {}^t\bar{y})$, then the diagonal idempotent $e = e_{11} + \cdots + e_{rr}$ $(1 \leqq r < p \leqq q)$ has $J_{1/2} = J_{10} \boxplus J_{01}$. In the simplest case $p = q = 2$, $r = 1$ we have $J_{1/2} = De_{12} \boxplus De_{21}$. Note, however, that these proper ideals $K_{1/2} = J_{10}$, $L_{1/2} = J_{01}$ are invariant under J_1 and J_0 but not under brackets. It is still an open question whether $J_{1/2}$ is simple as a bracket algebra (it is if $J_0 = 0$), or whether it is always simple or a direct sum of two ideals as a triple system.

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HYPERSPACES OF COMPACT CONVEX SETS

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The purpose of this paper is to develop in detail certain aspects of the space of nonempty compact convex subsets of a subset X (denoted $\mathrm{cc}(X)$) of a metric locally convex T.V.S. It is shown that if X is compact and $\dim(X) \geqq 2$ then $\mathrm{cc}(X)$ is homeomorphic with the Hilbert cube (denoted $\mathrm{cc}(X) \cong I_\infty$). It is shown that if $n \geqq 2$, then $\mathrm{cc}(R^n)$ is homeomorphic to I_∞ with a point removed. More specialized results are that if $X \subset R^2$ is such that $\mathrm{cc}(X) \cong I_\infty$ then X is a two cell; and that if $X \subset R^3$ is such that $\mathrm{cc}(X) \cong I_\infty$ and X is not contained in a hyperplane then X must contain a three cell.

For the most part we will be restricting ourselves to compact spaces X although in the last section of the paper, § 7, we consider some fundamental noncompact spaces.

We will be using the following definitions and notation. For each $n = 1, 2, \cdots, R^n$ will denote Euclidean n-space, $S^{n-1} = \{x \in R^n: \|x\| = 1\}$, $B^n = \{x \in R^n: \|x\| \leqq 1\}$, and ${}^0B^n = \{x \in R^n: \|x\| < 1\}$. A *continuum* is a nonempty, compact, connected metric space. An *n-cell* is a continuum homeomorphic to B^n. The symbol I_∞ denotes the *Hilbert cube*, i.e., $I_\infty = \prod_{i=1}^\infty [-1/2^i, 1/2^i]$. By I_∞^0 we will denote the *pseudo interior* of the Hilbert cube, $I_\infty^0 = \prod_{i=1}^\infty (-1/2^i, 1/2^i)$. We let I^+ denote the set of natural numbers. We use cl and $\overline{\mathrm{co}}$, respectively, to denote closure and closed convex hull. If Y is a subset of a space Z, then $\mathrm{int}[Y]$ means the union of all open subsets of Z which are contained in Y. The notation $X \cong Y$ will mean that the space X is homeomorphic to the space Y.

All spaces are considered in this paper to be subsets of a real topological vector space. Since we are restricting our attention in this paper to separable metric spaces this is no restriction topologically or geometrically (cf. Vol. I of [14, p. 242]). If X is a space, by *cc*(X) we will mean the hyperspace of all nonempty compact convex subsets of X (with the Hausdorff metric). We will call *cc*(X) the *cc-hyperspace of* X.

If x and y are points in a real topological vector space V, then $\widehat{xy}$ or $[x, y]$ denotes the *convex segment or point* (if $x = y$) *determined by* x and y, i.e., $\widehat{xy} = \{tx + (1-t)y: 0 \leqq t \leqq 1\} = [x, y]$. Let $X \subset V$. If $x \in X$, we let $S(x)$ denote $\{y \in X: \widehat{xy} \subset X\}$, and we let $\mathrm{Ker}(X)$ denote $\bigcap_{x \in X} S(x)$; the set $\mathrm{Ker}(X)$ is called the *kernel of* X. We say X is *starshaped* if and only if $\mathrm{Ker}(X) \neq \varnothing$. For $A \subset Y$, a point p in A is called an *extreme point of* A if and only if no convex segment lying in A has p in its (relative) interior. The symbol $ext[A]$ denotes the set of all extreme points of A. If X is a subset of R^n, for some n, a point $p \in X$ is said to be a *point of local nonconvexity of* X if every neighborhood of p in X fails to be convex. We will denote the set of all points of local nonconvexity of a set X by $LN(X)$. For spaces X and Y with $X \subset Y$ the boundary of X, denoted $\mathrm{Fr}(X)$, is defined by $\mathrm{Fr}(X) = \mathrm{cl}(X) \cap \mathrm{cl}(Y - X)$. A closed subset A of a metric space X is a *Z-set in* X if for any nonnull and homotopically trivial open set $U \subset X$ it is true that $U - A$ is nonnull and homotopically trivial (see [1]).

The paper is organized as follows: In § 2 we give some general results which are closely related to early work of Klee. One of the results of this section establishes that if K is a compact convex subset of a metrizable locally convex topological vector space and $\dim[K] \geqq 2$, then $\mathrm{cc}(K) \cong I_\infty$. This sets the stage for the remainder of the paper, as one of our major concerns becomes obtaining answers to the following question:

(1.1) For what continua K is $\mathrm{cc}(K) \cong I_\infty$? In § 3, we show that if $K \subset R^2$ is as in (1.1), then K is a 2-cell. Thus, for R^2, a complete answer to (1.1) becomes a matter of determining which 2-cells K in R^2 have their cc-hyperspace homeomorphic to I_∞. Results about this are in § 5, where we show that there is a 2-cell in R^2 whose cc-hyperspace is not homeomorphic to I_∞ and we obtain some geometric results which give sufficient conditions on a continuum X in order that $\mathrm{cc}(X) \cong I_\infty$. Many of the results in § 5 are for continua more general than 2-cells in the plane.

Though $K \subset R^2$ as in (1.1) must be a 2-cell, $K \subset R^3$ as in (1.1) need not be a 2-cell or 3-cell (see (4.7)). However, in § 4, we show that if $K \subset R^3$ is as in (1.1) and K is not contained in a 2-dim hyperplane in R^3, then K must *contain* a 3-cell (see (4.1)). Some lemmas about arcs of convex arcs in R^2 and arcs of convex 2-cells in R^3, which we use to prove (4.1), seem to be of interest in themselves.

In § 6 we give some examples and state some problems. Many of these help to delineate the status of the problem of which 2-cells in R^2 have their cc-hyperspace homeomorphic to I_∞. The technique used in (6.4) is particularly noteworthy since using it in combination with suitable results for 2-cells with polygonal boundary can, perhaps, lead to a satisfactory solution of (1.1).

The final section, § 7, begins to touch on the problems connected with determining the topological type of the cc-hyperspace of some noncompact subsets of topological vector spaces. The main result of this section is that, for $n \geqq 2$, $\mathrm{cc}(R^n) \cong I_\infty - \{p\}$ for $p \in I_\infty$. Several open questions are also posed in this section.

2. Some basic results.

(2.1) LEMMA. *Let K be a compact convex subset of a metrizable locally convex real topological vector space L, $\dim[K] \geqq 2$. Then there exists a countable family $\{\zeta_i: i = 1, 2, \cdots\}$ of continuous linear functionals ζ_i such that given $A \in \mathrm{cc}(K)$ and $x \in [K - A]$, there exists a $j \in I^+$ such that $\zeta_j(x) \notin \zeta_j(A)$.*

Proof. The compact metric space K in the relative topology has a countable base of convex sets $Q = \{V_i\}_{i=1}^{\infty}$. Define a family F by $F = \{(V_1, V_2, \cdots, V_n) | n \in I^+, V_i \in Q \text{ and } \overline{\mathrm{co}}(\bigcup_{i=1}^{n-1} V_i) \cap \mathrm{cl}[V_n] = \varnothing\}$. Given any $(V_1, V_2, \cdots, V_n) \in F$, by a (well known) separation theorem there exists a continuous linear functional strictly separating $\overline{\mathrm{co}}(\bigcup_{i=1}^{n-1} V_i)$ and $\mathrm{cl}[V_n]$. For each member of F, select one such functional thus obtaining a countable family $\{\zeta_i\}_{i=1}^{\infty}$ of functionals. The proof is completed by noting that for $x \in K$ and $A \in \mathrm{cc}(K)$ with $x \notin A$ there exists a $(V_1, V_2, \cdots, V_n) \in F$ with $A \subset \overline{\mathrm{co}}(\bigcup_{i=1}^{n-1} V_i)$ and $x \in \mathrm{cl}[Y_n]$.

(2.2) THEOREM. *Let K be a compact convex subset of a metrizable locally convex real topological vector space L, $\dim[K] \geqq 2$. Then $\mathrm{cc}(K) \cong I_\infty$.*

Proof. For each $A \in \mathrm{cc}(K)$, let $\zeta_i(A) = [a_i, b_i]$ where the ζ_i are as in (2.1) such that, without loss of generality, $\sup\{|\zeta_i(x)|: x \in K\} \leqq 1$ for each i. Let $F: \mathrm{cc}(K) \to I_\infty$ be defined by

$$F(A) = (a_1/2, b_1/2^2, a_2/2^3, b_2/2^4, \cdots, a_n/2^{2n-1}, b_n/2^{2n}, \cdots) .$$

Since $\{\zeta^2\}_{i=1}^{\infty}$ is a separating family, F is one-to-one. Furthermore, for each j, the co-ordinate functions $F_{2j-1} = a_j/2^{2j-1}$ and $F_{2j} = b_j/2^{2j}$ are continuous since ζ_j is continuous. Thus, F is continuous (we are mapping into I_∞). Let $A^1, A^2 \in \mathrm{cc}(K)$, $\lambda \in [0, 1]$, and $j \in I^+$; then, using the linearity of ζ_j,

$$\begin{aligned} \zeta_j(\lambda A^1 + (1-\lambda)A^2) &= \lambda\zeta_j(A^1) + (1-\lambda)\zeta_j(A^2) \\ &= \lambda[a_j^1, b_j^1] + (1-\lambda)[a_j^2, b_j^2] \\ &= [\lambda a_j^1 + (1-\lambda)a_j^2, \lambda b_j^1 + (1-\lambda)b_j^2] , \end{aligned}$$

where $[a_j^k, b_j^k] = \zeta_j(A^k)$ for $k = 1$ and 2. Thus, $F_t(\lambda A^1 + (1-\lambda)A^2) = \lambda F_t(A^1) + (1-\lambda)F_t(A^2)$ where $t = 1, 2, \cdots,$. This says that the set $F(\mathrm{cc}(K))$ is convex. Now, since $\dim[K] \geqq 2$ K contains a convex 2-cell, say D. Thus, for each n, K contains a regular n-sided polygon P_n with sides $s_1, s_2, \cdots, s_n$ which lies in the "interior" of the 2-cell D. For each i, let A_i be a convex arc which lies in the exterior of P_n along the perpendicular bisector of s_i in D. For each n-tuple $(t_1, t_2, \cdots, t_n)$ in $\prod_{i=1}^n A_i$ let $G((t_1, t_2, \cdots, t_n)) = \overline{\text{co}}(\{t_1, t_2, \cdots, t_n\})$. It is clear that the mapping G is a homeomorphism of the n-cell $\prod_{i=1}^n A_i$ into $\text{cc}(K)$. Thus, $\text{cc}(K)$ contains an n-cell for every n and, tnerefore, is infinite dimensional. Thus, $F(\text{cc}(K))$ is a compact and infinite dimensional convex subset of l_2. Hence, by Keller's theorem [10], $F(\text{cc}(K)) \cong I_\infty$. Therefore, $\text{cc}(K) \cong I_\infty$.

We point out that the proof of Theorem 2.2 is a slight modification of a proof used by Klee [12] to generalize Keller's theorem. Also Klee, in a conversation with the authors, has pointed out a different proof of Theorem 2.2 in the case when L is a normed linear space. This consists of using a theorem in [17] to embed the compact convex subsets of a normed linear space into a normed linear space, noting that for a fixed $K \subset L$, $\text{cc}(K)$ is embedded convexly, and then using Klee's generalization [12] of Keller's theorem.

Let L be as in (2.2) and let $F \subset \text{cc}(L)$. We say that the *family F is convex* if and only if for all $A, B \in F$ and λ, $0 \leqq \lambda \leqq 1$, $(\lambda A + (1 - \lambda)B) \in F$ (where λA means $\{\lambda \cdot a\colon a \in A\}$).

(2.3) THEOREM. *Let L be as in* (2.2) *and let $F \subset \text{cc}(L)$ be such that F is compact, convex, and infinite dimensional. Then, $F \cong I_\infty$.*

Proof. By (2.2) $\text{cc}(L)$ and hence F can be affinely embedded into l_2. But then F is a compact, convex, infinite dimensional subset of l_2 and Keller's theorem applies to give $F \cong I_\infty$ (see [10]).

As a consequence of (2.3) and the part of the proof of (2.2) showing that $\text{cc}(K)$ is infinite dimensional, we have the following two corollaries.

(2.4) COROLLARY. *Let K and L be as in* (2.2). *Let Q be a given compact subset of K such that $\overline{\text{co}}[Q] \neq K$. Then, $\{A \in \text{cc}(K)\colon Q \subset A\} \cong I_\infty$.*

(2.5) COROLLARY. *Let K and L be as in* (2.2). *Let K_0 be a given nonempty compact convex subset of K. Then $\{A \in \text{cc}(K)\colon A \cap K_0 \neq \varnothing\} \cong I_\infty$.*

It follows, in particular, from (2.3) or (2.4) that the space of compact convex subsets of the unit disc in R^2 which contain the origin is homeomorphic to I_∞.

3. A topological converse to (2.2) for the plane. In the plane, (2.2) says that the cc-hyperspace of a convex 2-cell is homeomorphic to the Hilbert cube. The question arises as to which subsets of the plane have their cc-hyperspaces homeomorphic to I_∞. A complete answer to this problem will involve both topological and geometric considerations. The topological considerations are the subject of this section. Our result is

(3.1) THEOREM. *If X is a continuum in R^2 such that $\mathrm{cc}(X) \cong I_\infty$, then X is a two cell.*

To prove (3.1) we will make use of the following lemmas. The first three lemmas are stated in more generality than explicitly needed for proving (3.1).

(3.2) LEMMA. *Let E be a Banach space which admits a topologically equivalent norm that is strictly convex. Then there is a continuous selection from $\mathrm{cc}(E)$ to E. Thus, for any separable Banach space, there is such a selection.*

Proof. Let $\|\cdot\|$ denote a strictly convex norm on E and let $p \in E$. Define η: $\mathrm{cc}(E) \to E$ by letting $\eta(A)$ denote the unique point $a_0 \in A$ such that $\inf\{\|p - a\| \, a \in A\} = \|p - a_0\|$ (see [3, p. 19]). It is easy to see that η is continuous and is a selection. The second part of (3.2) follows from the fact that any separable Banach space admits an equivalent strictly convex norm [3, p. 18].

(3.3) LEMMA. *Let X be a dendrite. Then* $\dim[\mathrm{cc}(X)] \leqq 2$.

Proof. Let X be a dendrite (in some real topological vector space) and note that any member of $\mathrm{cc}(X)$ is either a (convex) arc or a singleton. Hence, the barycenter map g: $\mathrm{cc}(X) \to X$ is continuous where g is defined by: if a and b are the endpoints of a convex arc A in X or if $a = b$, in which case let $A = \{a\}$, then $g(A) = (a + b)/2$. Let $p \in X$. Since p belongs to arbitrarily small open subsets of X with finite boundaries [21, p. 99], there are at most countably many convex arcs $A_i = [a_i, b_i]$, $i = 1, 2, \cdots$, maximal with respect to the property that $g(A_i) = p$. For each p let $D_i = \{[s_i, t_i] \subset A_i\colon g([s_i, t_i]) = p\}$. Since the map $s_i \to [s_i, t_i]$ is a homeomorphism of $[a_i, p]$ onto D_i, $D_i \cong [a_i, p]$ (note: D_i could be just $\{p\}$). Also, it is clear that $g^{-1}(p) = \bigcup_{i=1}^{\infty} D_i$. Hence, by III 2 of [9], $\dim[g^{-1}(p)] \leqq 1$. Therefore, from the statement on p. 92 of [9] which is verified in order to prove VI 7 of [9], $\dim[\mathrm{cc}(X)] \leqq 1 + \dim[X] = 2$.

(3.4) LEMMA. *Let X be a continuum lying in a Banach space*

E. If $\mathrm{cc}(X) \cong I_\infty$, *then* X *is an absolute retract and* $\dim[X] \geqq 2$.

Proof. Let F denote the closed linear span of X. Since X is separable, F is a separable subspace of E. Hence, by (3.2), we have a continuous selection $\eta\colon \mathrm{cc}(F) \to F$. Since the restriction of η to $\mathrm{cc}(X)$ is a retraction of $\mathrm{cc}(X)$ onto X, the fact that X is an AR now follows from the well known fact that [**14**, Vol. II, Th. 7, p. 341] a retract of I_∞ is an AR. For the remainder of the proof, suppose $\dim[X] \leqq 1$. If $\dim[X] = 0$, in which case X consists of only one point, then $\mathrm{cc}(X) \cong X$. So, for the purpose of proof, assume $\dim[X] = 1$. Then X is a one-dimensional AR and, hence, a dendrite (cf. Brosuk's "Theory of Retracts" p. 138). By (3.3) this implies $\dim[\mathrm{cc}(X)] \leqq 2$ which contradicts the assumption that $\mathrm{cc}(X) \cong I_\infty$.

(3.5) *Conjecture.* If A is a dendrite, then $\mathrm{cc}(A)$ is embeddable in the plane.

(3.6) LEMMA. *The space of singletons and convex arcs in* $R^n (n \geqq 2)$ *denoted* $AS(R^n)$, *is homeomorphic to* $R^n \times ([0, \infty) \times P^{n-1}/0 \times P^{n-1})$. *In the special case that* $n = 2$, $AS(R^2) \cong R^4$.

Proof. We note that the space of lines through the origin in R^n is homeomorphic to projective $n-1$ space P^{n-1}. For each convex arc or point $\widehat{ab}$ in R^n define $F(\widehat{ab})$ in $R^n \times ([0, \infty) \times p^{n-1}/0 \times p^{n-1})$ by $F(\widehat{ab}) = (a+b)/2, [(\|b-a\|, s)]$ where s is the point of p^{n-1} determined by the line parallel to $\widehat{ab}$ if $\widehat{ab}$ is nondegenerate and s is the point of p^{n-1} determined by the first axis if $\widehat{ab}$ is a singleton. In this proof we have used $[\circ]$ to denote "equivalence class." It is a straightforward matter to check that F is a homeomorphism. If $n = 2$, then $R^2 \times ([0, \infty) \times p^1/0 \times p^1) \cong R^2 \times ([0, \infty) \times S^1/0 \times S^1) \cong R^2 \times R^2 \cong R^4$. The lemma is proved.

(3.7) LEMMA. *If* X *is a continuum in* R^2 *such that* $\mathrm{cc}(X) \cong I_\infty$, *then* $\mathrm{int}[X] \neq \varnothing$ *and* $X = \mathrm{cl}(\mathrm{int}[X])$.

Proof. Suppose there is a point p in $X - \mathrm{cl}(\mathrm{int}(X))$. Clearly, we may then choose a neighborhood N in $\mathrm{cc}(X)$ about $\{p\}$ such that N consists only of singletons and convex arcs. Hence, N is embeddable in R^4 (by (3.6)) and, therefore, finite dimensional. This contradicts the assumption that $\mathrm{cc}(X) \cong I_\infty$.

(3.8) LEMMA. *If* X *is a continuum in* R^2 *such that* $\mathrm{cc}(X) \cong I_\infty$, *then* $\mathrm{int}[X]$ *is connected.*

Proof. Let p and q be distinct points of int$[X]$. We show that there is an arc in int$[X]$ from p to q. Let $\Lambda = \{A \in \mathrm{cc}(X) | A$ is a singleton or a convex arc$\}$. By virtue of (3.6), Λ is finite dimensional. Therefore, since $\mathrm{cc}(X) \cong I_\infty$ and Λ is compact, $\mathrm{cc}(X) - \Lambda$ is arcwise connected (that no finite dimensional continuum can separate I_∞) (arc separate is equivalent to separate for locally connected continua) follows from the fact that, for each n, I_n is a Cantor manifold (see Corollary 2 on p. 48 of [9]) and the set of all points of the form $\bigcup_{n=1}^{\infty} I_n^1$ is dense in I_∞ (here $I_n^1 = \prod_{i=1}^{n} I_i \times (1/2, 1/2, \cdots))$. Let $K, L \in \mathrm{cc}(X)$ be 2-cells with $[K \cup L] \subset \mathrm{int}[X]$ and $\beta(K) = p$ and $\beta(L) = q$ (where $\beta\colon \mathrm{cc}(X) \to X$ is the barycenter map). Now, let α be an arc in $\mathrm{cc}(X) - \Lambda$ with endpoints K and L. Since $\alpha \subset [\mathrm{cc}(X) - \Lambda]$ each point of α is a 2-cell and thus, the restriction of β to α is continuous. Thus, $\beta(\alpha)$ is a locally connected continuum and hence $\beta(\alpha)$ is arcwise connected. Since $X \subset R^2$ and each member M of α is a 2-cell, it follows that $\beta(M) \in \mathrm{int}(M) \subset \mathrm{int}[X]$. Therefore, we now have that $\beta(\alpha)$ is arcwise connected and $p, q \in \beta(\alpha) \subset \mathrm{int}[X]$. The lemma follows.

Proof of Theorem 3.1. By (3.4), X is an absolute retract and therefore $R^2 - X$ is connected [7, p. 364]. Therefore, (since X is a locally connected continuum in R^2), $\mathrm{Bd}[R^2 - X]$ is a locally connected continuum (see 2.2 of [21, p. 106]). Let N denote $\mathrm{Bd}[R^2 - X]$. Direct computation using only definitions yields

$$R^2 - N = [R^2 - X] \cup \mathrm{int}\, X\,. \tag{$*$}$$

Thus we have that N is a locally connected continuum and, by (3.9), and ($*$) $E^2 - X$ and int$[X]$ are the components of $E^2 - N$. It now follows from 2.51 of [21, p. 107] that there is a simple closed curve $J \subset N$. Let G denote the bounded component of $E^2 - J$. By (3.8), $\mathrm{int}[X] \subset G$, and hence, $\mathrm{cl}(\mathrm{int}[X]) \subset [G \cup J]$. Therefore, by (3.7), $X \subset [G \cup J]$. However, since $E^2 - X$ is connected and $J \subset X$, we have $G \subset X$, i.e., $[G \cup J] \subset X$. This proves $X = G \cup J$ and, thus, X is a 2-cell. This proves (3.1).

REMARK. The part of the proof of Theorem 3.1 which follows the lemmas is devoted entirely to showing that if Z is a planar compact absolute retract such that $Z = \mathrm{cl}(\mathrm{int}[Z])$ and int$[Z]$ is connected, then Z is a 2-cell. This characterization of 2-cells among continua in the plane does not seem to be explicitly stated in the literature.

4. Analogue to the 2-cell theorem for 3-space. In this section we will establish

(4.1) THEOREM. *If X is a continuum in R^3 such that* $\mathrm{cc}(X) \cong I_\infty$ *and X is not contained in any 2-dimensional hyperplane, then* $\mathrm{int}[X] \neq \varnothing$.

We use the following lemmas to prove (4.1).

(4.2) LEMMA. *Let $\sigma: [0, 1] \to cc(R^2)$ be an arc of convex arcs in R^2. Suppose that L is a straight line in R^2 such that, for $0 \leqq t \leqq s$ where $s > 0$, $L \cap \sigma(t)$ consists of only one point. Then the convex segment with noncut points $\sigma(0) \cap L$ and $\sigma(s) \cap L$ is contained in $\bigcup_{0 \leqq t \leqq s} \sigma(t)$.*

(4.3) REMARK. It is easy using (4.2) to prove that if $\sigma[0, 1] \to \mathrm{cc}(R^2)$ is a one-to-one continuous mapping such that, for each $x \in [0, 1]$, $\sigma(s)$ is a convex arc and such that there exist s_1 and s_2 such that $\sigma(s_1)$ and $\sigma(s_2)$ are not co-linear, then $\bigcup_{s \in [0,1]} \sigma(s)$ contains a 2-cell.

Proof. Consider the mapping $\tilde{\sigma}$: $[0, s] \to L$ defined by $\tilde{\sigma}(t) = \sigma(t) \cap L$. Using the single valuedness of $\tilde{\sigma}$, it is easy to show that $\tilde{\sigma}$ is continuous. Thus, $\tilde{\sigma}([0, s])$ is connected in L and the result follows.

(4.4) LEMMA. *Let $\sigma: [0, 1] \to \mathrm{cc}(R^3)$ be an arc of convex 2-cells in R^3 such that there is a sequence $s_r \to 0$ such that $\sigma(s_r)$ and $\sigma(0)$ are not co-planar. Then, $\bigcup_{s \in [0,1]} \sigma(s)$ contains a 3-cell.*

Proof. Let $\prod_i$ $(i = 1, 2, 3)$ be the standard projection onto the ith factor of R^3. Since $\sigma(0)$ is nondegenerate, there exist i_1 and i_2 such that neither $\prod_{i_1}[\sigma(0)]$ nor $\prod_{i_2}[\sigma(0)]$ is a single point. Without loss of generality, we will assume that $i_1 = 1$ and $i_2 = 2$. Let $[a_1, a_2] \subset \mathrm{int}[\prod_1(\sigma(0))]$. Note that, for $x \in [a_1, a_2]$, $\prod_1^{-1}(x) \cap \sigma(0)$ is a nondegenerate arc. Let c be chosen so that $\prod_2^{-1}(c) \cap \prod_1^{-1}((a_1 + a_2)/2) \cap \sigma(0)$ is an interior point of the arc $\sigma(0) \cap \prod_1^{-1}((a_1 + a_2)/2)$. Let $a_1 \leqq a_1' < (a_1 - a_2)/2 < a_2' \leqq a_2$ be chosen so that, for each $x \in [a_1', a_2']$, $\prod_2^{-1}(c) \cap \prod_1^{-1}(x) \cap \sigma(0)$ is an interior point of the arc $\prod_1^{-1}(x) \cap \sigma(0)$. Let $c_1 < c < c_2$ be chosen so that, for $y \in [c_1, c_2]$ and $x \in [a_1', a_2']$ it is true that $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(0)$ is an interior point of the arc $\prod_1^{-1}(x) \cap \sigma(0)$. Let $t > 0$ be chosen so that:

(1) for $s \in [0, t]$ and $x \in [a_1', a_2']$, $\prod_1^{-1}(x) \cap \sigma(s)$ cuts $\sigma(s)$, and

(2) for $s \in [0, t]$, $x \in [a_1', a_2']$ and $y \in [c_1, c_2]$, $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(s)$ is an interior point of the arc $\prod_1^{-1}(x) \cap \sigma(s)$.

Let $0 < t' < t$ be chosen so that $\sigma(0)$ and $\sigma(t')$ arc not co-planar. Note, since there can be at most one x in $[a_1', a_2']$ for which $\sigma(0) \cap$

$\prod_1^{-1}(x)$ and $\sigma(t') \cap \prod_1^{-1}(x)$ are co-linear, we may assume without loss of generality that, for $x \in [a_1', a_2']$, $\prod_1^{-1}(x) \cap \sigma(0)$ and $\prod_1^{-1}(x) \cap \sigma(t')$ are not co-linear. Since, for each $x \in [a_1', a_2']$, there can be at most one $y \in [c_1, c_2]$ such that $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(0) \cap \sigma(t') \neq \varnothing$, we may now choose $a_1' \leqq a_1'' < a_2'' \leqq a_2'$ and $c_1 \leqq c_1' < c_2' \leqq c_2$ so that, for $x \in [a_1'', a_2'']$ and $y \in [c_1', c_2']$, $(*)$ $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(0) \cap \sigma(t') = \varnothing$. Consider now the set of points $D = \{\prod_2^{-1}(c_i') \cap \prod_1^{-1}(a_j'') \cap \sigma(z)\colon i, j = 1, 2, z = 0 \text{ or } z = t'\}$. We claim that $\overline{\mathrm{co}}(D) \subset \bigcup_{s \in [0,1]} \sigma(s)$. To see this, note first that if $D_0 = \{\prod_2^{-1}(c_i) \cap \prod_1^{-1}(a_j'') \cap \sigma(0)\}$ where $i, j = 1, 2\}$ and $D_{t'} = \{\prod_2^{-1}(c_i) \cap \prod_1^{-1}(a_j'') \cap \sigma(t')\colon i, j = 1, 2\}$ then $\overline{\mathrm{co}}(D_z) \subset \sigma(z) \subset [\bigcup_{s \in [0,1]} \sigma(s)]$ where $z \in \{0, t'\}$. Now, if $p \in \overline{\mathrm{co}}(D)$ then, for some $x \in [a_1'', a_2'']$, we have that $p \in \prod_1^{-1}(x)$. Also, for some $y \in [c_1', c_2']$ we have that $p \in \prod_2^{-1}(y)$. Since $p \in \overline{\mathrm{co}}(D)$ we have that p is on the convex segment in $\prod_2^{-1}(y) \cap \prod_1^{-1}(x)$ which joins $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(0)$ and $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(t')$. This is true because $\overline{\mathrm{co}}(D_0) \cap \overline{\mathrm{co}}(D_t) = \varnothing$ (otherwise we would contradict $(*)$). Now, the mapping $\sigma_x\colon [0, t'] \to \mathrm{cc}(\prod_1^{-1}(x))$ defined by $\sigma_x(s) = \sigma(s) \cap \prod_1^{-1}(x)$ is easily seen to be continuous. Also, $\sigma_x(0)$ and $\sigma_x(t')$ are not co-linear and the line $\prod_2^{-1}(y) \cap \prod_1^{-1}(x)$ in $\prod_1^{-1}(x)$ cuts each of the arcs $\sigma_x(s)$ for $s \in [0, t']$. It now follows from (4.2) that $p \in \bigcup_{s \in [0,t']} \sigma_x(s)$. The lemma is proved.

The following lemma is a simple consequence of (4.4).

(4.5) LEMMA. *Let $\sigma\colon [0, 1] \to \mathrm{cc}(R^3)$ be a one-to-one continuous mapping of* $[0, 1]$ *into $cc(R^3)$ such that $\sigma(s)$ is a (convex) 2-cell for each s and such that there exist s_1 and s_2 such that $\sigma(s_1)$ and $\sigma(s_2)$ are not co-planar. Then, $\bigcup_{s \in [0,1]} \sigma(s)$ contains a 3-cell. We are now ready to establish* (4.1).

Proof of (4.1). It can be seen that the space of convex arcs and points in a compact subset of R^3 is of dimension less than or equal to 6 (see (3.6)). If X satisfies the conditions of (4.1) and $AS(X)$ denotes the space of arcs and singletons in $\mathrm{cc}(X)$ then $\mathrm{cc}(X) - AS(X)$ must be arcwise connected (see the remark in the proof of (3.8)). Let p_1 and p_2 be points in X which lie in the interior of two cells P_1 and P_2, respectively, such that P_1 and P_2 are not co-planar. Now, $[\mathrm{cc}(X) - AS(X)] \supset \{P_1, P_2\}$ and, hence, there is a one-to-one continuous mapping $\sigma\colon [0, 1] \to [\mathrm{cc}(X) - AS(X)]$ such that $\sigma(0) = P_1$ and $\sigma(1) = P_2$. If $\sigma(s)$ is not a 2-cell for some s, then $\sigma(s)$ is a 3-cell and we are done. Hence, without loss of generality, we may assume $\sigma(s)$ is a 2-cell for each $s \in [0, 1]$. Thus, by virtue of (4.5), $X \supset \bigcup_{s \in [0,1]} \sigma(s)$ contains a 3-cell. The theorem is proved.

(4.6) EXAMPLE. We show that the natural analogue to (4.1) does not hold in R^n, $n > 3$. Let Y be the continuum in R^4 defined by $Y = Y_1 \cup Y_2$ where $Y_1 = \{(x, y, z, w): |x| \leqq 1, |y| \leqq 1, |z| \leqq 1, w=0\}$ and $Y_2 = \{(x, y, z, w): |x| \leqq 1, |y| \leqq 1, z = 0, |w| \leqq 1\}$. Now, $\mathrm{cc}(Y) = \mathrm{cc}(Y_1) \cup \mathrm{cc}(Y_2)$ and $\mathrm{cc}(Y_1 \cap Y_2) = \mathrm{cc}(Y_1) \cap \mathrm{cc}(Y_2) \cong I_\infty$. A theorem of Anderson [20] asserts that the union of two Hilbert cubes which intersect in a Hilbert cube is a Hilbert cube provided the intersection has property Z in each. We thus want to see that $\mathrm{cc}(Y_1 \cap Y_2)$ has property Z in $\mathrm{cc}(Y_1)$ and $\mathrm{cc}(Y_2)$.

To this end, let U be a homotopically trivial subset of $\mathrm{cc}(Y_1)$. Let $g: S^{k-1} \to U - \mathrm{cc}(Y_1 \cap Y_2)$ and let $\bar{g}: B^k \to U$ be an extension of g. For each $p \in U$ let $d(p) = \inf\{d(p, q): q \in \mathrm{cc}(Y_1) - U\}$. For each $t \in [0, 1]$ and each b in the sphere of radius t in B^k, let $G(b) = \mathrm{co}(N((1-t)(d(\bar{g}(b)))/2, g(b)))(N(\varepsilon, \bar{g}(b))) = \{x\colon$ for some $a \in \bar{g}(b)$, $||x - a|| < \varepsilon\})$. Clearly $G(b) \in U$ for each $b \in B^k$ and, even more, since $G(b)$ is a 3-cell for each b, we have $G(b) \in U - \mathrm{cc}(Y_1 \cap Y_2)$. Also $G|S^{k-1} = g$. We have established that $\mathrm{cc}(Y_1) \cap \mathrm{cc}(Y_2)$ has property Z in $\mathrm{cc}(Y_1)$. The proof for $\mathrm{cc}(Y_2)$ is the same. It now follows that $\mathrm{cc}(Y) = I_\infty$. This shows that the analogue to (4.1) does *not* hold in R^4. Actually, it is clear that similar examples exist in dimensions $n > 4$ as well.

This next example is of a 3-dimensional continuum in R^3 which is not a 3-cell but whose cc-hyperspace is homeomorphic to I_∞.

(4.7) EXAMPLE. Let X be the continuum in R^3 defined by $X = X_1 \cup X_2$ where

$$X_1 = \{(x, y, z)\colon ||(x, y, z)|| \leqq 1\}$$

and

$$X_2 = \{(x, y, 0)\colon \max\{|x|, |y|\} \leqq 1\}.$$

Now, $\mathrm{cc}(X) = \mathrm{cc}(X_1) \cup \mathrm{cc}(X_2)$ is a union of two convex Hilbert cubes. Also, $\mathrm{cc}(X_1) \cap \mathrm{cc}(X_2) = \mathrm{cc}(X_1 \cap X_2)$ is a convex Hilbert cube. Using the same techniques as were used in Example (4.6) one can easily show that $\mathrm{cc}(X_1) \cap \mathrm{cc}(X_2)$ is a Z-set in $\mathrm{cc}(X_1)$. By Handel's result [8], it follows that $\mathrm{cc}(X_1) \cap \mathrm{cc}(X_2) = \mathrm{cc}(X)$ is a Hilbert cube.

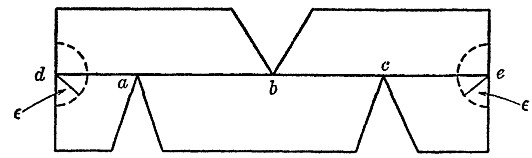
5. Some geometric considerations.

In view of Theorem (3.1), it is natural to ask the question:

Which 2-cells X in R^2 have the property that $\mathrm{cc}(X) \cong I_\infty$?

The following example shows that not every 2-cell in R^2 has this property.

(5.1) EXAMPLE. Let X be the 2-cell in R^2 pictured below.



The three points a, b and c of local nonconvexity of X all lie on the convex arc $\widehat{de}$. It is clear that any compact convex subset of X which is within ε of the arc $\widehat{de}$ (in the Hausdorff metric) must be a subarc of $\widehat{de}$. Hence, it follows that $\widehat{de}$ has small 2-cell neighborhoods in $\mathrm{cc}(X)$. Therefore, $\mathrm{cc}(X)$ is 2-dimensional at $\widehat{de}$ and, thus, $\mathrm{cc}(X) \not\cong I_\infty$.

The remainder of this section is devoted to proving two results which can be used to establish that some rather wide classes of 2-cells do have the property that their hyperspaces of nonempty compact convex subsets are topologically I_∞. We begin with some definitions.

(5.2) DEFINITION. Let K be a starshaped subset of l^2 and let $p \in \mathrm{Ker}(K)$. The point $x \in K$ will be called a *p-relative interior point of K* if there exists an $x^* \in K$ such that, for some $\lambda \in (0, 1)$, $\lambda x^* + (1 - \lambda)p = x$. A point in K which is not a p-relative interior point will be called a *p-relative extreme point of K.*

(5.3) DEFINITION. Let $K_1 \subseteqq K_2$ be two starshaped subsets of l_2 such that $\mathrm{Ker}(K_1) \cap \mathrm{Ker}(K_2) \neq \emptyset$. Let $p \in [\mathrm{Ker}(K_1) \cap \mathrm{Ker}(K_2)]$. Then p is called a *K_2 inside point of K_1* if, for every $x \in K_2$, $\{\lambda p + (1-\lambda)x\colon \lambda \in (0, 1)\} \cap K_1 \neq \emptyset$.

(5.4) THEOREM. *Let $K_1 \subseteqq K_2$ be two compact, starshaped subsets of l_2 and suppose that there exists a point $p \in K_1$ such that:*

(i) $p \in \mathrm{Ker}(K_1) \cap \mathrm{Ker}(K_2)$,

(ii) *p is a K_2-inside point of K_1,*

(iii) *the set of all p-relative interior points of K_1 (resp., K_2) is an open subset of K_1 (resp., K_2). Then, K_1 and K_2 are homeomorphic.*

Proof. Let the hypothesis of the theorem be satisfied. We will assume without loss of generality that $p = (0, 0, 0, \cdots)$. For each point $x \in K_2 - \{p\}$ (clearly, the theorem is valid if $K_2 - \{p\} = \emptyset$) let $\bar{x}$ be that p-relative extreme point of K_2 defined by $\bar{x} = \alpha_x x$ where $\alpha_x = \sup\{\alpha \in (0, \infty)\colon \alpha x \in K_2\}$. To each p-relative extreme point y of K_2, let $\lambda_y = \sup\{\lambda \in [0, 1]\colon \lambda y \in K_1\}$. Let $f\colon K_2 \to K_1$ be the function defined by

$$f(x) = \begin{cases} \lambda_2 x, & \text{if } x \in K_2 - \{p\} ; \\ p, & \text{if } x = p . \end{cases}$$

It is easy to see that f is one-to-one. We wish now to show that f is onto and continuous. To see that f is onto, let $x \in K_1$. If $x = p$, we are done since $f(p) = p$. If $x \neq p$, then $1/\lambda_2 \leqq \alpha_x$. Hence, $y = x/\lambda_2 \in K_2$ and, clearly, $f(y) = x$. We have seen that f is onto. To see that f is continuous, let $\{x_i\}_{i=1}^{\infty}$ be a sequence in K_2 such that $\lim_{i\to\infty} x_i = x \in K_2$. If $x = p$, it is clear that $\lim_{i\to\infty} f(x_i) = f(p) = p$. So, assume that $x \neq p$. We may then assume that $x_i \neq p$ for all i. We will first show that $\lim_{i\to\infty} \bar{x}_i = \bar{x}$. Since K_2 is compact, we must have that some subsequence $\{\bar{x}_{i_j}\}_{j=1}^{\infty}$ of the sequence $\{\bar{x}_i\}_{i=1}^{\infty}$ converges to an $x_0 \in K_2$. Without loss of generality, we may assume that the sequence $\{\bar{x}_i\}_{i=1}^{\infty}$ converges to x_0. Now, it follows from condition (iii) that x_0 must be a p-relative extreme point of K_2. To see that $x_0 = \bar{x}$, we need now only show that, for some $\lambda > 0$, $\lambda x_0 = x$. Let λ_i be such that $\lambda_i x_i = \bar{x}_i$ and consider $\lambda_i x$. Now, the λ_i's are bounded and since $||\lambda_i x - \lambda_i x_i|| = |\lambda_i|\, ||x - x_i||$ we have that $\lim_{i\to\infty} \lambda_i x = x_0$. It is now not difficult to see that, for some $\lambda_0 > 0$, $\lim_{i\to\infty} \lambda_i = \lambda_0$ and $\lambda_0 x = x_0 = \bar{x}$. To establish the continuity of f, we need now only show that $\lim_{i\to\infty} \lambda_{\bar{x}_i} = \lambda_{\bar{x}}$. First consider $\{\lambda_{\bar{x}_i} \bar{x}_i\}_{i=1}^{\infty}$. Since, for each i, $\lambda_{\bar{x}_i} \bar{x}_i$ is a p-relative extreme point of K_1, we have that some subsequence converges to a p-relative extreme point of K_1. Without loss of generality, we will assume that $\lim_{i\to\infty} \lambda_{\bar{x}_i} \bar{x}_i = x'$ where x' is a p-relative extreme point of K_1. But, $||\lambda_{\bar{x}_i} \bar{x} - \lambda_{\bar{x}_i} \bar{x}|| = |\lambda_{\bar{x}_i}|\, ||\bar{x} - \bar{x}_i|| \leqq ||\bar{x} - \bar{x}_i||$. Hence, $\lim_{i\to\infty} \lambda_{\bar{x}_i} \bar{x} = x'$. But, the fact that the sequence $\{\lambda_{\bar{x}_i} \bar{x}\}_{i=1}^{\infty}$ is Cauchy implies that $\{\lambda_{\bar{x}_i}\}_{i=1}^{\infty}$ is Cauchy and, hence, that there exists a λ' such that $\lim_{i\to\infty} \lambda_{\bar{x}_i} = \lambda'$. Thus, $\lambda' \bar{x} = x'$ which says that $\lambda' = \lambda_{\bar{x}}$. We have now established the continuity of f. Since K_1 and K_2 are compact, it follows that f is a homeomorphism.

(5.5) COROLLARY. *Let X be a compact starshaped subset of R^n such that* $\operatorname{int}[\operatorname{Ker}(X)] \neq \varnothing$. *Then,* $\operatorname{cc}(X) \cong I_\infty$.

Proof. For simplicity, we will assume that the origin $0 \in \operatorname{int}[\operatorname{Ker}(X)]$. Let $\varepsilon > 0$ be such that $\bar{B}_\varepsilon = \{x \in R^n : ||x|| \leqq \varepsilon\}$ is contained in $\operatorname{Ker}(X)$. Since X is compact, there exists an $r > 0$ such that $X \subset \bar{B}_r$. Let F be an affine embedding of $\operatorname{cc}(\bar{B}_r)$ into l_2 such that $F(0) = 0$ (as in the proof of (2.2)). Let $K_1 = F(\operatorname{cc}(\bar{B}_\varepsilon))$ and let $K_2 = F(\operatorname{cc}(X))$. Then, $K_1 \subseteqq K_2$. Since we have already seen that $\operatorname{cc}(\bar{B}_\varepsilon) \cong I_\infty$ (Theorem (2.2)), the result will now follow provided conditions (i), (ii) and (iii) of (4.4) are shown to be satisfied for $p = 0$. It is easy to see that conditions (i) and (ii) are satisfied. That condition (iii) is satisfied will follow if we can show that the

p-relative extreme points of K_1 (resp., K_2) are precisely those elements of the form $F(G)$ where $G \cap \mathrm{Fr}(\bar{B}_\varepsilon) \neq \varnothing$ (resp., $G \cap \mathrm{Fr}(X) \neq \varnothing$). We will show this only for K_2 since it is obvious for K_1. It is clear that if $G \in \mathrm{cc}(X)$ is such that $G \cap \mathrm{Fr}(X) = \varnothing$ then $F(G)$ is not a p-relative extreme point of K_2. It remains only to show that if $G \in \mathrm{cc}(X)$ is such that $G \cap \mathrm{Fr}(X) \neq \varnothing$ then $F(G)$ is a p-relative extreme point of K_2. Suppose not, then there exists a $\lambda > 1$ such that $\lambda F(G) \in K_2$. Let $G' \in \mathrm{cc}(X)$ be such that $F(G') = \lambda F(G)$. By the one-to-oneness and the convexity of F, it follows that $\lambda G = G'$. If $c \in G \cap \mathrm{Fr}(X)$, then $\lambda c \in X$. But $\overline{\mathrm{co}}(\lambda c, \bar{B}_\varepsilon) \subset X$ and contains c as an interior point. This contradicts the fact that $c \in \mathrm{Fr}(X)$. The corollary now follows. T. A. Chapman showed (see Theorem 10 of [5]) that a compact Hilbert cube manifold is homeomorphic to the Hilbert cube if and only if it is homotopically trivial. This enables one to "localize" the problem of showing the cc-hyperspace of a given 2-cell is homeomorphic to I_∞.

(5.6) THEOREM. (1) *If X is a contractible continuum lying in a Banach space, then* $\mathrm{cc}(X)$ *is contractible.*

(2) *Thus, in particular, if X is a 2-cell (or n-cell),* $\mathrm{cc}(X) \cong I_\infty$ *if and only if* $\mathrm{cc}(X)$ *is a Hilbert cube manifold.*

Proof. The closed linear span L of X is a separable Banach space. By (3.2), there is a continuous selection η from $\mathrm{cc}(X)$ to X. Define $g\colon \mathrm{cc}(X) \times [0, 1] \to \mathrm{cc}(X)$ by $g(A, t) = t\{\eta(A)\} + (1 - t)A$. It follows using g and the contractibility of X that $\mathrm{cc}(X)$ is contractible. This proves (1). The proof of (2) uses (1) and Theorem 10 of [5].

These next results will show that a fairly large class of 2-cells have the property that their hyperspaces of compact convex subsets are homeomorphic to I_∞. We begin with a notational agreement and a definition.

If A is a nondegenerate, convex arc in the plane then by $A^\sim$ we will denote the unique line in R^2 which contains A. If $p \in R^n$ and $\varepsilon > 0$ then $B(\varepsilon, p) = \{x \in R^n\colon \|x - p\| < \varepsilon\}$.

(5.7) DEFINITION. Let X be a 2-cell in R^2 and let $A \in \mathrm{cc}(X)$ be an arc. Suppose that one complementary domain of $A^\sim$ has been designated the right side of $A^\sim$ and the other the left side of $A^\sim$. A point $p \in LN(X) \cap A$ will be said to lie on the left side (right side) of A if, for every $\varepsilon > 0$, $B(\varepsilon, p) - X$ contains points on the left side (right side) of $A^\sim$. If for some $\varepsilon > 0$, $B(\varepsilon, p) - X$ contains no points on the right side (left side) of $A^\sim$ then p will be said to lie strictly on the left side (right side) of A.

(5.8) LEMMA. *Let X be an n-cell. If $A \in \mathrm{cc}(X)$ is an n-cell then A is contained in a closed starshaped subset N of X with* $\mathrm{int}[\mathrm{Ker}(N)] \neq \emptyset$ *such that* $\mathrm{cc}(N)$ *is a neighborhood of A in* $\mathrm{cc}(X)$.

Proof. Let $A \in \mathrm{cc}(X)$ be an n-cell and let $q \in \mathrm{int}[A]$. Let $\varepsilon > 0$ be chosen so that $\mathrm{cl}(B(\varepsilon, q)) \subset \mathrm{int}[A]$. Let $\Gamma = \{K \in \mathrm{cc}(X) : \mathrm{cl}(B(\varepsilon, q)) \subset K\}$ and let $D = \cup \Gamma$. It is not difficult to see that D is a closed starshaped subset of X and that $\mathrm{Ker}(D) \supseteqq \mathrm{cl}(B(\varepsilon, q))$. It is also not difficult to see that $\mathrm{cc}(D)$ is a neighborhood of A in $\mathrm{cc}(X)$. The lemma is proved.

(5.9) LEMMA. *If X is an n-cell in R^n then the following are equivalent:*

(i) *Every $A \in \mathrm{cc}(X)$ lies in a starshaped subset of X whose kernel has nonvoid interior.*

(ii) *Every maximal convex subset of X is an n-cell.*

Proof. Suppose (i) is satisfied. Let $A \in \mathrm{cc}(X)$ be maximal. By (i) there exists an n-ball $B \subset X$ such that $\overline{\mathrm{co}}\{B, A\} \subset X$. But, by maximality of A, $\overline{\mathrm{co}}\{B, A\} = A$. Hence A is an n-dimensional compact convex subset of R^n and thus must be an n-cell. We have that (i) implies (ii). Now, if (ii) holds and $A \in \mathrm{cc}(X)$, then let $M(A)$ be a maximal convex subset of X which contains A. As $M(A)$ is a starshaped set whose kernel has nonvoid interior, we are done.

(5.10) LEMMA. *Let X be a 2-cell in R^2. Let $A \in \mathrm{cc}(X)$ be an arc with noncut points p and q. Suppose there exists a closed ball $D \subset X$ and neighborhoods P of p and Q of q in X such that for each $d \in D$ we have $P \cup Q \subset S(d)$. Then A is contained in a closed starshaped subset Y of X with* $\mathrm{int}[\mathrm{Ker}(Y)] \neq \emptyset$ *such that* $\mathrm{cc}(Y)$ *is a neighborhood of A in* $\mathrm{cc}(X)$.

Proof. We can assume that D lies in the interior of a convex 2-cell $B \subset X$ such that A is on the boundary of B. We may also assume that $A - (P \cup Q) \neq \emptyset$ (we would be done in this case anyway as will become evident at the end of the proof). Let P' and Q' be balls in R^2 centered at p and q, respectively, which satisfy

(a) the radii of P' and Q' are less than 1/2 min {radius of P, radius of Q}, and

(b) for each $a \in [A - (P \cup Q)]$, $r \in \mathrm{cl}(P')$, $s \in \mathrm{cl}(Q')$ and $d \in D$, the ray through a from d must intersect the segment $\widehat{rs}$ in a cut point. Now, for each $a \in A - (P \cup Q)$, choose a ball B_a about a such that

(**) if $r \in \mathrm{cl}(P')$, $s \in \mathrm{cl}(Q')$, $t \in B_a$ and $d \in D$, then the ray from

d through t must intersect the segment $\widehat{rs}$ in a cut point.

Let $\sum$ be the collection of all convex sets C in X such that C inter sects both P' and Q' and is contained in the union of P, Q and the balls B_a for $a \in A - (P \cup Q)$. It is clear that $\sum$ is a neighborhood of A in $\mathrm{cc}(X)$. We wish to show now that if $d \in D$, then d sees each point of any C in $\sum$. So, let $C \in \sum$ and let $r \in [P' \cap C]$ and $s \in [Q' \cap C]$. Let $\alpha \in C - (P \cup Q)$ (note, if $\alpha \in [P \cup Q]$ we are done) and let $a \in A - (P \cup Q)$ be such that $\alpha \in B_a$. Since $\alpha \in B_a$, by (**) we have that the ray from d through $\alpha (d \in D)$ must intersect $\widehat{rs}$. By simple connectivity of X, it follows that the 2-cell (rds) and $(rs\alpha)((rs\alpha)$ may be an arc) lie in X. If the segment $\widehat{d\alpha}$ intersects $\widehat{rs}$ then $\widehat{d\alpha} = [\widehat{d\alpha} \cap (rsd)] \cup [\widehat{d\alpha} \cap (rs\alpha)] \subset X$. If the segment $\widehat{d\alpha}$ does not intersect $\widehat{rs}$, then $\widehat{d\alpha} \subset (rsd) \subset X$. Thus, $\widehat{d\alpha} \subset X$ and we have the desired conclusion. Now, let $\Gamma = \{K \in \mathrm{cc}(X)\colon\ K \supset D\}$. Let $Y = \cup \Gamma$. We have just seen that the starshaped set Y has the property that $\mathrm{cc}(Y) \supset \sum$. Also, we have that $\mathrm{Ker}[Y] \supset \mathrm{int}[D]$ and hence $\mathrm{int}[\mathrm{Ker}(Y)] \neq \varnothing$. The lemma is proved.

(5.11) LEMMA. *Let X be a polygonal 2-cell in R^2 and let $A \in \mathrm{cc}(X)$ be an arc such that no two points in $LN(X) \cap A$ lie strictly on opposite sides of A. Then there exists a closed starshaped subset N of X with* $\mathrm{int}[\mathrm{Ker}(N)] \neq \varnothing$ *such that* $\mathrm{cc}(N)$ *is a neighborhood of A in* $\mathrm{cc}(X)$.

Proof. Let A be an arc in $\mathrm{cc}(X)$ such that no two points of $LN(X) \cap A$ lie strictly on opposite sides of A. Consider the noncut points, say p and q, of A. If at least one of p and q is not a point in $LN(X)$ which lies strictly on one side of A then it can be seen that there is a closed ball D in X and neighborhoods $B(\alpha, p) \cap X$ and $B(\gamma, q) \cap X$ such that, for any $d \in D$, $(B(\alpha, p) \cup B(r, q)) \cap X \subset S(d)$. The result now follows from (5.10). Suppose now that both p and q are points in $LN(X)$ which lie strictly on one side of A. It is geometrically clear that, in this event, ono can obtain balls P, Q and M such that

(a) $p \in P$, $q \in Q$ and $\mathrm{cl}(M) \subset \mathrm{int}[X]$,

(b) $\mathrm{cl}(M) \cap A = \varnothing$, and

(c) if C is a convex set in X such that $C \cap P \neq \varnothing$ and $C \cap Q \neq \varnothing$ then $C \cap (P \cup Q) \subset S(m)$ for every $m \in \mathrm{cl}(M)$.

The proof from here proceeds as it did in the proof of (5.10).

(5.12) THEOREM. *Let X be a polygonal 2-cell in R^2. Then the following are equivalent*:

(i) *Every maximal convex subset of* X *is a* 2-*cell.*

(ii) *Each* $A \in \mathrm{cc}(X)$ *is contained in a closed starshaped subset* N *of* X *for which* $\mathrm{int}[\mathrm{Ker}(N)] \neq \emptyset$ *and* $\mathrm{cc}(N)$ *is a neighborhood of* A *in* $\mathrm{cc}(X)$.

Furthermore, if (i) *or* (ii) *holds then* $\mathrm{cc}(X) \cong I_\infty$.

Proof. That condition (ii) implies condition (i) follows from (5.9). Now, assume that (i) holds. If $A \in \mathrm{cc}(X)$ is a singleton then it is easy to see that A is contained in a closed starshaped neighborhood N in X. But then $\mathrm{cc}(N)$ is a neighborhood of A in $\mathrm{cc}(X)$ and we are done in this case. If $A \in \mathrm{cc}(X)$ is a 2-cell, then we are done by virtue of (5.8). If A is an arc, then by (5.11) we will be done if we can show that no two points in $LN(X) \cap A$ lie strictly on opposite sides of A. Let $p_1, p_2 \in LN(X) \cap A$ lie strictly on opposite sides of A. If both p_1 and p_2 are cut points of A then it is clear that no convex 2-cell in X can contain A and this contradicts (i). If one or more of p_1 and p_2 are noncut points of A then one can obtain an arc $A' \supset A$ with $A' \in \mathrm{cc}(X)$ for which both p_1 and p_2 are cut points. This again leads to a contradiction of condition (i). Thus, no two points of $LN(X) \cap A$ can lie strictly on opposite sides of A and we have the desired result. We have now established the equivalence of (i) and (ii).

To complete the proof we need only see that if (ii) holds then $\mathrm{cc}(X) \cong I_\infty$. So, suppose that (ii) holds. Let $A \in \mathrm{cc}(X)$ by virtue of (ii) there exists a closed starshaped subset N of X with $\mathrm{int}[\mathrm{Ker}(N)] \neq \emptyset$ for which $\mathrm{cc}(N)$ is a neighborhood of A in $\mathrm{cc}(X)$. But, $\mathrm{cc}(N) \cong I_\infty$ by (5.5). Thus, $\mathrm{cc}(X)$ is homeomorphic to I_∞ by virtue of (5.6). The theorem is proved.

(5.13) THEOREM. *Let* X *be a* 2-*cell in* R^2 *such that* $(*)$ *whenever* $p, q \in X$ *are such that* $p \in S(q)$ *and* N *is a neighborhood of* p *in* X, *then there exists an open set* $M \subset N$ *and a neighborhood* Q *of* q *such that for each point* m *in* M *we have* $S(m) \supset Q$.

The following are equivalent:

(i) *Every maximal convex subset of* X *is a* 2-*cell.*

(ii) *Each* $A \in \mathrm{cc}(X)$ *is contained in a starshaped subset* N *of* X *for which* $\mathrm{int}[\mathrm{Ker}(N)] \neq \emptyset$ *and* $\mathrm{cc}(N)$ *is a neighborhood of* A *in* $\mathrm{cc}(X)$.

Furthermore, if (i) *or* (ii) *holds then* $\mathrm{cc}(X) \cong I_\infty$.

Proof. All aspects of the proof for this result are the same as the proof of (5.12) with the exception of showing that condition (i) implies condition (ii). So, suppose that condition (i) holds and let $A \in \mathrm{cc}(X)$. If A is a singleton, it is easy to use $(*)$ to obtain the desired set N. If A is a 2-cell we are again done by virtue of (5.8). Suppose, that $A = [p, q]$ is an arc. Let B be a 2-cell in $\mathrm{cc}(X)$ which contains A (condition (i) implies B exists). Let $b \in \mathrm{int}(B)$. Since $p \in S(b)$ there is by $(*)$ a ball $C \subset B$ and a neighborhood P of p such that for each $m \in C$ we have $S(m) \supset P$. Let $m_1 \in C$. Since $m_1 \in B$ we have $S(m_1) \supset q$. Thus, by $(*)$, there exists a closed ball $D \subset C$ and a neighborhood Q of q such that, for any $d \in D$. $S(d) \supset Q$. Now application of (5.10) gives the existence of the starshaped subset N of X with the desired properties. The result is established.

6. Some problems and examples. While at present we have some large classes of nonconvex 2-cells whose cc-hyperspaces are homeomorphic to I_∞, we still do not know exactly which 2-cells have their cc-hyperspaces homeomorphic to I_∞. The following problems are connected with this.

(6.1) *Problem.* Let X be a 2-cell in R^2. If every point of $\mathrm{cc}(X)$ has arbitrarily small infinite dimensional neighborhoods, is it true that $\mathrm{cc}(X) \cong I_\infty$?

(6.2) *Problem.* Let X be a 2-cell in R^2. If every maximal convex subset of X is either a point or a 2-cell, is it true that $\mathrm{cc}(X) \cong I_\infty$?

(6.3) *Problem.* Let X be a 2-cell in R^2. If every maximal convex subset of X is a 2-cell, is it true that $\mathrm{cc}(X) \cong I_\infty$?

An affirmative answer to (6.1) would provide a satisfactory characterization. This is true since it would then follow that Example 5.1 is, in a sense, canonical. An affirmative answer to (6.1) would imply an affirmative answer to (6.2) and an affirmative answer to (6.2) would imply an affirmative answer to (6.3).

The following two examples give a bit more insight into the above problems. The technique used in this next example is one which has become standard in infinite dimensional topology. It was first used by Schori and West in [18]. For the difinition of shape see [4]. An onto map $f: X \to Y$ where X and Y are homeomorphic metric spaces, is a *near homeomorphism* if f can be uniformly approximated by homeomorphisms. For terminology related to inverse limits it is suggested that the reader see [13] or [18]. In the discussion of the example we use a characterization by T. A. Chapman of near homeomorphisms between Hilbert cubes as being those continuous surjections for which point inverses have trivial shape.

(6.4) EXAMPLE. Consider the planar 2-cell X formed by intersecting the planar regions A, B and C where $A = \{(x, y): x \leqq 1/2, y \geqq 0\}$, $B = \{(x, y): (x + 1/2)^2 + y^2 \geqq 1/4\}$ and $C = \{(x, y): x^2 + y^2 \leqq 1\}$ (see Fig. 6.6 below).

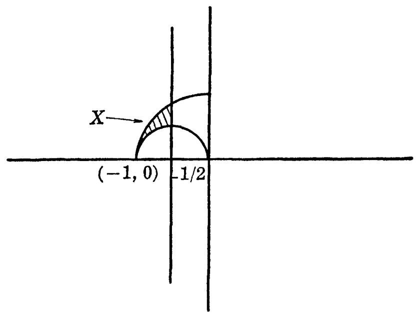


FIGURE 6.6

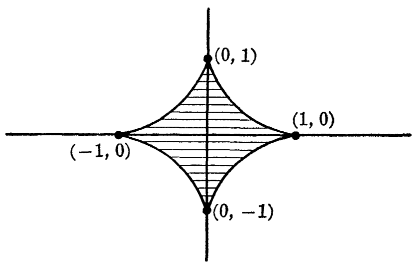


FIGURE 6.7

Note that the point $(-1, 0)$ is a maximal convex subset of X. Now, for each $3\pi/4 \leqq \theta \leqq \pi$ let $X_\theta = X \cap \{(r, \varphi): \pi/2 \leqq \varphi \leqq \theta\}$. For each pair (θ_1, θ_2) with $\pi/2 \leqq \theta_1 \leqq \theta_2 \leqq \pi$, let the mapping $g_{\theta_2\theta_1}: X_{\theta_2} \to X_{\theta_1}$ be defined by $g_{\theta_2\theta_1}(r, \varphi) = (r, \theta_1)$ for $\theta_1 \leqq \varphi \leqq \theta_2$, and $g_{\theta_2\theta_1}(r, \varphi) = (r, \varphi)$ if $\pi/2 \leqq \varphi \leqq \theta_1$. Define, for (θ_1, θ_2) as above, the retraction $r_{\theta_2\theta_1}$: $\mathrm{cc}(X_{\theta_2}) \to \mathrm{cc}(X_{\theta_1})$ by $r_{\theta_2\theta_1}(A) = \overline{\mathrm{co}}(g_{\theta_2\theta_1}(A))$. Also, for a compact convex subset A of X_θ which intersects $\{(r, \theta): r \geqq 0\}$ define $p_1(A, \theta) = \inf\{r: (r, \theta) \in A\}$. For each $n = 1, 2, \cdots$, let $\theta_n = \pi - \pi/2^{n+1}$ and let $r_n = r_{\theta_{n+1}\theta_n}$ and $X_n = X_{\theta_n}$. For $A \in \mathrm{cc}(X_n)$, let $Y \in r_n^{-1}(A)$ and define $\theta_Y = \sup\{\theta: r_n(r_{\theta_{n+1}\theta}(Y)) = A\}$. For each $\theta \in [\theta_n, \theta_{n+1}]$, let

$$H(Y, \theta) = r_{\theta_{n+1},\theta}(Y) \quad \text{if} \quad \theta_Y \leqq \theta \leqq \theta_{n+1} ,$$

$$\mathrm{co}(r_{\theta_{n+1}\theta_Y}(Y) \cap X_\theta) \cup \{(p_1(r_{\theta_{n+1}\theta}(r_{\theta_{n+1}\theta_Y}(Y)), \theta)\} \quad \text{if} \quad \theta_n \leqq \theta \leqq \theta_Y .$$

It is geometrically clear that $H: r_n^{-1}(A) \times [\theta_n, \theta_{n+1}] \to r_n^{-1}(A)$ is a homotopy of the identity on $r_n^{-1}(A)$ to a constant map. Thus, for each $A \in \mathrm{cc}(X_n)$, $r_n^{-1}(A)$ is contractible and, hence [4, (5.5) p. 28], of trivial shape. It now follows that r_n is a near homeomorphism and, hence, (since each X_n satisfies the conditions of Theorem 5.13) that $\varprojlim_n(\mathrm{cc}(X_n), r_n) = \mathrm{cc}(X_1) = I_\infty$. Furthermore, the inverse sequence

$\{(\mathrm{cc}(X_n), r_n)\}$ also satisfies the conditions that

(a) $\mathrm{cc}(X_n) \subset \mathrm{cc}(X_{n+1})$ and $\overline{\bigcup_n \mathrm{cc}(X_n)} = \mathrm{cc}(X)$,

(b) $\sum_{n=1}^{\infty} d(r_n, id_{\mathrm{cc}(X_{n+1})}) < \infty$,

(c) for each j, $\{r_j \circ \cdots \circ r_i : \mathrm{cc}(X_{i+1}) \to \mathrm{cc}(X_j) \mid i \geqq j\}$ is an equi-uniformly continuous family of functions.

That condition (a) holds is immediate. The fact that condition (b) holds rests on the fact that if $d(A, B) < \varepsilon$ and B is convex then $d(\overline{\mathrm{co}}(A), A) \leqq \varepsilon$.

To see that (c) holds, let, for each n, $r^n: X \to X_n$ be the retraction $g_{\pi\theta_n}$.

Let $j \in I^+$ be given and let $\varepsilon > 0$. Choose j_0 so that if $A \notin \mathrm{cc}(\mathrm{int}[X_{j_0+1}])$ then $A \cap X_j = \varnothing$. Choose $\delta_1 > 0$ so that if $d(A, B) < \delta_1$ then $d(r^n(A), r^n(B)) < \varepsilon$. Let $\delta_2 > 0$ be chosen so that, if $d(A, B) < \delta_2$ and $A, B \in \mathrm{cc}(X_{j_0+1})$, then $d(r_j \circ \cdots \circ r_{j_0}(A), r_j \circ \cdots \circ r_{j_0}(B)) < \varepsilon$. Let δ_3 be chosen so that, if $A \notin \mathrm{cc}(\mathrm{int}[X_{j_0+1}])$ and $d(A, B) < \delta_3$, then $B \cap X_j = \varnothing$. Now, if $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and $d(A, B) < \delta$ then, either $A, B \in \mathrm{cc}(X_{j_0+1})$ in which case $d(r_j \circ \cdots \circ r_k(A), r_j \circ \cdots \circ r_k(B)) \leqq d(r_j \circ \cdots \circ r_{j_0}(A), r_j \circ \cdots \circ r_{j_0}(B)) < \varepsilon$ or $A \cap X_j = \varnothing$ and $B \cap X_j = \varnothing$ in which case $r_j \circ \cdots \circ r_k(A) = r^j(A)$ and $r_j \circ \cdots \circ r_k(B) = r^j(B)$ and, hence, $d(r_j \circ \cdots \circ r_k(A), r_j \circ \cdots \circ r_k(B)) < \varepsilon$. We have established that condition (c) holds. Thus, by [13, Lemma B], $\mathrm{cc}(X) \cong \lim_i(\mathrm{cc}(X_i), r_i)$ and thus $\mathrm{cc}(X) \cong I_\infty$.

(6.5) EXAMPLE. Consider the 2-cell X in R^2 which is the closure of the bounded complementary domain of $\bigcup_{i=1}^4 C_i$, where

$$C_1 = \{(x, y): (x-1)^2 + (y-1)^2 \leqq 1\}, \quad C_2 = \{(x, y): (x-1)^2 + (y+1)^2 \leqq 1\}$$
$$C_3 = \{(x, y): (x+1)^2 + (y+1)^2 \leqq 1\} \text{ and } C_4 = \{(x, y): (x+1)^2 + (y-1)^2 \leqq 1\}.$$

(Fig. 6.7.) Note, the convex segment with noncut points $(0, -1)$ and $(0, 1)$ is a maximal convex subset of X and the kernel of X consists only of the origin $(0, 0)$. In spite of this, if one takes $Y = \{(x, y): x^2 + y^2 \leqq 1/4\}$ and sets $K_1 = \mathrm{cc}(Y)$, $K_2 = \mathrm{cc}(X)$ and $p = (0, 0)$ then all the conditions of Theorem 4.4 are satisfied. It follows that $\mathrm{cc}(X) \cong \mathrm{cc}(Y) \cong I_\infty$.

The 2-cell of Example (6.4) illustrates the validity of (6.1) and (6.2) for a specific 2-cell. The 2-cell of Example (6.5) illustrates that though the hypotheses in (6.2) and (6.3) may be sufficient, they are definitely not necessary.

7. The cc-hyperspaces of ${}^0B^n$ and R^n, $n \geqq 2$. In this section we show that $\mathrm{cc}({}^0B^n)$ and $\mathrm{cc}(R^n)$, $n \geqq 2$, are homeomorphic to the Hilbert cube with a point removed. We also state some problems.

Let U be a nonempty proper open subset of $\mathrm{cc}(B^n)$. For each

$A \in U$ let $Au = \inf\{d(A, D) \mid D \in [\mathrm{cc}(X) - U]\}$, where d denotes the Hausdorff metric. Note that $0 < Au \leqq 2$.

(7.1) LEMMA. *Let U be a proper open subset of* $\mathrm{cc}(B^n)$. *Let $A \in U$ and let α be real, $0 < \alpha \leqq 1$. Then* $(1 - \alpha Au/2)A \in [U \cap \mathrm{cc}({}^0B^n)]$.

Proof. For any $a \in A$ and $\beta > 0, \beta \neq 1$, note that $\|a - \beta a\| = |1 - \beta| \, \|a\| \leqq |1 - \beta| < 2|1 - \beta|$. Thus, setting $\beta = 1 - \alpha Au/2$, it follows that

$$d\left(A, \left(1 - \frac{\alpha Au}{2}\right)A\right) < 2\left|1 - \left(1 - \frac{\alpha Au}{2}\right)\right| = \alpha Au \leqq Au \,,$$

which implies $(1 - \alpha Au/2)A \in U$. Note that $(1 - \alpha Au/2)A \in \mathrm{cc}({}^0B^n)$ since $(1 - \alpha Au/2) < 1$.

(7.2) THEOREM. *If $n \geqq 2$, then* $\mathrm{cc}({}^0B^n) \cong I_\infty - \{p\}$ *for* $p \in I_\infty$.

Proof. Let $K = \{A \in \mathrm{cc}(B^n) \mid A \cap S^{n-1} \neq \varnothing\}$. We show K has property Z in $\mathrm{cc}(B^n)$. Let U be a nonempty homotopically trivial open subset of $\mathrm{cc}(B^n)$. Let $f\colon S^{k-1} \to U - K$ be continuous, and let $F\colon B^k \to U$ be a continuous extension of f. Let $h\colon [0, 1] \to [0, 1]$ be a homeomorphism such that $h(0) = 1$ and $h(1) = 0$. Define a function F^* on B^k by $F^*(x) = (1 - [h(\|x\|\, F(x)u/2)])F(x)$. Note F^* is continuous and F^* extends f since if $\|x\| = 1$, $F^*(x) = F(x) = f(x)$. If $\|x\| < 1$ note that $F^*(x) \in [U \cap \mathrm{cc}({}^0B^n)]$ by (7.1), and hence $F^*(x) \in [U - K]$. Thus, K has property Z in $\mathrm{cc}(B^n)$. Hence, by (2.2) above and a theorem of Anderson [1], we assume without loss of generality that $K \subset I^0_\infty$. For each $t \in [0, 2]$ and $A \in K$ let $g(A, t) = \mathrm{cl}(N(t, A) \cap B^n)(N(t, A) = \bigcup_{a \in A} \{x \mid \|x - a\| < t\})$. Note g is continuous and that $g(A, 0) = A$ and $g(A, 2) = B^n$. (See Borsuk [4].) By a result of Chapman [6] it follows that $\mathrm{cc}(B^n) - K \cong \mathrm{cc}(B^n) - \{M\}$ for $M \in \mathrm{cc}(B^n)$. Hence, by (2.2) above, $\mathrm{cc}({}^0B^n) \cong I_\infty - \{p\}$, and this completes the proof.

(7.3) THEOREM. *If $n \geqq 2$,* $\mathrm{cc}(R^n) \cong I_\infty - \{p\}$ *for* $p \in I_\infty$.

Proof. Using the proof of (5.4), it is easy to see that $\mathrm{cc}(R^n) \cong \mathrm{cc}({}^0B^n)$. Therefore, by (7.2) $\mathrm{cc}(R^n) \cong I_\infty - \{p\}$. Theorem 7.3 suggests the following.

(7.4) *Problem.* If H is a separable Hilbert space, is $\mathrm{cc}(H) \cong H$?

We will now discuss and state two problems which arise out of our previous work. Problem 7.5 is motivated in part by the result of Schori and West [16] that $2^I \cong I_\infty$.

Let D be the semidisc in R^2 given by $\{(x, y)|x^2 + y^2 \leqq 1, y \geqq 0\}$ and let K be the semicircle $D \cap S^1$. Let $R = \{A \in \mathrm{cc}(D)|\mathrm{ext}[A] \subset K\}$. The mapping $f: 2^K \to R$ given by $f(E) = \overline{\mathrm{co}}(E)$ is a homeomorphism. Let $R^* = \mathrm{cc}(D) - R$. Note that R^* is an open convex subset of $\mathrm{cc}(D)$ and that $I_\infty \cong R = \mathrm{cc}(D) - R^*$. This suggests the following problem:

(7.5) *Problem.* Let M be an open convex subset of a convex Hilbert cube Q. What are necessary and sufficient conditions on M in order that $I_\infty \cong Q - M$?

Several times in our work we encountered infinite dimensional compact convex subsets P of I_∞ such that $P \cong \mathrm{ext}[P] \cong I_\infty$. The countable product of semidiscs is such an example. This suggests the following problem.

(7.6) Let Q be a convex Hilbert cube. What are necessary and sufficient conditions for Q to be homeomorphic with $\mathrm{ext}[Q]$?

We remark that a theorem answering the above question may by considered as a compact analogue of the theorem of Klee [11] that in separable Hilbert space the unit sphere is homeomorphic with the closed unit ball.

REMARK. After this paper was written, certain developments occurred which may be of interest to the reader. D. W. Curtis in a forthcoming paper entitled "Growth hyperspaces" investigates, among other things, subspaces G of the cc-hyperspace having the property that if $A \in G$ and $A \subset B$ then $B \in G$. D. W. Curtis, J. Quinn and R. M. Schori in a forthcoming paper entitled "On the cc-hyperspace of a polyhedral two-cell" show that the cc-hyperspace of a polyhedral two cell in R^2 is I_∞ with perhaps a finite number of two cell flanges. J. Quinn and R. Y. T. Wong in a forthcoming paper entitled "Unions of convex Hilbert cubes" show that the union of finitely many convex Hilbert cube manifolds each subcollection of which intersects vacuously or in a Hilbert cube is a Hilbert cube manifold, and, as a corollary, obtain the result that if A and B are infinite dimensional compact convex sets in l_2 such that $A \cap B$ is infinite dimensional then $A \cup B \cong I_\infty$. Reiter and Stavrakas in a forthcoming paper entitled "On the compactness of the hyperspace of faces" and Quinn and Stavrakas in a forthcoming paper "Selections in the hyperspace of faces" investigate certain topological aspects of the hyperspace of faces of a compact convex set.

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AN EXPLICIT FORMULA FOR THE FUNDAMENTAL UNITS OF A REAL PURE SEXTIC NUMBER FIELD AND ITS GALOIS CLOSURE

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The object of this paper is to give a set of fundamental units of a real pure sextic number field $K=\boldsymbol{Q}(\sqrt[6]{a^6-1})$, where a is a special type of natural number and a^6-1 is not necessarily 6th power free. It is also shown that a set of fundamental units of the galois closure $L=K(\sqrt{-3})$ of K is formed by a real unit and its conjugates.

Let d be a 6th power free natural number which is not a perfect square or a perfect cube in the rational number field $\boldsymbol{Q}$. Put $\theta=\sqrt[6]{d}$; then $K=\boldsymbol{Q}(\theta)$ is a real pure sextic number field. We investigate the group of units of K for a special type of d as follows. Let d be given by

$$d=c(b^6c\pm 2)(b^{12}c^2\pm b^6c+1)(b^{12}c^2\pm 3b^6c+3) \tag{1}$$

with natural numbers b and c. Put

$$a=b^6c\pm 1\ . \tag{2}$$

(The $\pm$ signs correspond respectively throughout this paper.) Then

$$b^6d=a^6-1 \tag{3}$$

and $K=\boldsymbol{Q}(\sqrt[6]{a^6-1})$.

THEOREM 1. *The notation being as above, we assume that $d>1$ and d is square free. Then*

$$\xi_1=a-b\theta\ ,\quad \xi_2=a+b\theta\ ,\quad \xi_3=a^2+ab\theta+b^2\theta^2 \tag{4}$$

form a set of fundamental units of K.

As to explicit formulas for the fundamental units of number fields, G. Degert [2] has given one for certain real quadratic fields. As an application of the Jacobi-Perron algorithm (J.P.A.), L. Bernstein, H.-J. Stender and R. J. Rudman has extended Degert's result to certain real pure cubic, quartic and sextic fields (see [9] and [10]). On the other hand, H. Yokoi has given a different formula for the fundamental units of real quadratic and pure cubic number fields in [11], [12] and [13]. Theorem 1 is an extension of Yokoi's result to real pure sextic fields. A similar formula can be obtained for the fundamental units of real pure quartic fields (see [7]). Theorem 1 is not included in Stender's result when $b > 1$ (see Remark 4).

THEOREM 2. *Under the same assumption as in Theorem* 1, *any* 5 *of* 6 *conjugates of* ξ_1 *form a set of fundamental units of* $K(\sqrt{-3})$.

Theorem 2 gives an example of a real *Minkowski unit* of a non-abelian galois extension $K(\sqrt{-3})/\boldsymbol{Q}$ (see [1]).

To prove Theorem 1, we use the same method as in Stender [8]. Let K_2 and K_3 be the quadratic and cubic subfields of K respectively, and let E be the group of units of K. Define the group H of positive relative units of K with respect to K_2 and K_3 by

$$H = \{\xi \in E \mid N_2(\xi) = N_3(\xi) = 1\}, \tag{5}$$

where N_2 and N_3 denote the relative norm maps from K to K_2 and K_3 respectively. Then H is a free abelian group of rank 1. The fundamental units of the subfields will be determined in §1. A generator of H will be determined in §2. In §3, we shall prove Theorem 1 and show the existence of infinitely many fields which satisfy the condition of Theorem 1. In §4, we shall prove Theorem 2.

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1. Fundamental units of the subfields. Let d be a natural number given by (1) with natural numbers b and c, and define a as in (2). Assume that d is neither a perfect square nor a perfect cube in $\boldsymbol{Q}$. Then $K = \boldsymbol{Q}(\theta)$, where $\theta = \sqrt[6]{d}$, is of degree 6 over $\boldsymbol{Q}$, and it contains the quadratic subfield $K_2 = \boldsymbol{Q}(\theta^3)$ and the cubic subfield $K_3 = \boldsymbol{Q}(\theta^2)$. Denote respectively by η_2 and η_3 the fundamental units of K_2 and K_3 which are larger than 1. Define the algebraic integers ξ_1, ξ_2, ξ_3 as in (4). Then it immediately follows from (3) that their absolute norms are all equal to 1; hence they belong to the group E of units of K. We also see that $1/\xi_1\xi_3 = a^3 + b^3\theta^3$ belongs to $E \cap K_2$, and that $1/\xi_1\xi_2 = a^4 + a^2b^2\theta^2 + b^4\theta^4$ belongs to $E \cap K_3$.

PROPOSITION 1. *If* $d > 1$ *and is square free, then* $\eta_2 = 1/\xi_1\xi_3 = a^3 + b^3\theta^3$.

Proof. Since $1/\xi_1\xi_3 > 1$, we have $\eta_2^n = a^3 + b^3\theta^3$ with a natural number n. Let us assume $n \geqq 2$. We can write $\eta_2 = (t + u\theta^3)/2$

with nonzero rational integers t and u, because d is square free. Then $u = (\eta_2 - \eta_2')/\theta^3$, where $\eta_2' = (t - u\theta^3)/2$. Taking into account that $u \neq 0$, $|\eta_2'| = 1/\eta_2 < 1$, $n \geqq 2$ and $a^3 + b^3\theta^3 > 1$, we see that

$$1 \leqq |u| \leqq (\eta_2 + |\eta_2'|)/\theta^3 < \sqrt{(a^3 + b^3\theta^3)/d} + \sqrt{1/d} .$$

From (3), $b\theta < a$ and $1/d = b^6/(a^6 - 1)$. Therefore

$$1 < \sqrt{2a^3b^6/(a^6 - 1)} + \sqrt{b^6/(a^6 - 1)} .$$

From (2), $b^6 \leqq a + 1$. Then

$$1 < \sqrt{2a^3(a + 1)/(a^6 - 1)} + \sqrt{(a + 1)/(a^6 - 1)} .$$

However, the right side of the last inequality is smaller than 1 for $a \geqq 3$, which is a contradiction. When $a = 2$, we see from (3) that $b = 1$, and then $d = 63$ is not square free. Since $a \geqq 2$ by (3), $n = 1$ under our assumption, and the proposition follows.

REMARK 1. When d has a square factor, the conclusion of Proposition 1 does not necessarily hold. For example, set $b = 1$ and $c = 22$ in (1) and (2) for the plus case, i.e.,

$$d = 22(22 + 2)(22^2 + 22 + 1)(22^2 + 3\cdot 22 + 3) , \quad a = 22 + 1 .$$

Then $d = 2^4\cdot 3^2\cdot 7\cdot 11\cdot 13^2\cdot 79$, $a = 23$, and

$$\eta_2 = 2\cdot 3\cdot 13 + \sqrt{7\cdot 11\cdot 79}, \eta_2^2 = a^3 + b^3\theta^3 .$$

When the square factor of d is small, Proposition 1 is also true as is seen from the proof.

PROPOSITION 2. *If* $d > 1$ *and is cube free, then* $\eta_3 = 1/\xi_1\xi_2 = a^4 + a^2b^2\theta^2 + b^4\theta^4$.

Proof. It follows from T. Nagell [5] (see also [13]), that the binomial unit $\xi_1\xi_2 = a^2 - b^2\theta^2$ is either fundamental unit of K_3 or its square, and the latter occurs only for $d = 20, 50$ and a finite number of $d \equiv \pm 1 \pmod 9$. Now we assume $1/\eta_3^2 = a^2 - b^2\theta^2$. Let $d = fg^2$ with relatively prime natural numbers f and g, and write $1/\eta_3 = \{x + y\theta^2 + (z/g)\theta^4\}/3$ with rational integers x, y and z. Then

$$|y| < \{1 + 2\sqrt[4]{1(a^2 - b^2\theta^2)}\}/\theta^2$$

follows similarly as in [5]. Note I, 1. Here

$$1/(a^2 - b^2\theta^2) = a^4 + a^2b^2\theta^2 + b^4\theta^4 < 3a^4$$

and

$$1/\theta^2 = \sqrt[3]{1/d} = \sqrt[3]{b^6/(a^6-1)} \leqq \sqrt[3]{(a+1)/(a^6-1)}$$

are obtained as before, and hence

$$|y| < \sqrt[3]{(a+1)/(a^6-1)} + 2\sqrt[4]{3}\ \sqrt[3]{a^3(a+1)/(a^6-1)}\ .$$

When $a \geqq 6$, the right side of the last inequality is smaller than 1. Therefore $y=0$ and $1/\eta_3 = \{x + (z/g)\theta^4\}/3$. This is a contradiction, because the square of a binomial unit cannot be binomial. When $a = 2, 3, 4$ or 5, a^6-1 is 6th power free, and, by (3), $b=1$ and $d = a^6 - 1$. For $a = 2, 4$ or 5, we have $d \equiv 0 \not\equiv \pm 1 (\mathrm{mod}\ 9)$. For $a=3$, we see that d is not cube free. This completes the proof.

REMARK 2. By the same method as in the proof of Proposition 2, we can verify that the exceptional case of Theorem 6(iii) of [10] occurs only when $(u, n) = (1, 4)$, i.e., $d = 28$.

REMARK 3. As we have seen in the end of the proof of Proposition 2, we have $a \geqq 6$ when $b \geqq 2$. This fact will be used in the next section.

2. Relative fundamental unit.

Let d, a and K be as in §1. We keep the notation as before. Let H be the group of positive relative units of K with respect to K_2 and K_3 which is defined by (5). Then, as in [8], §1, II, 8, the group H is a free abelian group of rank 1. We denote by ε_1 the generator of H which is larger than 1.

Suppose $d>1$ and is square free. Then, by Propositions 1 and 2,

$$(6)\qquad \eta_2 = 1/\xi_1\xi_3 = a^3 + b^3\theta^3\ ,\quad \eta_3 = 1/\xi_1\xi_2 = a^4 + a^2b^2\theta^2 + b^4\theta^4\ .$$

The field belongs to *Klasse I* of [8], because

$$(7)\qquad N_2(1/\xi_1) = \eta_2\ ,\quad N_3(1/\xi_1) = \eta_3\ .$$

Put now $\varepsilon = 1/\xi_1^6\eta_2^2\eta_3^3$, then $\varepsilon \in H$ and

$$\varepsilon = \xi_2^3\xi_3^2/\xi_1 = (a+b\theta)^3(a^2+ab\theta+b^2\theta^2)^2(a^5+a^4b\theta+\cdots+b^5\theta^5)$$

by (3) and (6).

PROPOSITION 3. *If $d>1$ and is square free, then $\varepsilon_1 = \varepsilon = \xi_2^3\xi_3^2/\xi_1$.*

Proof. When $b=1$, $d = a^6-1$ by (3), and then Stender has shown that $\varepsilon_1 = \varepsilon$ in [8], Hilfssatz 7. Let $b \geqq 2$. Since $\varepsilon > 1$ and $\varepsilon \in H$, $\varepsilon_1^n = \varepsilon$ with a natural number n. Assume $n \geqq 2$. The relative unit $\varepsilon = 1/\xi_1^6\eta_2^2\eta_3^3$ can be neither a square nor a cube in K by [8], Hilfssatz 1. Therefore $n \geqq 5$. Now we can write $\varepsilon_1 = 1/6\sum_{j=0}^{5} x_j\theta^{j-1}$

with rational integers $x_j (j = 0, 1, \cdots, 5)$ according to [8], Hilfssatz 2. Note that d divides x_0 and that either x_0 or x_5 is distinct from zero by [9], Hilfssatz 3. On the other hand, by [8], (1.6),

$$|x_j| < \theta^{1-j} A (j = 0, 1, \cdots, 5) \quad \text{with} \quad A = \sqrt[5]{\varepsilon} + 2\sqrt[10]{\varepsilon} + 3 ,$$

since $n \geqq 5$ and $\varepsilon > 1$. Hence either

$$d = \theta^6 \leqq |x_0| < \theta A$$

or

$$1 \leqq |x_5| < A/\theta^4$$

should hold. From the fact that $\theta > 1$, we obtain

$$1 < A/\theta^4 = A \sqrt[3]{1/d^2} .$$

Taking into account that $b\theta < a$ and $1/d = b^6/(a^6 - 1) \leqq (a + 1)/(a^6 - 1)$ as before, we can derive

$$1 < \sqrt[3]{(a+1)^2/(a^6-1)^2}(\sqrt[5]{2^4 \cdot 3^3 a^{12}} + 2\sqrt[10]{2^4 \cdot 3^3 a^{12}} + 3) .$$

However, since $a \geqq 6$ as we have mentioned in Remark 3, the right side of the last inequality is smaller than 1. This is a contradiction. Thus $\varepsilon_1 = \varepsilon$ for $b \geqq 2$, too.

3. Fundamental units of K. For natural numbers b and c, let d and a be given by (1) and (2). Let $K = \boldsymbol{Q}(\theta)$, where $\theta = \sqrt[6]{d}$. Further let ξ_1, ξ_2, ξ_3 be given by (4).

THEOREM 1. (i) *If $d > 1$ and is square free, then ξ_1, ξ_2, ξ_3 form a set of fundamental units of K.*

(ii) *For a fixed natural number b, there are infinitely many values of c which make d square free.*

Proof. (i) Recall that K belongs to *Klasse I* of [8] by (7). It follows from Propositions 1, 2 and 3 that

$$\varepsilon_1 = \xi_2^3 \xi_3^2 / \xi_1 , \quad \sqrt[3]{\eta_2/\varepsilon_1} = 1/\xi_2\xi_3 , \quad \sqrt{\eta_3\varepsilon_1} = \xi_2\xi_3/\xi_1 .$$

These three units form a set of fundamental units of K by [8], Satz 1′. Hence the assertion is obvious.

(ii) Let

$$f(X) = X(b^6 X \pm 2)(b^{12}X^2 \pm b^6 X + 1)(b^{12}X^2 \pm 3b^6 X + 3) .$$

We shall find infinitely many square free natural numbers in the sequence $\{f(c)\}_{c=1}^{\infty}$ by the help of Nagell [2], §2. Evidently, (I) the degrees of the irreducible factors of $f(X)$ are at most 2; (II) the discriminant of $f(X)$ is not zero. For a prime number p, there is a natural number c such that $b^6 f(c) = (b^6 c \pm 1)^6 - 1 \not\equiv 0 (\bmod p^2)$ if $b \not\equiv 0 (\bmod p)$, and there is a c' such that $f(c') \equiv 6c' \not\equiv 0 (\bmod p^2)$ if $b \equiv 0 (\bmod p)$. This implies that (IV) there is no prime number p such that $f(c) \equiv 0 (\bmod p^2)$ for all natural numbers $c = 1, 2, \cdots$. Now let us assume that b is prime to 6. Then (III) the polynomial $f(X)$ is primitive. From (I), (II), (III) and (IV), we can apply [2], §2, I, and find infinitely many square free natural numbers in $\{f(c)\}_{c=1}^{\infty}$. When b is not prime to 6, we apply Nagell's result to $(1/2)f(2X+1)$, $(1/3)f(3X+1)$ or $1/6f(6X+1)$ in a similar but slightly different manner from the above in order to prove the assertion.

REMARK 4. Stender has given in [10] an explicit formula for the fundamental units of $\boldsymbol{Q}(\sqrt[6]{M})$, where $M = N^6 \pm n (>1)$ with natural numbers N and n such that n is 6th power free and divides N^5, assuming that $(N^6/n) \pm 1$ or N^6/n is square free. We will see that Theorem 1 is contained in his result only if $b = 1$. Let $n = p_1^{v_1} \cdots p_s^{v_s} (v_j = 1, 2, \cdots, 5)$ with distinct prime numbers $p_1, \cdots, p_s$. Write $(N^6/n) \pm 1 = mx^6$ with natural numbers m and x, where m is 6th power free. Put $m' = (p_1 \cdots p_s)^6/n$; then m' is also 6th power free. When $M = N^6 + n$, the diophantine equation $mX^6 - m'Y^6 = 1$ belongs to the field $\boldsymbol{Q}(\sqrt[6]{M})$ in the sense of [10], Definition 1, and has a solution $(X, Y) = (x, N/p_1 \cdots p_s)$(see also [10], Satz 10). On the other hand, the equation $X^6 - dY^6 = 1$ belongs to K and has a solution $(X, Y) = (a, b)$. Suppose $K = \boldsymbol{Q}(\sqrt[6]{M})$; then it follows from [10], Satz 7 that

$$m = 1, \quad m' = d, \quad x = a, \quad N/p_1 \cdots p_s = b.$$

Then $(N^6/n) + 1 = x^6$ cannot be square free. If N^6/n is square free, $n = N^5$ and N is square free. Therefore $N = p_1 \cdots p_s$, i.e., $b = 1$. When $M = N^6 - n$, we similarly obtain

$$m = d, \quad m' = 1, \quad x = b, \quad N/p_1 \cdots p_s = a,$$

if $K = \boldsymbol{Q}(\sqrt[6]{M})$. If $(N^6/n) - 1$ is square free, then $x = b = 1$. If N^6/n is square free, then $n = N^5 = 1$, and this is a contradiction. Thus, we have seen that Theorem 1 is not contained in Satz 22 of Stender [10] if $b > 1$.

4. Real Minkowski unit. Let $K = \boldsymbol{Q}(\theta) (\theta = \sqrt[6]{d})$ be a real pure sextic field, and $L = K(\zeta)$ its galois closure, where $\zeta = \exp(2\pi\sqrt{-1}/3)$. According to A. Brumer [1], we say a unit ξ of L is a *Minkowski unit* of L if we can take 4 conjugates $\xi^{(1)}, \cdots, \xi^{(4)}$ of ξ such that

$\xi, \xi^{(1)}, \cdots, \xi^{(4)}$ form a set of fundamental units of L. The galois group of L over $\boldsymbol{Q}$ is generated by the two automorphisms σ and τ which satisfy

$$\theta^\sigma = -\zeta\theta , \quad \theta^\tau = \theta ; \quad \zeta^\sigma = \zeta , \quad \zeta^\tau = \zeta^{-1} .$$

The defining relations of σ and τ are $\sigma^6 = \tau^2 = (\sigma\tau)^2 = 1$. We will give an example of a real Minkowski unit of the non-abelian, galois, totally imaginary field L. Since K is a maximal real subfield of L, it suffices to find a unit ξ of K such that $\xi, \xi^\sigma, \cdots, \xi^{\sigma^4}$ form a set of fundamental units of L. Now let d, a and K be as in §3. Assume $d > 1$ and is square free. Using the same notation as before, we first study the subfields of L.

PROPOSITION 4. *The assumptions being as above,* (i) $\xi_1^{1+\sigma^2+\sigma^4}$ *is a fundamental unit of* $K_2(\zeta)$, (ii) $\xi_1^{1+\sigma^3}, \xi_1^{(1+\sigma^3)\sigma}$ *form a set of fundamental units of* $K_3(\zeta)$, (iii) $\xi_1^{\sigma+\sigma^2}, \xi_1^{\sigma^4+\sigma^5}$ *form a set of fundamental units of the fixed field* $F = \boldsymbol{Q}(\sqrt[6]{-27d})$ *of* $\sigma^3\tau$.

Proof. (i) On acconut of (6), $\eta_2 = 1/\xi_1\xi_3 = a^3 + b^3\theta^3$ is a fundamental unit of K_2. Suppose that η_2 is not a fundamental unit of $K_2(\zeta)$. Then since $d \neq 3$, it follows from S.-K. Kuroda [4], Satz 14, that $3\eta_2 = \alpha^2$ with an integer α of K_2 Since $d \not\equiv 1 (\mathrm{mod}\, 4)$, we have $\alpha = x + y\theta^3$ with rational integers x and y. Therefore

$$3(a^3 + b^3\theta^3) = (x + y\theta^3)^2 .$$

Comparing the coefficients and taking the norms of both sides of this equation, we see

$$3a^3 = x^2 + dy^2 , \qquad 9 = (x^2 - dy^2)^2 .$$

This leads us to a contradiction after an easy calculation using the fact that d is square free. Hence $\eta_2 = 1/\xi_1\xi_2 = \xi_1^{-(1+\sigma^2+\sigma^4)}$ is a fundamental unit of $K_2(\zeta)$. (ii) On account of (6), $\eta_3^{-1} = \xi_1\xi_2 = a^2 - b^2\theta^2$ is a fundamental unit of K_3. Suppose that η_3^{-1} and $\eta_3^{-\sigma}$ does not form a set of fundamental units of $K_3(\zeta)$. Then we have

$$(8) \qquad \beta^{1+\tau} = \mathrm{Tr}_{\boldsymbol{Q}}^{K_3}(1 + \eta_3^{-1} + \eta_3) = 3(a^4 + a^2 + 1)$$

with an integer β of $\boldsymbol{Q}(\zeta)$ such that $(\gamma/\beta) + (\gamma/\beta)^\tau$, where $\gamma = 1 + \eta_3^{-1} + \eta_3^{-(1+\sigma)}$, is an integer of K_3 (see K. Iimura [3], Theorem 1 and Proposition). Put $\beta = x + y\zeta$ with rational integers x and y; then we can compute $(\gamma/\beta) + (\gamma/\beta)^\tau$ by (8), and see that the coefficient of θ^4 is equal to $(x + y)b^4/3(a^4 + a^4 + 1)$. Since d is square free, it follows that $(x + y)b^4/(a^4 + a^2 + 1)$ is a rational integer. By (2), b and $a^4 + a^2 + 1$ have no common divisor except 3. Moreover, since $(x + y)^2 -$

$3xy = 3(a^4 + a^2 + 1)$ by (8), $x + y$ and $a^4 + a^2 + 1$ have no common divisor except 3, because $a^4 + a^2 + 1$ is square free as a divisor of d. Therefore $a^4 + a^2 + 1 = 3$, i.e., $a = 1$ follows. This is a contradiction. Hence $\eta_3^{-1} = \xi_1\xi_2 = \xi_1^{1+\sigma^3}$ and $\eta_3^{-\sigma} = \xi_1^{(1+\sigma^3)\sigma}$ form a set of fundamental units of $K_3(\zeta)$. (iii) Let H' be the subgroup of the group E_F of units of F given by

$$H' = \{\xi \in E_F \mid \xi^{1+\tau} = 1\} .$$

Then H' is generated by a unit ε_2 and the roots of unity in F (see [10], §4, II). It is easy to see that $\xi_1^{(\sigma+\sigma^2)(1+\tau)} = \xi_1^{-(1+\sigma^3)} = \eta_3$, and that $\xi_1^{(\sigma+\sigma^2)(1-\sigma^3)(1+\tau)} = 1$. Therefore $\xi_1^{(\sigma+\sigma^2)(1-\sigma^3)} = \omega\varepsilon_2^n$ with a rational integer n and a root of unity ω. Applying $\sigma + \sigma^2$ to both sides, we obtain $\xi_1^{-1+3\sigma^3+2(\sigma^2+\sigma^4)} = \omega^{\sigma+\sigma^2}\varepsilon_2^{n(\sigma+\sigma^2)}$. Since F is the fixed field of $\sigma^3\tau$, $\varepsilon_2^{\sigma+\sigma^2}$ is a unit of K, and hence $\omega^{\sigma+\sigma^2}$ also belongs to K. Recall that ξ_1, $\xi_1^{\sigma^3}$, $\xi_1^{\sigma^2+\sigma^4}$ form a set of fundamental units of K by Theorem 1. Consequently $n = \pm 1$, and $\xi_1^{(\sigma+\sigma^2)(1-\sigma^3)}$ and the roots of unity of F generate H'. As we have seen above, $\xi_1^{(\sigma+\sigma^2)(1+\tau)} = \eta_3$. According to [10], Satz 24, $\xi_1^{(\sigma+\sigma^2)(1-\sigma^3)}$ and $\xi_1^{(\sigma+\sigma^2)}$ form a set of fundamental units of F. This completes the proof of (iii).

THEOREM 2. *Under the same assumptions as in Theorem* 1, *the galois closure* $L = K(\zeta)$ *of* K *has a real Minkowski unit* $\xi_1 = a - b\theta$.

Proof. Let E' be the subgroup of the group of units of L which is generated by all the units of K, K^{σ^2}, K^{σ^4} and $K_2(\zeta)$. Then for every unit ξ of L, $\xi^3 = \xi^{1+\tau}\xi^{1+\sigma^2\tau}\xi^{1+\sigma^4\tau}\xi^{-\tau(1+\sigma^2+\sigma^4)}$ belongs to E'. On the other hand, by Proposition 4(i) and Theorem 1, E' is generated by the roots of unity in L and $\xi_1, \xi_1^{\sigma}, \cdots, \xi_1^{\sigma^4}$. Hence

$$\xi^3 = \omega\xi_1^{n_0+n_1\sigma+\cdots+n_4\sigma^4} ,$$

where ω is a root of unity and $n_0, n_1, \cdots, n_4$ are rational integers. By applying $1 + \tau, 1 + \sigma^3\tau$ and $1 + \sigma^3$ to both sides, we get

$$\begin{aligned}
\xi^{3(1+\tau)} &= \xi_1^{(2n_0-n_1)}\xi_1^{(2n_3-n_1)\sigma^3}\xi_1^{((n_2+n_4-n_1)(\sigma^2+\sigma^4)} , \\
\xi^{3(1+\sigma^3\tau)} &= \omega'\xi_1^{(n_1+n_2-n_0-n_3)(\sigma+\sigma^2)}\xi_1^{(n_4-n_0-n_3)(\sigma^4+\sigma^5)} , \\
\xi^{3(1+\sigma^2)} &= \omega''\xi_1^{(n_0+n_3-n_1-n_4)(1+\sigma^3)}\xi_1^{(n_2-n_1-n_4)(1+\sigma^3)\sigma} ,
\end{aligned}$$

where ω' and ω'' are roots of unity. By Theorem 1 and Proposition 4(ii) and (iii), we see that $n_0 \equiv n_1 \equiv \cdots \equiv n_4 \equiv 0 (\mathrm{mod}\ 3)$. This implies that ξ^3 is already a cube in E' modulo the roots of unity, and hence ξ belongs to E'. This shows that E' is the group of all units of L, and that ξ_1 is a real Minkowski unit of L.

CONCLUDING REMARK. Stender's method is based on the group of relative units of a non-galois number field which has proper subfields. We can generalize this to a field whose galois closure is a dihedral extension over $\boldsymbol{Q}$ (see [7]).

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INNER FUNCTIONS INVARIANT CONNECTED COMPONENTS

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The inner functions $d = \exp\{(z+1)/(z-1)\}$ and zd belong to the same connected component in the space of inner functions under uniform topology. Therefore, simplification is not possible in general but it is always possible to simplify by a finite Blaschke product.

0. **Introduction.** This work deals with the inner functions of one variable. A complex, holomorphic function f, bounded on the open unit disk D of the complex plane is called inner if $|f(e^{i\theta})| = 1$ a.e.; where $f(e^{i\theta}) = \lim_{\rho\to 1} f(\rho e^{i\theta})$.

In the set F of the inner functions we consider the topology induced by the Banach space H^∞; that is, we consider F with the topology of uniform convergence.

In this work, related to a publication of D. Herrero [2], we are interested in the connected components of the space F, mainly with respect to multiplication of inner functions.

Let us denote by $f \sim g$ the fact that the inner functions f and g belong to the same connected component. The questions that motivate this work are the following:

(a) For the identity function z, is there an inner function f such that $f \sim zf$?

(b) Is simplification permitted? That is, does relation $f\omega \sim g\omega$ imply $f \sim g$ for any three inner functions f, g, ω?

The results of this work can be summarized as follow:

(1) "Simplification" by a finite Blaschke product is always possible.

(2) "Simplification" is not possible in general.

(3) If the singular measure μ associated with a singular function S contains at least one atom, then relation $S \sim zS$ holds.

(4) For any nonconstant inner function g, the inner functions $\exp\{(g+1)/(g-1)\}$ and $g\exp\{(g+1)/(g-1)\}$ belong to the same connected component.

(5) For any nonconstant singular function S, there exists a nonconstant inner function g such that: $S \sim gS$.

In order to prove that simplification by a finite Blaschke product is possible, we first show that the set $zF = \{zh: h \in F\} = \{x \in F: x(0) = 0\}$ is a retract of F.

In order to give an example of an inner function f such that $f \sim zf$, we shift the zeros of an infinite Blaschke product in such a way that the Blaschke product moves continuously with respect to the uniform topology.

The following problems seem to be open:

(1) Does relation $S \sim zS$ hold for any singular function?

(2) Find all inner functions such that $f \sim zf$.

(3) Characterize the inner functions ω such that $\omega f \sim \omega g \Rightarrow f \sim g$ for all $f, g \in F$.

(4) Find a necessary and sufficient condition for two inner functions f and g to belong to the same connected component.

1. **Preliminaries.** A complex, holomorphic function f, bounded on the open unit disk D of the complex plane is called inner if its boundary values have almost everywhere absolute value one; that is, relation $|f(e^{i\theta})|=1$ holds almost everywhere (with $f(e^{i\theta})=\lim_{\rho\to 1} f(\rho e^{i\theta})$).

It is well-known that a function f is inner if and only if f is of the form:

$$f(z) = cz^k \prod_{i\in I} \frac{\bar{\alpha}_i}{|\alpha_i|} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z} \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right\}$$

where c is a complex constant of modulus one ($|c| = 1$), k is a nonnegative integer, μ is a positive singular measure on the unit circle and the points $\alpha_i \in D$ are such that $\sum_{i\in I} 1 - |\alpha_i| < \infty$.

If $\mu = 0$, then f is a Blaschke product, finite if the set I is finite or infinite if the set I is infinite (countable).

In the case $I = \varnothing$ and $k = 0$, the function f is called singular.

The topology of the uniform convergence on the set F of the inner functions is induced by the following metric:

$$d(f, g) = \|f - g\|_\infty = \sup_D |f(z) - g(z)| = \sup_{\theta\in R} \mathrm{ess}\, |f(e^{i\theta}) - g(e^{i\theta})| .$$

Let us denote by $f \sim g$ the fact that the inner functions f and g belong to the same connected component in the space F.

In what follows we make use of the well-known facts below:

(1) For any three inner functions f, g and ω the relation $f \sim g$ implies $\omega f \sim \omega g$. This is due to the continuity of the multiplication of inner functions.

(2) For any inner function f and any complex number α, with $|\alpha| < 1$, we have the relation:

$$f \sim f_\alpha = \frac{f - \alpha}{1 - \bar{\alpha} f} ;$$

for the mapping $D \ni \alpha \to f_\alpha \in F$ is continuous.

(3) For every nonnegative integer n, the set of all finite Blaschke products with exactly n zeros forms a connected component

and an open and closed subset of F. In particular the set of the constant inner functions is connected and open and closed in F.

This fact is an easy application of Rouche's theorem.

2. Simplification by z. Let us begin with the question, does the relation $\omega f \sim \omega g$ implies $f \sim g$. This is the problem of "Simplification". In the case of a finite Blaschke product ω, the answer to this question is affirmative.

PROPOSITION 1. *Let ω be a finite Blaschke product. Then for any two inner functions f and g, the relation $\omega f \sim \omega g$ implies $f \sim g$.*

Proof. The general case easily follows from the case $\omega = z$, to which we will limit ourselves from now on.

Let us consider the set:

$$zF = \{zh: h \in F\} = \{x \in F: x(0) = 0\} .$$

The maps $z^*: zF \to F$ and $\Phi: F \to zF$, where $z^*(x) = x/z$, $\Phi(f) = (f - f(0))/(1 - \overline{f(0)}f)$ for $f \in F$ nonconstant and $\Phi(f) = z$ for $f \in F$ constant, are both continuous. (The set of the constant inner functions is, both, open and closed!).

Therefore the mapping $z^* \circ \Phi: F \to F$ is continuous and the relation $zf \sim zg$ implies: $f = z^* \circ \Phi(zf) \sim z^* \circ \Phi(zg) = g$, as $\Phi(x) = x$ for any $x \in zF$; that is, Φ is a retraction map and zF is a retract of F. The proof is complete now.

3. The main result. The following theorem implies in particular that we cannot "simplify" by any inner function.

THEOREM 1. *For any nonconstant inner function g, the inner functions* $\exp\{(g + 1)/(g - 1)\}$ *and* $g \exp\{(g + 1)/(g - 1)\}$ *belong to the same connected component.*

This theorem applied for the identity function $g = z(z(a) = a$ for all $a \in D)$ implies the following:

PROPOSITION 2. *The inner functions $d =$* $\exp\{(z + 1)/(z - 1)\}$ *and zd belong to the same connected component* (*that is*: $d \sim zd$).

Proposition 2 is equivalent to Theorem 1; for Proposition 2 implies also Theorem 1. The point is that the range of the continuous map $T_g: F \to H^\infty$, $T_g(f) = f \circ g$ is contained in F; that is, the composition of two inner functions is an inner function ([6] or [8]). Therefore relation $d \sim zd$ implies:

$$\exp\frac{g+1}{g-1} = T_g(d) \sim T_g(zd) = g\exp\frac{g+1}{g-1}\,.$$

Hence, it remains to prove Proposition 2, which will be a consequence of the following lemma, which is of a concrete geometric nature on the half-plane:

LEMMA 1. *Let*

$$K_1 = \prod_{n=1}^{\infty}\frac{\bar{\alpha}_n}{|\alpha_n|}\frac{\alpha_n - z}{1-\bar{\alpha}_n z} \quad \text{and} \quad K_2 = \prod_{n=1}^{\infty}\frac{|\bar{\beta}_n|}{\bar{\beta}_n}\frac{\beta_n - z}{1-\bar{\beta}_n z}\,.$$

Be two infinite Blaschke products such that $K_1(0) > 0$ and $K_2(0) > 0$. If we denote $\varphi(z) = (1+z)/(1-z)$ then we have the following inequality:

$$\|K_1 - K_2\|_\infty \leqq \sum_{n=1}^{\infty}\left|\alpha\text{ rg}\frac{\alpha_n}{\beta_n}\right| + 2\sum_{n=1}^{\infty}\left|\alpha\text{ rg}\frac{1-\alpha_n}{1-\beta_n}\right|$$
$$+ 2\operatorname*{sup\,ess}_{y\in R}\sum_{n=1}^{\infty}\left|\alpha\text{ rg}\frac{\varphi(\alpha_n) - iy}{\varphi(\beta_n) - iy}\right|\,.$$

Proof of Lemma 1. The pointwise convergence $f_n \to f$ implies trivially the inequality:

$$\|f\|_\infty \leqq \liminf_n \|f_n\|_\infty$$

We have therefore:

$$\|K_1 - K_2\|_\infty \leqq \liminf_N \left\|\prod_{n=1}^{N}\frac{\bar{\alpha}_n}{|\alpha_n|}\frac{\alpha_n - z}{1-\bar{\alpha}_n z} - \prod_{n=1}^{N}\frac{\bar{\beta}_n}{|\beta_n|}\frac{\beta_n - z}{1-\bar{\beta}_n z}\right\|_\infty$$
$$= \liminf_N \left\|\prod_{n=1}^{N}\frac{\bar{\alpha}_n}{|\alpha_n|}\frac{1-\alpha_n}{1-\bar{\alpha}_n}\frac{\varphi(\alpha_n) - \varphi(z)}{\bar{\varphi}(\alpha_n) + \varphi(z)}\right.$$
$$\left. - \prod_{n=1}^{N}\frac{\bar{\beta}_n}{|\beta_n|}\frac{1-\beta_n}{1-\bar{\beta}_n}\frac{\varphi(\beta_n) - \varphi(z)}{\bar{\varphi}(\beta_n) + \varphi(z)}\right\|_\infty\,.$$

We notice that $|\alpha| = |\beta| = |\alpha'| = |\beta'| = 1 \Rightarrow |\alpha\beta - \alpha'\beta'| \leqq |\alpha - \alpha'| + |\beta - \beta'|$. Consequently, for almost every z, with $|z| = 1$, we have:

$$\left|\prod_{n=1}^{N}\frac{\bar{\alpha}_n}{|\alpha_n|}\frac{1-\alpha_n}{1-\bar{\alpha}_n}\frac{\varphi(\alpha_n) - \varphi(z)}{\bar{\varphi}(\alpha_n) + \varphi(z)} - \prod_{n=1}^{N}\frac{\bar{\beta}_n}{|\beta_n|}\frac{1-\beta_n}{1-\bar{\beta}_n}\frac{\varphi(\beta_n) - \varphi(z)}{\bar{\varphi}(\beta_n) + \varphi(z)}\right|$$
$$\leqq \sum_{n=1}^{N}\left|\frac{\bar{\alpha}_n}{|\alpha_n|} - \frac{\bar{\beta}_n}{|\beta_n|}\right| + \sum_{n=1}^{N}\left|\frac{1-\alpha_n}{1-\bar{\alpha}_n} - \frac{1-\beta_n}{1-\bar{\beta}_n}\right|$$
$$+ \sum_{n=1}^{N}\left|\frac{\varphi(\alpha_n) - \varphi(z)}{\bar{\varphi}(\alpha_n) + \bar{\varphi}(z)} - \frac{\varphi(\beta_n) - \varphi(z)}{\varphi(\beta_n) + \varphi(z)}\right|$$

$$\leqq \sum_{n=1}^{N} \left|\alpha\ \mathrm{rg}\frac{\alpha_n}{\beta_n}\right| + 2\sum_{n=1}^{N} \left|\alpha\ \mathrm{rg}\frac{1-\alpha_n}{1-\beta_n}\right| + 2\sum_{n=1}^{N} \left|\alpha\ \mathrm{rg}\frac{\varphi(\alpha_n)-\varphi(z)}{\varphi(\beta_n)-\varphi(z)}\right|$$

$$\leqq \sum_{n=1}^{\infty} \left|\alpha\ \mathrm{rg}\frac{\alpha_n}{\beta_n}\right| + 2\sum_{n=1}^{\infty} \left|\alpha\ \mathrm{rg}\frac{1-\alpha_n}{1-\beta_n}\right| + 2\sup_{y\in R}\mathrm{ess}\sum_{n=1}^{\infty} \left|\arg\frac{\varphi(\alpha_n)-iy}{\varphi(\beta_n)-iy}\right| .$$

The required inequality is now implied.

Proof of Proposition 2. Let $\alpha_n(t)$ be the unique point of D such that $\varphi(\alpha_n(t)) = 1 + i(n+t)\pi$, where $t\in[0, 1]$, $n\in N^* = \{1, 2, \cdots\}$ and $\varphi(z) = (1+z)/(1-z)$.

One, then, verifies easily that:

$$d_{1/e} = \frac{d-\dfrac{1}{e}}{1-\dfrac{1}{e}d} = f\cdot\prod_{n=4}^{\infty}\frac{\overline{\alpha_n(0)}}{|\alpha_n(0)|}\frac{\alpha_n(0)-z}{1-\overline{\alpha_n(0)}z}, \quad \text{with} \quad f\in F .$$

It is enough to prove that

$$B_1 = \prod_{n=4}^{\infty}\frac{\overline{\alpha_n(0)}}{|\alpha_n(0)|}\frac{\alpha_n(0)-z}{1-\overline{\alpha_n(0)}z} \sim B_0 = \prod_{n=3}^{\infty}\frac{\overline{\alpha_n(0)}}{|\alpha_n(0)|}\frac{\alpha_n(0)-z}{1-\overline{\alpha_n(0)}z};$$

for, then we have

$$d \sim d_{1/e} = fB_1 \sim fB_0 = fB_1\frac{\overline{\alpha_3(0)}}{|\alpha_3(0)|}\frac{\alpha_3(0)-1}{1-\overline{\alpha_3(0)}z} \sim fB_1 z = d_{1/e}\cdot z \sim dz ,$$

and we obtain the result.

In order to prove $B_1 \sim B_0$, it is sufficient to prove the continuity of the following map:

$$[0, 1]\ni t \xmapsto{B} B_t = \prod_{n=3}^{\infty}\frac{\overline{\alpha_n(t)}}{|\alpha_n(t)|}\frac{\alpha_n(t)-z}{1-\overline{\alpha_n(t)}z}\in F;$$

that is, $\lim_{t\to t_0}\|B_t - B_{t_0}\|_\infty = 0$ for all $t_0\in[0, 1]$. Using Lemma 1 we essentially have to prove the following fact:

$$\lim_{t\to t_0}\sup_{y\in \boldsymbol{R}}\sum_{n=3}^{\infty}\left|\arg\frac{\varphi(\alpha_n(t))-iy}{\varphi(\alpha_n(t_0))-iy}\right| = 0 .$$

This relation follows immediately from the observation that:

$$\sum_{n=3}^{\infty}\left|\arg\frac{\varphi(\alpha_n(t))-iy}{\varphi(\alpha_n(t_0))-iy}\right|$$
$$\leqq 2\sum_{n=3}^{\infty}\left|\arg\frac{1+2i\pi(n+t_0+|t-t_0|)-2i\pi(t_0+3)}{1+2i\pi(n+t_0-|t-t_0|)-2i\pi(t_0+3)}\right| \underset{t\to t_0}{\longmapsto} 0 .$$

4. **Consequences.** Theorem 1 yields trivially the following:

COROLLARY 1. *For any inner function g, there exists an inner function f such that $f \sim gf$.*

Proposition 2 implies the following more general result:

COROLLARY 2. *Let f be an inner function whose singular measure μ contains at least one atom. Then $f \sim zf$.*

Proof of Corollary 2. We have $f = f_1 \exp K(z+\alpha)/(z-\alpha)$, with $f_1 \in F$, $|\alpha| = 1$ and $K > 0$. Thus, it is enough to establish the relation $\exp K(z+\alpha)/(z-\alpha) \sim z \exp K(z+\alpha)/(z-\alpha)$. By a rotation this becomes:

$$\exp K\frac{z+1}{z-1} \sim z \exp K\frac{z+1}{z-1}\,.$$

If $K \geqq 1$, using the known relation $d \sim zd$ (Proposition 2) we have

$$\begin{aligned}\exp K\frac{z+1}{z-1} &= d \cdot \exp{(K-1)}\frac{z+1}{z-1} \sim zd \exp{(K-1)}\frac{z+1}{z-1}\\ &= z \exp\frac{z+1}{z-1}K\,.\end{aligned}$$

If $0 < K < 1$, let us consider the transformation[1]:

$$-w(z) = \frac{\dfrac{1-K}{1+K} - z}{1 - \dfrac{1-K}{1+K}z}\,.$$

Evidently $w \in F$ and $w \sim z$. From the known relation $d \sim zd$ we obtain:

$$\exp K\frac{z+1}{z-1} = d\circ w \sim (zd)\circ w = w\cdot(d\circ w) \sim z\cdot(d\circ w) = z \exp K\frac{z+1}{z-1}\,.$$

REMARK. Corollary 2 implies that if the singular measure μ associated with a singular function S contains some atoms, then the relation $S \sim zS$ holds. If the measure $\mu \neq 0$ does not contain any atoms, then we do not know if the relation $S \sim zS$ is true. It seems that this problem (probably not difficult) is still open and we offer the following conjecture:

"Every nonconstant singular inner function S belongs to the same connected component as zS".

[1] This trick is found in [**2**].

In this direction we have the following proposition, which follows from Theorem 1 combined with a remark suggested to the author by K. Stephenson.

PROPOSITION 3. *For any nonconstant singular inner function S, there exists a nonconstant inner function g such that $S \sim gS$.*

Proof. The point is that any singular inner function S is of the form $S = \exp (g + 1)/(g - 1)$, with $g \in F$. Theorem 1 gives, then, the result.

In an obvious manner Proposition 3 implies the following:

COROLLARY 3. (i) *For every nonconstant singular inner function S, there exist inner functions f and g such that $fS \sim gS$ but $f \not\sim g$.*

(ii) *Let ω be an inner function such that the relation $f_1\omega \sim f_2\omega$ implies $f_1 \sim f_2$ for every couple (f_1, f_2) of inner functions f_1 and f_2. Then the connected component of ω contains only Blaschke products. In particular ω is a Blaschke product.*

(iii) *If the connected component of an inner function f does not contain any proper multiple of f, then this component contains only Blaschke products. In particular f is a Blaschke product.*

The existence of infinite Blaschke products satisfying the hypothesis of Corollary 3 (iii) follows from the proof of a theorem of D. Herrero ([3], Theorem 1.1). Later, the present author proved in [6] that if the zeros α_n, $n = 1, 2, \cdots$ of a Blaschke product B satisfy the condition

$$\lim_n \prod_{m \neq n} \left| \frac{\alpha_n - \alpha_m}{1 - \bar{\alpha}_n \alpha_m} \right| = 1$$

then, the connected component for B does not contain any proper multiple of B.

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ON COMPACT SUBMANIFOLDS WITH NONDEGENERATE PARALLEL NORMAL VECTOR FIELDS

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In this paper we obtain characterizations of spherical submanifolds in Euclidean space of codimension $\geqq 1$. Such characterizations are given here in terms of certain relationships involving the elementary symmetric functions of principal radii of curvature and the support function of a submanifold.

1. **Introduction.** For hypersurfaces similar characterizations are well known. For example, let M be a closed convex hypersurface in Euclidean space, h the support function of M, and S_l the elementary symmetric function of order l of principal curvatures. It has been proved by several authors (see Simon [8], and further references given there) that if for some integer $l(1 \leqq l \leqq \dim M)$ everywhere on M $h^l S_l = \text{const}$, then M is a hypersphere. Other results of this type are also known [8], [9].

Our proofs are based on a differential analogue of the Minkowski-Hsiung formulas, relating the support function and elementary symmetric functions of various orders of the principal radii of curvature. Those formulas are obtained for submanifolds which possess a nondegenerate normal vector field parallel in the normal bundle.

Finally, we note that characterizations of spherical submanifolds in terms of the elementary symmetric functions of principal curvatures are obtained by Chen [2] and Chen and Yano [4] (see also Chen [3], Chapter 6).

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2. **Preliminaries.** In this section we shall present local formulas relating the second fundamental form and the support function of a submanifold in Euclidean space. We shall use the following convention on the ranges of indices:

$$1 \leqq i, j, k, l, r \leqq m, \qquad 1 \leqq \alpha \leqq n,$$

and as usual, it is agreed that repeated lower and upper indices are summed over the respective ranges. We denote by E the Euclidean space of dimension $m+n$, and we fix the origin at some point O. Consider a smooth, orientable submanifold M of dimension $m(\geqq 2)$ immersed in E, and represented by the position vector field

$$X = X(u^1, \cdots, u^m) ,$$

where $\{u^i\}$ are the local coordinates on M. Let x be a point of M. We denote by $T_x(M)$ and $N_x(M)$ the restrictions of the tangent bundle $T(M)$ and normal bundle $N(M)$ at x.

Put

$$X_i = \partial_i X , \qquad \partial_i = \partial/\partial u^i .$$

The metric I, or the first fundamental form induced on M from E via X, is $G_{ij} = \langle X_i, X_j \rangle$, where $\langle , \rangle$ denotes the inner product in E. Let ξ be an arbitrary unit normal vector field defined in a neighborhood U of $x \in M$. The second fundamental form at x with respect to ξ is $II(\xi) = b_{ij}(\xi) du^i du^j$, where $b_{ij}(\xi) = -\langle X_i, \xi_j \rangle$. Let η be a unit normal vector field in U not necessarily different from ξ. The mixed third fundamental form is $III(\xi, \eta) = g_{ij}(\xi, \eta) du^i du^j$, where $g_{ij}(\xi, \eta) = \langle \xi_i, \eta_j \rangle$. We write $III(\xi) \equiv III(\xi, \xi)$, and $g_{ij}(\xi) \equiv g_{ij}(\xi, \xi)$. Evidently, $g_{ij}(\xi, \eta) = g_{ji}(\eta, \xi)$, but, in general, no other symmetries exist. For a unit normal vector field $\xi \in N(M)$, $h(\xi)$ denotes the support function of M with respect to ξ, that is, $h(\xi) = -\langle X, \xi \rangle$.

Recall that a nondegenerate normal vector field on M is a unit normal vector field ξ such that $\det(b_{ij}(\xi)) \neq 0$ everywhere on M (see [2], and also [3], p. 59).

Vectors $\{X_i\}$ form a basis in $T_x(M)$, $x \in M$, and we denote by $\{N(\alpha)\}$ a field of orthonormal frames in $N(M)$. Put

$$X_{ij} = \partial_{ij} X , \quad \partial_{ij} = \partial^2/\partial u^i \partial u^j .$$

At first we note that $b_{ij}(\xi) = \langle X_{ij}, \xi \rangle$, and $b_{ij}(\xi) = b_{ji}(\xi)$ for a unit normal vector field ξ. Also, $g_{ij}(\xi) = -\langle \xi_{ij}, \xi \rangle = g_{ji}(\xi)$. Suppose that ξ is parallel in $N(M)$, that is, $\xi_i \in T(M)$, $i = 1, \cdots, m$, everywhere on M, and let η be an arbitrary unit normal vector field on M. Then $g_{ij}(\xi, \eta) = -\langle \xi_{ij}, \eta \rangle = g_{ji}(\xi, \eta)$.

In the frame $X_1, \cdots, X_m, N(1), \cdots, N(n)$ we have for an arbitrary unit vector field $\xi \in N(M)$:

$$\xi_i = -b_i^j(\xi) X_j + \sum_\alpha \langle \xi_i, N(\alpha) \rangle N(\alpha) , \tag{1}$$

where $b_i^j(\xi) = b_{il}(\xi) G^{lj}$, and G^{lj} being the inverse of G_{lj}. From here, for two unit normal vector fields ξ and η, we find

$$g_{ij}(\xi, \eta) = b_i^r(\xi) b_{rj}(\eta) + \sum_\alpha \langle \xi_i, N(\alpha) \rangle \langle \eta_j, N(\alpha) \rangle . \tag{2}$$

If ξ or η is parallel, then

$$g_{ij}(\xi, \eta) = b_i^r(\xi) b_{rj}(\eta) . \tag{3}$$

Note that when M, ξ and η are such that $II(\xi)$ and $II(\eta)$ are positive definite then so is $III(\xi, \eta)$. However, the form $III(\xi)$ is nonnegative definite for an arbitrary $II(\xi)$. If ξ is nondegenerate everywhere on M, then $III(\xi)$ induces a Riemannian metric on M. We denote by $dO(\xi)$ the corresponding volume-element. From formula (1) it follows that if ξ is nondegenerate and parallel, then vectors $\{\xi_i\}$ form a basis in $T_x(M)$, $x \in M$, and according to the Gauss equation we have:

$$\xi_{ij} = \Gamma^k_{ij}(\xi)\xi_k - \sum_\alpha g_{ij}(\xi, N(\alpha))N(\alpha) \ , \tag{4}$$

where $\Gamma^k_{ij}(\xi)$ denote the Cristoffel symbols of the second kind with respect to $III(\xi)$.

When ξ is nondegenerate, then translating it parallel to itself in E to the origin O we can define an immersion $\gamma_\xi\colon M \to \Sigma$, where Σ is a unit hypersphere in E centered at O. In codimension one γ_ξ is the standard Gauss map.

PROPOSITION 2.1. *Let M be a submanifold of E and ξ a nondegenerate parallel normal vector field on M. Then γ_ξ is an isometric immersion of M with the metric $III(\xi)$ into Σ.*

Proof. Let the symbol $\hookrightarrow$ denote an immersion, and $\to$ a pullback of the mertic from the ambient space. Then the following diagram is commutative in $\hookrightarrow$ and $\to$.

$$\begin{tikzcd}
(\Sigma, \bar{g}) \arrow[rr, hook, shift left, "\sigma"] \arrow[dr, shift left] & & E \arrow[ll, shift left] \arrow[dl, shift left] \\
& (M, III(\xi)) \arrow[ul, hook, shift left, "\gamma_\xi"] \arrow[ur, hook, shift left, "\xi"'] &
\end{tikzcd}$$

where σ is the standard imbedding of Σ in E, and $\bar{g}$ is the metric induced on Σ from E. The Proposition is proved.

For convenience we write $h(\alpha) \equiv h(N(\alpha))$. The position vector field X of a submanifold M can be decomposed into two parts:

$$X = X_T + X_N \ , \tag{5}$$

where $X_T \in T(M)$, $X_N \in N(M)$. In the frame $X_1, X_2, \cdots, X_m, N(1), \cdots, N(n)$ we have

$$\begin{aligned} X_T &= G^{ij}\langle X, X_i\rangle X_j \ , \\ X_N &= -\sum_\alpha h(\alpha)N(\alpha) \ . \end{aligned} \tag{6}$$

If ξ is nondegenerate and parallel, then from (1) we see that

$$X_T = -g^{ij}(\xi)h_i(\xi)\xi_j \tag{7}$$

where $g^{ij}(\xi)$ are the elements of $(g_{ij}(\xi))^{-1}$, and $h_i(\xi) = \partial_i h(\xi)$.

Put

$$h_{ij}(\xi) = \partial_{ij}h(\xi),\ \nabla_{ij}h(\xi) = h_{ij}(\xi) - \Gamma^k_{ij}(\xi)h_k(\xi) .$$

Under the above assumptions on ξ we obtain with the use of (4)

$$b_{ij}(\xi) = \nabla_{ij}h(\xi) + \sum_\alpha g_{ij}(\xi, N(\alpha))h(\alpha) . \tag{8}$$

3. The elementary symmetric functions of principal radii of curvature and the associated differential equations. Let ξ be a unit normal vector at a point $x \in M$. The principal radii of curvature associated with ξ are denoted by $R_{\xi 1}, \cdots, R_{\xi m}$ and defined as the roots of the determinantal equation

$$\det(b_{ij}(\xi) - Rg_{ij}(\xi)) = 0 .$$

If ξ is a restriction to x of a nondegenerate vector field, then $III(\xi)$ is positive definite, and in this case the $R_{\xi i}$ are well defined. Moreover, in this case they do not vanish. Let $g(\xi) \equiv \det(g_{ij}(\xi))$. The elementary symmetric function of order k in $R_{\xi i}$ (nonnormed)

$$S_{\xi k}(R) = \sum_{i_l \neq i_r} R_{\xi i_1} \cdots R_{\xi i_k} ,$$

and it is the coefficient at $(-R)^{m-k}$ of the polynomial

$$\frac{\det(b_{ij}(\xi) - Rg_{ij}(\xi))}{g(\xi)} = (-R)^m + S_{\xi 1}(R)(-R)^{m-1} + \cdots + S_{\xi m}(R) . \tag{9}$$

Set $a_{ij}(\xi) = b_{ij}(\xi) - \lambda g_{ij}(\xi)$, where λ is real. Consider a polynomial in λ defined by the equation

$$a^{ij}(\xi) = \sum_{k=1}^m (-\lambda)^{m-k} S^{ij}_{\xi k} , \tag{10}$$

where $a^{ij}(\xi)$ is the cofactor of the element $a_{ij}(\xi)$.

PROPOSITION 3.1. *Let M be a submanifold of E and ξ is a parallel unit normal vector field defined in a neighborhood of $x \in M$ and such that $II(\xi)>0$ at x. Then the quadratic forms $S^{ij}_{\xi k}\nu_i\nu_j$, $k=2, \cdots, m$, are positive definite at x. Here $\nu_1, \cdots, \nu_m$ are arbitrary real parameters, $\nu^2 = \nu_1^2 + \nu_2^2 + \cdots + \nu_m^2 \neq 0$. If M is compact, ξ is defined on M, parallel, and $II(\xi) \neq 0$ everywhere on M, then those quadratic forms are definite everywhere and by selecting a proper orientation of M and E, they can be made positive definite. When $k = 1$ this* *assertion is true under the only assumption that ξ is parallel and nondegenerate.*

The proof of this Proposition is standard and we omit it here.

Suppose now that ξ is a nondegenerate parallel vector field. Then in view of (8), (9), and (10) we put

$$P_{\xi k}(h) \equiv \frac{1}{g(\xi)} S^{ij}_{\xi k} \nabla_{ij} h(\xi) ,$$

$$Q_{\xi k} = \frac{1}{g(\xi)} \sum_{\alpha} S^{ij}_{\xi k} g_{ij}(\xi, N(\alpha)) h(\alpha) ,$$

$$M_{\xi k}(h) \equiv P_{\xi k}(h) + Q_{\xi k} . \tag{11}$$

It is not difficult to see that

$$M_{\xi k}(h) = k S_{\xi k}(R) . \tag{12}$$

PROPOSITION 3.2. *Let M be a submanifold of E and ξ a nondegenerate parallel normal vector field defined in a neighborhood of $x \in M$. Then*

$$Q_{\xi k} = (m - k + 1) S_{\xi k-1}(R) h(\xi) + \langle H_{\xi k}, X \rangle , \quad (S_{\xi 0} \equiv 1) , \tag{13}$$

where $H_{\xi k}$ is a uniquely defined vector in $N'_x(M) = N_x(M) \ominus \xi$ independent on the choice of basis in $N'_x(M)$. If $k = 1$, then $-H_{\xi 1}$ is the m times mean curvature vector of the submanifold $\gamma_\xi(M) \subset \Sigma$.

Proof. Since $\xi \in N_x(M)$, we can select an orthonormal basis in $N_x(M)$ so that ξ is one of the vectors in this basis. Let us preserve the old notation for the new basis, and let $\xi = N(1)$. Then

$$\begin{aligned} Q_{\xi k} &= \frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi) h(\xi) + \sum_{2 \leq \alpha \leq n} \frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi, N(\alpha)) h(\alpha) \\ &= (m - k + 1) S_{\xi k-1}(R) h(\xi) - \left\langle \sum_{2 \leq \alpha \leq n} \frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi, N(\alpha)) N(\alpha), X \right\rangle . \end{aligned}$$

The form $-(S^{ij}_{\xi k}/g(\xi)) g_{ij}(\xi, \eta)$, where $\eta \in N'_x(M)$, is linear in η. Therefore, there exists a unique element $H_{\xi k}$ in $N'_x(M)$ such that

$$-\frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi, \eta) = \langle H_{\xi k}, \eta \rangle$$

for any $\eta \in N'_x(M)$. (Strictly speaking, the inner product in the last formula should be taken in $N'_x(M)$. But it is induced in $N'_x(M)$ from E, and, therefore, it is the same in either sense.) Thus, we conclude that

$$-\left\langle \sum_{2\leq\alpha\leq n} \frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi, N(\alpha))N(\alpha), X \right\rangle = \langle H_{\xi k}, X\rangle .$$

The rest of the Proposition follows from Proposition 2.1 and the fact that

$$\frac{S^{ij}_{\xi 1}}{g(\xi)} = g^{ij}(\xi) .$$

This completes the proof.

COROLLARY 3.1. *Let M be a submanifold of E and ξ a nondegenerate parallel vector field on M. Then* (13) *holds everywhere on M and*

$$M_{\xi k} = P_{\xi k}(h) + (m - k + 1)S_{\xi k-1}(R)h(\xi) + \langle H_{\xi k}, X\rangle \quad \text{for all } x \in M . \tag{14}$$

REMARK 1. The functions $\langle H_{\xi k}, X\rangle$ are similar to the functions $F_k(\xi)$ constructed in [4]. However, the latter are related to principal curvatures and depend on the first and the second fundamental forms, while $\langle H_{\xi k}, X\rangle$ depend on the second and the third fundamental forms in the direction ξ. It is not difficult to point out situations where $H_{\xi k}$ or $\langle H_{\xi k}, X\rangle$ vanish. For example, if $\dim E - \dim M = 1$, then $H_{\xi k} \equiv 0$ for all k. Another example is when the normal component of X has the direction ξ. Then $h(\alpha) = -\langle X, N(\alpha)\rangle \equiv 0$ for $\alpha = 2, \cdots, m$. In these examples the functions $F_k(\xi)$ introduced in [4] also vanish. One more example is given by the case where $III(\xi, N(\alpha)) \equiv 0$ for $\alpha = 2, \cdots, m$, $(N(1) = \xi)$.

REMARK 2. Let M be a submanifold of E and ξ a nondegenerate parallel normal field on M. Let f and f' be two smooth functions defined on M. Put

$$b^f_{ij}(\xi) = \nabla_{ij} f - \sum_{\alpha} g_{ij}(\xi, N(\alpha))\langle X, N(\alpha)\rangle .$$

Similarly to (9), (10), construct $S^{ij}_{\xi k}(f)$ and consider

$$M_{\xi k}(f, f') \equiv \frac{1}{g(\xi)} S^{ij}_{\xi k}(f)\nabla_{ij} f' + \frac{(m - k + 1)}{k - 1} M_{\xi k-1}(f, f)f' + \langle H_{\xi k}(f), X\rangle , \quad \text{for} \quad k > 1 ,$$

$$M_{\xi 1}(f, f') \equiv g^{ij}(\xi)\nabla_{ij} f' + mf' + \langle H_{\xi 1}(f), X\rangle , \quad \text{for} \quad k = 1 .$$

These differential operators proved to be useful in the study of uniqueness Theorems for convex hypersurfaces in Euclidean space [7]. (In this case they are elliptic, and the last term in the right-hand side vanishes.) It is plausible that they have applications in establishing uniqueness Theorems for submanifolds of E in codimension >1. We hope that we come back to it again elsewhere.

4. **Applications.** We begin with a slight generalization of the formula (14), which leads to an integral formula relating the elementary symmetric functions of arbitrary order. This formula is of Minkowski-Hsiung type, and in the form involving two consecutive elementary symmetric functions of principal curvatures it was derived and studied by many authors (see Chen and Yano [4], and also [3], Chapter 6; in both sources further references can be found). However, the methods of those authors do not seem to generalize so as to obtain the following formulas (16) and (17).

In what follows, unless stated otherwise, it is assumed that M is a compact submanifold without boundary.

The following Lemma is a version of E. Hopf's Lemma on Laplace-Beltrami operator.

LEMMA. *Suppose that M is a submanifold of E, ξ is a nondegenerate parallel normal vector field on M, and h' is a smooth function on M. Put*

$$P_{\xi k}(h') \equiv \frac{1}{g(\xi)} S^{ij}_{\xi k} \nabla_{ij} h' \,,$$

where the coefficients $S^{ij}_{\xi k}$ are the same as in (11). *Then*

$$P_{\xi k}(h') = \frac{1}{\sqrt{g(\xi)}} \partial_i \left(\frac{S^{ij}_{\xi k}}{\sqrt{g(\xi)}} \partial_j h \right) . \tag{15}$$

If $k=1$ and, in addition, we assume that $P_{\xi k}(h')$ does not change its sign on M, then h' is a constant function on M. The same is true when $k>1$ provided there exists at least one point on M where $II(\xi) \neq 0$.

Proof. It is easy to see, with the use of formula (4), that $b_{ij}(\xi)$ is a Codazzi tensor with respect to $\Gamma^k_{ij}(\xi)$. Therefore, $P_{\xi k}(h')$ can be written in the divergence form (15) (see [5, 7]). When $k>1$ and $II(\xi) \neq 0$ at some point of M then $II(\xi) \neq 0$ everywhere on M because ξ is nondegenerate. By Proposition 3.1 the operator $P_{\xi k}(h')$ is uniformly elliptic. Now the rest of the proof runs similarly to the standard proof of E. Hopf's Lemma on the Laplace-Beltrami operator on a compact Riemannian submanifold ([6], p. 338). The Lemma is proved.

THEOREM 4.1. *Let M be a submanifold of E and ξ a nonde-*

generate parallel normal vector field on M. Then for arbitrary k and s, $k=1,\cdots,m$, $s=1,\cdots,k$,

$$kS_{\xi k}(R)=\sum_{l=0}^{s-1}\frac{(m-k+l)!\ (k-l-1)!}{(m-k)!\ (k-1)!}[P_{\xi k-l}(h)+\langle H_{\xi k-l},X\rangle]h^l \\ +\frac{(m-k+s)!\ (k-s)!}{(m-k)!\ (k-1)!}S_{\xi k-s}(R)h^s\ , \tag{16}$$

and

$$\begin{aligned} k\int_M S_{\xi k}(R)dO(\xi) &= -\sum_{l=1}^{s-1} l\cdot\frac{(m-k+l)!\ (k-l-1)!}{(m-k)!\ (k-1)!}\int_M h^{l-1}\frac{S^{ij}_{\xi k-l}}{g(\xi)}h_ih_jdO(\xi) \\ &\quad+\sum_{l=0}^{s-1}\frac{(m-k+l)!\ (k-l-1)!}{(m-k)!\ (k-1)!}\int_M\langle H_{\xi k-l},X\rangle h^l dO(\xi) \\ &\quad+\frac{(m-k+s)!\ (k-s)!}{(m-k)!\ (k-1)!}\int_M S_{\xi k-s}(R)h^s dO(\xi)\ , \end{aligned} \tag{17}$$

where $h\equiv h(\xi)$ is the support function of M with respect to ξ.

Proof. Formula (16) follows from the formulas (12) and (14); and (17) is obtained from (16) by integrating, applying Green's formula, and the preceding Lemma.

COROLLARY 4.1. *If in Theorem 4.1 $s=k$, then*

$$k\int_M S_{\xi k}(R)dO(\xi)=(m-k+1)\int_M S_{\xi k-1}(R)h(\xi)dO(\xi) \\ +\int_M\langle H_{\xi k},X\rangle dO(\xi)\ . \tag{18}$$

This formula is an analogue of an integral formula due to Chen and Yano [4].

We recall that if a submanifold M (not necessarily compact) of E is contained in a hypersphere of E centered at the origin, then it is called a spherical submanifold (see [2]).

In the following we often make use of a Theorem due to Chen [2].

THEOREM A. *Let M be a submanifold (not necessarily compact) of E. If there exists a nondegenerate parallel normal vector field ξ such that $h(\xi)=$ const everywhere on M, then M is a spherical submanifold of E.*

From now on always when $k>1$ it is assumed that there exists a point on M where $II(\xi) \neq 0$, and the orientation is such that $II(\xi) > 0$.

Examples of submanifolds with this property can be constructed as follows. Let M_1 and M_2 be two strictly convex hypersurfaces. Then the natural imbedding of $M_1 \times M_2$ in Euclidean space of dimension $= \dim M_1 + \dim M_2 + 2$ gives such example.

The next Theorem is an immediate consequence of formulas (12), (14), the Lemma, and Theorem A.

THEOREM 4.2. *Let M be a submanifold in E and ξ is a non-degenerate parallel normal vector field on M. Assume further that for some k, $k = 1, 2, \cdots, m$, at every point of M*

$$cS_{\xi k}(R) = S_{\xi k-1}(R)h(\xi) \qquad (S_{\xi 0} \equiv 1)\ , \tag{19}$$

where c is a constant such that the expression

$$[k - c(m - k + 1)]S_{\xi k}(R) - \langle H_{\xi k}, X\rangle$$

is either nonnegative or nonpositive. Then M is a spherical submanifold.

Proof. In the formula (16) set $s = 1$. Then by (19)

$$[k - c(m - k + 1)]S_{\xi k}(R) - \langle H_{\xi k}, X\rangle = P_{\xi k}(h)\ ,$$

and the Theorem follows from the Lemma and Theorem A.

In case $k = 1$ a result similar to this Theorem has been given by Wegner [9], Satz 2. His result can be also obtained by our method, and furthermore, it can be generalized for $k > 1$.

Let $M = S^m$, where S^m is a standard m-sphere lying in $m + 1$-dimensional Euclidean space $E^{m+1} \subset E$. Then, evidently, $H_{\xi k} \equiv 0$ for all k, and ξ is the unit normal vector field on S^m in E^{m+1}. With this fact in mind we state the following

COROLLARY 4.2. *Let M be a submanifold of E and ξ a non-degenerate parallel normal vector field on M. If for some k, $k = 2, \cdots, m$, at every point of M*

$$\langle H_{\xi k}, X\rangle = 0\ ,$$

and

$$cS_{\xi k}(R) = S_{\xi k-1}(R)h(\xi)\ ,$$

where c is a constant $\neq 0$, then M is a spherical submanifold. Furthermore, in this case it is necessary that $c = k/(m - k + 1)$. The *assertion is also true when $k = 1$, provided $II(\xi) \neq 0$ at some point of M.*

Proof. We show at first that the function $S_{\xi k}(R)$ does not change its sign on M. Let A be a point on M where $II(\xi)$ is definite. Then the principal radii of curvature $R_{\xi i}$, $i = 1, \cdots, n$, must all be of the same sign at A. Since ξ is nondegenerate $R_{\xi i}$ will all have the same sign everywhere on M. Hence, the function $S_{\xi k}(R)$ can not change its sign on M, and moreover it does not vanish on M.

Now it is clear that the expression

$$[k - c(m - k + 1)]S_{\xi k}(R)$$

is either nonnegative or nonpositive and therefore by Theorem 4.2. M is a spherical submanifold. On the other hand,

$$\int_M S_{\xi k}(R)dO(\xi) \neq 0 \ ;$$

hence, the formula (18) implies that $c = k/(m - k + 1)$. The Corollary is proved.

A Theorem similar to Theorem 4.2 can be stated with the use of Theorem 4.1.

We point out only a particular case of it.

THEOREM 4.3. *Let M be a submanifold in E and ξ a nondegenerate parallel normal vector field on M. Suppose that for some k and s, $k = 1, \cdots, m$, $s = 1, \cdots, k$, the following conditions are satisfied*:

(a) $$kS_{\xi k}(R) \geqq \frac{(m - k + s)!\ (k - s)!}{(m - k)!\ (k - 1)!} S_{\xi k-s}(R)h^s(\xi) \ ;$$

(b) $$\int_M \langle H_{\xi k-l}, X\rangle h^l(\xi)dO(\xi) \leqq 0 \quad \textit{for} \quad l = 0, \cdots, s - 1 \ ;$$

(c) $$h(\xi) > 0 \ .$$

Then M is a spherical submanifold.

Proof. The conditions (a), (b), (c) and Proposition 3.1 imply that all integrals in formula (17) must vanish. Hence $h(\xi) = \text{const}$, that is, M is a spherical submanifold.

THEOREM 4.4. *Let M be a submanifold in E and ξ a nondegenerate parallel normal vector field on M. Suppose that for some k and s, $k = 1, \cdots, m$, $s = 1, \cdots, k$, the following conditions are* *satisfied*:

(a) $cS_{\xi k}(R) = S_{\xi k-s}(R)h^s(\xi)$ *everywhere on* M, *where* c *is a constant* $\neq 0$;

(b) $\langle H_{\xi k-l}, X\rangle = 0$ *for* $l = 0, \cdots, s-1$;

(c) $h(\xi) > 0$.

In case $k = 1$ *assume also that* $II(\xi) \neq 0$ *at some point of* M.

Then M *is a spherical submanifold and* $c=(m-k)!k!/(m-k+s)!(k-s)!$.

Proof. At first we show that c can have only the value indicated in the assertion. In showing that we follow Blaschke [1], p. 233. Let A be a point on M where $h(\xi)$ $(=h)$ attains its maximum. Then at A,

$$\nabla_{ij}h \leqq 0 .$$

By Proposition 3.1 the forms $S^{ij}_{\xi k}\nu_i\nu_j$, $k = 1, \cdots, m$, are definite, and therefore at the point A the expressions

$$P_{\xi k-l}(h) = \frac{1}{g(\xi)} S^{ij}_{\xi k-l}\nabla_{ij}h \quad l = 0, 1, \cdots, k-1 ,$$

are all of the same sign, and namely nonpositive. On the other hand, by Theorem 4.1 (formula (16)) in view of the conditions (a) and (b), we obtain

$$\left[k - c\frac{(m-k+s)!\ (k-s)!}{(m-k)!\ (k-1)!}\right]S_{\xi k}(R)$$
$$= \sum_{l=0}^{s-1}\frac{(m-k+l)!\ (k-l-1)!}{(m-k)!\ (k-1)!} h^l P_{\xi k-l}(h) .$$

The right-hand side is nonpositive at A, and similar to the proof of Corollary 4.2 one shows that $S_{\xi k}(R) > 0$ everywhere on M. Therefore,

$$k - c\frac{(m-k+s)!\ (k-s)!}{(m-k)!\ (k-1)!} \leqq 0 .$$

Considering the point where h attains its minimum we arrive at the opposite inequality. Thus, $c = (m-k)!\ k!/(m-k+s)!\ (k-s)!$.

Now, making use of the second part of Theorem 4.1 (formula 17)) and the conditions (a), (b), (c) with constant c taken as above, we obtain

$$-\sum_{l=1}^{s-1}\frac{(m-k+l)!\ (k-l-1)!}{(m-k)!\ (k-1)!}\int_M h^{l-1}\frac{S^{ij}_{\xi k-l}}{g(\xi)} h_i h_j dO(\xi) = 0 .$$

From here, it follows that $h = \text{const}$. Hence, M is a spherical submanifold. The Theorem is proved.

COROLLARY 4.3. *Let M be a closed strictly convex hypersurface in Euclidean space E and ξ is the unit normal vector field on M. Suppose that for some k and s, $k = 1, \cdots, m$, $s = 1, \cdots, k$,*

$$cS_k(R) = S_{k-s}(R)h^s(\xi)$$

everywhere on M, where c is a constant$\neq 0$. Then M is a hypersphere, and c is as in Theorem 4.4. (In the last equality the subscript ξ is omitted for the obvious reason.)

Proof. For a hypersurface in E, ξ is always parallel, and since M is strictly convex, ξ is nondegenerate. Also $H_{\xi l} \equiv 0$ for $l = 1, \cdots, k$. The support function $h(\xi)$ can always be made strictly positive by placing the origin of the coordinate system in E inside M. Now the Corollary follows from Theorem 4.4.

REMARK 1. As was mentioned in the introduction, this Corollary is known. In particular, the condition quoted earlier can be expressed in terms of the elementary symmetric functions of principal radii of curvature as follows:

$$cS_m(R) = S_{m-s}(R)h^s(\xi) .$$

If in Corollary 4.3 we take $k = m$, then we obtain the above result. It is due to Süss; see [8], Korollar 6.3, and other references there.

REMARK 2. Theorem 4.4 does not contain Corollary 4.2, since in the latter it is not required that $h(\xi) > 0$.

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FANS AND EMBEDDINGS IN THE PLANE

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We prove that every fan which is locally connected at its vertex can be embedded in the plane. This gives a solution to a problem raised by J. J. Charatonik and Z. Rudy.

1. Introduction and definitions. In 1963, K. Borsuk [4] constructed a fan which is not embeddable in the plane. Hence, the question arises to characterize those fans which are embeddable in the plane. In particular, in [5] it was asked whether each contractible fan is embeddable in the plane. In an attempt to solve this problem in the negative, J. J. Charatonik and Z. Rudy constructed a contractible fan which is locally connected at its vertex. They conjectured ([6], p. 215) that this fan is not embeddable in the plane. We show in this paper that each fan, which is locally connected at its vertex, is embeddable in the plane (see Theorem 5.2). We will also establish, for fans, several equivalences between the local connectedness at the vertex and other conditions. In a forthcoming paper [11] the author has shown that each contractible fan is locally connected at its vertex, and hence embeddable in the plane.

By a *continuum* we mean a compact connected metric space. A *dendroid* is an arc-wise connected and hereditarily unicoherent continuum. By a *fan* we understand a dendroid which has exactly one branch-point, and we call this branch-point the *vertex* of the fan. If x, y are points in a dendroid X, then we denote by $[x, y]$ the unique arc in X having x and y as end-points. The weak-cut order $\leqq$, with respect to a point p, in a dendroid X is given by

$$x \leqq y \text{ if and only if } [p, x] \subset [p, y] .$$

We denote by I the unit closed interval $[0, 1]$ of reals, and the symbol $B(x, \varepsilon)$ denotes the open ball of radius ε about the point x. We use the symbol $\cong$ to denote that two spaces are homeomorphic. The symbol R, as used in Lemma 3.1, denotes a set of indices.

2. Embeddings in the plane. A cover $U = \{U_1, U_2, \cdots, U_n\}$ of a space is called an *ε-chain* if the nerve (see [8], p. 318) of U is an arc and $\operatorname{diam}(U_i) < \varepsilon$ for $i = 1, 2, \cdots, n$. A continuum X is said to be *arc-like* if for each $\varepsilon > 0$ there exists an ε-chain covering X. A point e of an arc-like continuum X is called an *end-point* provided for each $\varepsilon > 0$ there exists an ε-chain $U_1, U_2, \cdots, U_n$ covering X such that

$$(1) \qquad e \in U_1 \Big\backslash \bigcup_{i=2}^{n} U_i \ .$$

It is known (see [9], p. 148) that every 0-dimensional compact metric space K is homeomorphic to a subset of the Cantor ternary set $C \subset [0, 1]$, and hence K possesses a natural order $\leqq$. We will call this ordering the *induced ordering* on K. The main result of this section is Theorem 2.2. We start with the following lemma.

LEMMA 2.1. *Let X be a compact metric space and let $\{J_\alpha\}$, $\alpha \in A$, be the decomposition of X into components. Let $\varepsilon > 0$ and let K be a 0-dimensional compact set in X, with induced ordering $\leqq$ such that:*

(2) *J_α is an arc-like continuum for each $\alpha \in A$,*

(3) *$J_\alpha \cap K = \{e_\alpha\}$, where e_α is an end-point of J_α for each $\alpha \in A$.*
Then there exists an open cover U of X such that U is a finite union of disjoint ε-chain $V_i (i = 1, 2, \cdots, t)$, where $V_i = \{U(i, j)\}(j = 1, 2, \cdots, k(i))$ such that:

(4) $K \subset \bigcup_{i=1}^{t} U(i, 1) \backslash \bigcup_{i=1}^{t} \bigcup_{j=2}^{k(i)} U(i, j)$,

(5) *all nonadjacent elements of U have positive distance,*

(6) *for each i, $1 \leqq i \leqq t$, there exist $a_i, b_i \in K$ such that:*

$$K \cap U(i, 1) = \{x \in K \,|\, a_i \leqq x \leqq b_i\} \ .$$

Proof. Denote by 0 the minimal and by 1 the maximum element of K. Let $g: X \to K$ be defined by $g(x) = e_\alpha$ if $x \in J_\alpha$, then g is a monotone retraction. Let

(7) $x_0 = \sup\{e \in K \,|$ for each $e' \leqq e$ there exists an open cover of $g^{-1}([0, e'])$ satisfying the conclusion of Lemma 2.1$\}$,

then $x_0 \geqq 0$. By (2) and (3) there exists an ε-chain $U_1, U_2, U_3, \cdots, U_k$ in X covering $g^{-1}(x_0)$ such that

$$K \cap \bigcup_{j=2}^{k} U_j = \varnothing \ .$$

Since $g^{-1}(x_0) \subset \bigcup_{j=1}^{k} U_j$ and K is 0-dimensional there exists a closed and open set $H \subset K$ such that $g^{-1}(H) \subset \bigcup_{j=1}^{k} U_j$. Moreover, we can choose H such that

$$H \cap K = \{x \in K \,|\, a \leqq x \leqq b\}$$

for some a and b in K. If $a > 0$, define $x_1 = \sup\{x \in K \,|\, x < a\}$, then $x_1 \notin U_1$ and $x_1 < a$. By (7) there exists a cover U of $g^{-1}([0, x_1])$ satisfying the conclusions of the lemma (if $a = 0$, take $U = \varnothing$). Since $g^{-1}([0, x_1])$ is open in X we may assume that $\cup\, U \subset g^{-1}([0, x_1])$. Hence

$$U \cup \{U_j \cap g^{-1}(H) \,|\, j = 1, 2, \cdots, k\}$$

is a cover of $g^{-1}([0, b])$ satisfying the conclusion of the lemma. It follows the definition of x_0 that $x_0 = b$.

If $x_0 = b = 1$, we are done, whence suppose $x_0 < 1$ and let $x_2 = \inf\{x \in K \mid x > x_0\}$. By repeating the argument above, replacing x_0 by x_2, one can show that $g^{-1}([0, x_2])$ can be covered with a cover satisfying the conclusion of the lemma, contrary to (7), since $x_2 > x_0$.

We will call a cover U that satisfies the conclusion of Lemma 2.1 an *ε-cover* of X.

THEOREM 2.2. *Let X be a compact metric space and K a closed subset of X. Let $\{J_\alpha\}$, $\alpha \in A$, be the decomposition of X into components such that:*

(8) *$J_\alpha \cap K = \{e\}$, where e is an end-point of J_α for each $\alpha \in A$,*

(9) *J_α is an arc-like continuum for each $\alpha \in A$. Then there exists an embedding $h: X \to I^2$ such that $h(K) = h(X) \cap l$, where $l = \{(x, y) \in I^2 \mid y = 0\}$.*

Proof. Notice that by (8) K is 0-dimensional. By Lemma 2.1, there exists for each $\varepsilon > 0$ an ε-cover of X. Let U_1 be a 1/2-cover of X and $\eta > 0$ such that η is the minimum distance between two nonintersecting elements of U_1. By induction we construct a sequence of covers $U_1, U_2, \cdots$ of X such that U_n refines U_{n-1}, U_n is a $(1/2)^n$-cover, no sub-chain of less than nine links of U_n connects two nonintersecting elements of U_{n-1}.

Given a cover U of X, satisfying the conclusion of Lemma 2.1, we label the chains $V_1, V_2, \cdots, V_t$ of U such that $\inf\{x \mid x \in K \cap V_i\} < \inf\{x \mid x \in K \cap V_j\}$ if $i < j$, and the links of the chain $V_i = \{U(i, 1), U(i, 2), \cdots, U(i, k(i))\}$ such that $K \cap V_i \subset U(i, 1)$. If U and U^* are both covers of X, satisfying the conclusion of Lemma 2.1, then we say that U follows the pattern $\{(a_1, b_1), (a_1, b_2), \cdots, (a_1, b_{k(1)}), \cdots, (a_t, b_{k(t)})\}$ in U^* if the jth link of the ith chain of U is contained in the b_jth link of the a_ith chain of U^*(i.e., $U(i, j) \subset U^*(a_i, b_j)$).

There exist in I^2 a sequence of open sets $D_1, D_2, \cdots$ such that D_n is a finite union of $(1/2)^n$-chains whose elements are interiors of rectangles, and such that D_n follows a pattern in D_{n-1} that U_n follows in U_{n-1}, each element of D_{n-1} contains the closure of an element of D_n, while the closure of each element of D_n lies in an element of D_{n-1} and the first link of each chain of D_n intersects l in a nondegenerate interval, while the closure of all other elements of D_n are contained in $I^2 \backslash l (n = 1, 2 \cdots)$.

The existence of the open sets D_n satisfying the above follows from an argument similar to one used by R. H. Bing (see [3], p. 654), the only difference being that in each cover D_{n-1} we insert, in the next step, finitely many, instead of one, new chains and we require the first link of each chain of D_n to intersect l in a nondegenerate interval, while the closures of all other elements of D_n are contained in $I^2\backslash l$.

The latter facts can be established by dividing each chain of D_{n-1} into finitely many "strips" in each of which we insert, in the next step, a new chain in such a way that we always insert new links on a predescribed "side" of already chosen previous links.

It follows from Theorem 11 of [2] that X is homeomorphic with the continuum $Y = D_1^* \cap D_2^* \cap \cdots$, where D_n^* denotes the union of the elements of D_n and moreover it follows from the choice of D_n that Y satisfies the conclusion of Theorem 2.2, and the proof is complete.

3. **Fans locally connected at the vertex.** A fan X has *property P*[1], if for each sequence of points $\{x_i\}$ in X $(i = 1, 2, \cdots)$ converging to the vertex v of X we have

$$Ls[v, x_i] = \{v\} \ . \tag{1}$$

THEOREM 3.1. *Let X be a fan with vertex v and*

(2) $X = \bigcup_{r \in R} \{J_r \mid J_r \cong [0, 1]$ *for each* $r \in R$ *and* $J_{r_1} \cap J_{r_2} = \{v\}$ *if* $r_1 \neq r_2 \in R\}$,

then the following are equivalent:

(3) *X has property P,*

(4) *for each $\varepsilon > 0$, there exists a connected open neighborhood U of v such that* $\operatorname{diam}(U) \leqq \varepsilon$ *and* $\operatorname{Bd}(U) \cap J_r$ *is connected for every $r \in R$,*

(5) *X is locally connected at v.*

Proof. (3) $\rightarrow$ (4). Let $\varepsilon > 0$ be given and let $\leqq$ be the weak-cut order of X with respect to v. Define $V = B(v, e)$,

$$x(r) = \inf \{x \in X \mid x \in J_r \cap \operatorname{Bd}(V)\} \text{ if } J_r \cap \operatorname{Bd}(V) \neq \varnothing \ ,$$

$$Q_r = \begin{cases} \{y \in J_r \mid y \geqq x(r)\} & \text{if } \ J_r \cap \operatorname{Bd}(V) \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases} \tag{6}$$

and $Q = \bigcup_{r \in R} Q_r$. It follows that $v \notin \bar{Q}$, since if $\{v_i\}$ is a sequence in Q converging to v, then $v_i \geqq x(r_i)$ for some $r_i \in R$, and hence $Ls[v, v_i] \cap \operatorname{Bd}(V) \neq \varnothing$, contrary to (3).

Let $U = X\backslash\bar{Q}$, then U is an open neighborhood of v and $\operatorname{diam}(U) \leqq$

[1] It follows from the definition that property P is related to the notion of a Q-point or a P-point (cf. [1] and [7], respectively).

$\operatorname{diam}(V) = \varepsilon$. We will show that U satisfies all conditions of (4). We claim that

(7) if $z \in U$ and $x < z$, then $x \in U$, or, equivalently, if $x \in \bar{Q}$ and $z \geqq x$, then $z \in \bar{Q}$.

To this end, suppose that (7) is false. Hence $x \in \bar{Q}$, let $\{x_i\}$ be a sequence in Q converging to x. Then $x_i \geqq x(r_i) \in \operatorname{Bd}(V)$ for some $r_i \in R (i = 1, 2, \cdots)$. We may assume that the sequence $\{x(r_i)\}$ converges to a point $x_0 \in J_{r_0} \cap \operatorname{Bd}(V)$ for some $r_0 \in R$.

By ([9], p. 171), $Ls[x_i, x(r_i)]$ is a continuum and since $[x_i, x(r_i)] \subset Q (i = 1, 2, \cdots)$ we have $Ls[x_i, x(r_i)] \subset \bar{Q} \subset X \backslash \{v\}$. Moreover, since X is hereditarily unicoherent, it follows that $[x, x_0] \subset Ls[x_i, x(r_i)] \subset \bar{Q} \subset X \backslash \{v\}$ and we consider two cases as follows:

Case 1. $z \in [x, x_0]$. Then $z \in \bar{Q}$.

Case 2. $z \notin [x, x_0]$. Then, since $z > x$, $z > \max\{x, x_0\}$ and consequently $z > x_0 \geqq x(r_0)$. Hence $z \in Q$ by (6) and the definition of Q.

In both case we conclude that $z \in \bar{Q}$, contrary to the assumptions in (7) and the proof of (7) is complete. It follows from (7) that U is connected. In order to show that $J_r \cap \operatorname{Bd}(U)$ is connected for each $r \in R$, we will show that if $x, y \in J_r \cap \operatorname{Bd}(U)$, say $x < y$, and $z \in [x, y]$, then $z \in J_r \cap \operatorname{Bd}(U)$.

Since $x \in J_r \cap \operatorname{Bd}(U) = J_r \cap \bar{U} \cap \bar{Q}$ and $z > x$, it follows from (7) that $z \in \bar{Q}$. Moreover, since $y \in \bar{U}$, there exists a sequence $\{y_i\}$ in U converging to y. Since $Ls[v, y_i]$ is a continuum ([9], *p.* 171), containing both y and v and X is hereditarily unicoherent, it follows that $[v, y] \subset Ls[v, y_i]$. As $z \in [v, y]$, we may assume that there exists a sequence $\{z_i\}$, where $z_i \in [v, y_i]$, converging to z. By (7), $z_i \in U$ and whence $z \in \bar{U}$. Obviously $z \in J_r$ and we conclude $z \in J_r \cap \bar{U} \cap \bar{Q} = J_r \cap \operatorname{Bd}(U)$.

(4) $\rightarrow$ (5): Trivial.

(5) $\rightarrow$ (3): Suppose X does not have property P. Let $\{x_i\}$ be a sequence of points in X converging to v such that $Ls[v, x_i] = K \neq \{v\}$.

Let $\varepsilon > 0$ be such that $\operatorname{diam}(K) > 3\varepsilon$ and let U be a connected neighborhood of v such that $\operatorname{diam}(U) < \varepsilon$. Then there exists an index $i > 0$ such that $x_i \in U$ and $[v, x_i] \cap [X \backslash B(v, 2\varepsilon)] \neq \emptyset$. But then $\bar{U}$ and $[v, x_i]$ are two continua in X whose intersection is not connected, contradicting the fact that X is hereditarily unicoherent, and the proof is complete.

4. Decompositions of fans. We say that a space X is a $(q = c)$-*space* if, in X, every quasi-component is connected. In other words, for $(q = c)$-spaces the quasi-components and the components coincide. We will show that if a fan is locally connected at the vertex v of

X, then $X\backslash\{v\}$ is a $(q = c)$-space.

THEOREM 4.1. *Let X be a fan which is locally connected at the vertex v of X and*

$$X = \bigcup_{r \in R} \{J_r \mid J_r \cong [0, 1] \text{ for each } r \in R \text{ and } J_{r_1} \cap J_{r_2} = \{v\} \text{ if } r_1 \neq r_2 \in R\} .$$

Then $X\backslash\{v\}$ is a $(q = c)$-space and $\{J_r\backslash\{v\}\}$, $r \in R$, is the decomposition of $X\backslash\{v\}$ into quasi-components.

Proof. It is sufficient to show that if $r_0 \neq r_1 \in R$, then there exists a closed and open set $G \subset X\backslash\{v\}$ such that

$$(1) \qquad J_{r_0}\backslash\{v\} \subset G \subset X\backslash J_{r_1} .$$

By Theorem 3.1 there exists for each $n (n = 1, 2, \cdots)$ a neighborhood U_n of v such that $\operatorname{diam}(U_n) < 1/n$, $\bar{U}_{n+1} \subset U_n$ and $\operatorname{Bd}(U_n) \cap J_r$ is connected for each $r \in R$. We may assume that $J_{r_0} \cap \operatorname{Bd}(U_1) \neq \varnothing \neq J_{r_1} \cap \operatorname{Bd}(U_1)$. Let $R_n = \{r \in R \mid \operatorname{Bd}(U_n) \cap J_r \neq \varnothing\} (n = 1, 2, \cdots)$, then $R_n \subset R_{n+1}$ and $\bigcup_{n=1}^{\infty} R_n = R$.

Let Y be the space obtained from $\operatorname{Bd}(U_1)$ by identifying all components of $\operatorname{Bd}(U_1)$ to a point and let $f\colon \operatorname{Bd}(U_1) \to Y$ be the natural projection. It follows ([9], p. 148) that $\dim Y = 0$. Since

$$f(J_{r_0} \cap \operatorname{Bd}(U_1)) \neq f(J_{r_1} \cap \operatorname{Bd}(U_1)) ,$$

there exists a closed and open set H_1^* in Y such that

$$f(J_{r_0} \cap \operatorname{Bd}(U_1)) \subset H_1^* \subset Y\backslash f(J_{r_1} \cap \operatorname{Bd}(U_1)) .$$

Let $H_1 = f^{-1}(H_1^*)$, then H_1 is a closed and open set in $\operatorname{Bd}(U_1)$. Define $A_1 = \{r \in R_1 \mid J_r \cap H_1 \neq \varnothing\}$ and $B_1 = \{r \in R_1 \mid J_r \cap H_1 = \varnothing\}$, then $A_1 \cap B_1 = \varnothing$ and $A_1 \cup B_1 = R_1$. Moreover, since H_1 is closed and open in $\operatorname{Bd}(U_1)$, we have that

$$P_1 = \bigcup_{r \in A_1} \{J_r\backslash\{v\}\} \quad \text{and} \quad Q_1 = \bigcup_{r \in B_1} \{J_r\backslash\{v\}\}$$

are disjoint and closed subsets of $X\backslash\{v\}$.

By induction we will construct sets A_n and B_n such that

$$(2) \qquad A_{n-1} \subset A_n,\ B_{n-1} \subset B_n,\ A_n \cap B_n = \varnothing \text{ and } A_n \cup B_n = R_n$$

and if $P_n = \bigcup_{r \in A_n} \{J_r\}$ and $Q_n = \bigcup_{r \in B_n} \{J_r\}$ then P_n and Q_n are disjoint and closed subsets of $X\backslash\{v\} (n = 1, 2, \cdots)$.

Suppose A_{n-1} and B_{n-1} have been constructed. Since $P_{n-1} \cap \operatorname{Bd}(U_n)$ and $Q_{n-1} \cap \operatorname{Bd}(U_n)$ are disjoint closed subsets of $\operatorname{Bd}(U_n)$ and

$J_r \cap \mathrm{Bd}(U_n)$ is connected for each $r \in R$, it follows as above, replacing U_1, $J_{r_0} \cap \mathrm{Bd}(U_1)$ and $J_{r_1} \cap \mathrm{Bd}(U_1)$ by U_n, $P_{n-1} \cap \mathrm{Bd}(U_n)$ and $Q_{n-1} \cap \mathrm{Bd}(U_n)$ respectively, that there exists a closed and open subset H_n of $\mathrm{Bd}(U_n)$ such that

$$P_{n-1} \cap \mathrm{Bd}(U_n) \subset H_n \subset \mathrm{Bd}(U_n) \backslash Q_{n-1} .$$

Let $A_n = \{r \in R_n \mid J_r \cap H_n \neq \emptyset\}$ and $B_n = \{r \in R_n \mid J_r \cap H_n = \emptyset\}$, then A_n and B_n satisfy (2).

Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$, then $A \cup B = R$ and $A \cap B = \emptyset$. Let $G = \bigcup_{r \in A} \{J_r \backslash \{v\}\}$ and $G_n = \bigcup_{r \in A_n} \{J_r \backslash \bar{U}_n\}$. Since G_n is open in X and $G = \bigcup_{n=1} G_n$, it follows that G is open in X. Similarly $X \backslash (G \cup \{v\}) = \bigcup_{r \in B} \{J_r \backslash \{v\}\}$ is open in X. Hence G is both open and closed in $X \backslash \{v\}$ and, since $r_0 \in A_1$ and $r_1 \in B_1$, (1) is proved.

5. Property P and embeddings in the plane.

The main result of this section is Theorem 5.2 where we prove that if a fan is locally connected at its vertex, then it can be embedded in the plane. This result gives a solution to problem 1015 of [6].

Since every fan is hereditarily decomposable and hence 1-dimensional ([9], p. 206), we can consider every fan as a subspace of I^3. We start with the following lemma.

LEMMA 5.1. *Let X be a fan, with vertex v and*

$$X = \bigcup_{r \in R} \{J_r \mid J_r \cong [0, 1] \text{ for each } r \in R \text{ and } J_{r_1} \cap J_{r_2} = \{v\} \text{ if } r_1 \neq r_2 \in R\}$$

such that $\{J_r \backslash \{v\}\}$, $r \in R$, is the decomposition of $X \backslash \{v\}$ into quasi-components, then there exists an embedding $f\colon X \backslash \{v\} \to C \times I^3$ such that each quasi-component of $X \backslash \{v\}$ is contained in $\{c\} \times I^3$ for some $c \in C$, and

$$(1) \qquad \overline{f(X \backslash \{v\})} \backslash f(X \backslash \{v\}) \subset C \times \{v\} ,$$

where $C \subset [0, 1]$ denotes the Cantor ternary set.

Proof. We may assume that $X \subset I^3$. By ([9], p. 148), there exists a continuous function $g\colon X \backslash \{v\} \to C$ such that the quasi-components of $X \backslash \{v\}$ coincide with the point-inverses of g. Then the function $f\colon X \backslash \{v\} \to C \times I^3$ defined by $f(x) = (g(x), x)$ is an embedding. Only (1) remains to be shown. Let

$$(2) \qquad (c_0, x_0) \in \overline{f(X \backslash \{v\})} \backslash f(X \backslash \{v\}) ,$$

and let $\{(c_i, x_i)\}(i = 1, 2, \cdots)$ be a sequence of points in $f(X \backslash \{v\})$ converging to (c_0, x_0). We may assume that the sequence $\{x_i\}$ in X, where $x_i = f^{-1}((c_i, x_i))$, converges to a point $y \in X$. We consider two cases as follows:

Case 1. $y \neq v$. Then the sequence $\{f(x_i)\}$, where $f(x_i) = (c_i, x_i)$, converges to $f(y)$. Hence $f(y) = (c_0, x_0)$, contrary to (2).

Case 2. $y = v$. Then $x_0 = v$ and whence (1) holds.

These two cases complete the proof of the lemma.

THEOREM 5.2. *Let X be a fan which is locally connected at the vertex v of X, then X is embeddable in the plane.*

Proof. Let

$$X = \bigcup_{r \in R} \{J_r \mid J_r \cong [0, 1] \text{ for each } r \in R \text{ and } J_{r_1} \cap J_{r_2} = \{v\} \text{ if } r_1 \neq r_2 \in R\}.$$

It follows from 4.1 that $\{J_r \backslash \{v\}\}$, $r \in R$, is the decomposition of $X \backslash \{v\}$ into quasi-components. Hence by Lemma 5.1 there exists an embedding $f: X \backslash \{v\} \to C \times I^3$ such that each quasi-component of $X \backslash \{v\}$ is contained in $\{c\} \times I^3$ for some $c \in C$ and

$$\overline{f(X \backslash \{v\})} \backslash f(X \backslash \{v\}) \subset C \times \{v\} .$$

It follows that $\overline{f(X \backslash \{v\})}$ satisfies all conditions of Theorem 2.2, where $K = \overline{f(X \backslash \{v\})} \cap (C \times \{v\})$. Hence there exists an embedding $h: \overline{f(X \backslash \{v\})} \to I^2$ such that $h(K) = h(\overline{f(X \backslash \{v\})}) \cap l$, where $l = \{(x, y) \in I^2 \mid y = 0\}$. Let $\pi: I^2 \to I^2/l$ be the natural projection. It follows (see [9], p. 533) that $I^2 \cong I^2/l$ and whence the mapping $g: X \to I^2/l$ defined by

$$g(x) = \begin{cases} \pi \circ h \circ f(x) & \text{if } x \neq v , \\ \pi(l) & \text{if } x = v \end{cases}$$

is the required embedding.

REMARK. J. J. Charatonik and Z. Rudy constructed a fan X which is locally connected at its vertex (see [6], p. 215). They conjectured that this fan is not embeddable in the plane. The above theorem disproves their conjecture and gives a solution to problem 1015 of [6].

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ON BANACH SPACES HAVING THE PROPERTY G. L.

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A Banach space E has the property G. L. if every absolutely summing operator defined on E factors through an L_1-space. Some properties of spaces having G. L. property are investigated, using methods of Banach ideals of operators.

1. Introduction and notations. The property G. L. is known to be shared by a number of important classes of Banach spaces: in [6] it is shown that if E'' is isomorphic to a complemented subspace of a Banach lattice (in particular, if E has local unconditional structure in the sense of [4]) then E has the G. L. property. Subspaces of L_1 spaces as well as quotients of $C(K)$ spaces have G. L. property. Moreover, in [17] it is shown that if E is a subspace of a Banach space F *s.t.* $\Pi_2(\mathscr{L}_\infty, F) = \mathscr{L}(\mathscr{L}_\infty, F)$ (in particular if F has cotype 2) and F has the property G. L. then E has the property G. L. In fact, it is easy to see that it is enough for E to be finitely represented in F. In this paper, we try to investigate the property G. L. using methods of Banach ideals of operators. It is shown that this property is characterized by a perfect ideal $[\Gamma, \gamma]$. We obtain a description of the conjugate ideal $[\Gamma^*, \gamma^*]$ and deduce that $[\Gamma, \gamma]$ is a symmetric ideal hence E has G. L. iff E' has it.

It is also shown that a number of properties, known to hold for spaces having *l.u.st.* in the sense of [4] are common to all the spaces having G. L. For example, if E is a space having G. L. which does not contain l_∞^n-*s* uniformly, then either E contains l_1^n-*s* uniformly and uniformly complementably, or E does not contain l_1^n-*s* uniformly at all.

It follows that if E is a space having G. L. and F a Banach space, then there exist compact nonnuclear operators from E to F and from F to E. These are partial generalizations to results of Davis and Johnson (see [2] and [9]). We show also that for spaces having G. L. the property $\Pi_2(\mathscr{L}_\infty, E) = \mathscr{L}(\mathscr{L}_\infty, E)$ implies that E is of cotype 2; we show a dual implication as well.

The paper is divided into two parts. In §2 we describe some tools in Banach ideals of operators; in §3 we use these tools in investigating spaces having G. L. It seems to us that these tools may be useful in other contexts.

The notations are of two kinds:

(1) *General notations.* We use standard notations of Banach

space theory. If E is a Banach space its dual space is E' and for $x \in E$, $x' \in E'$ we denote by $\langle x, x' \rangle$ the scalar product of x and x'.

We deal with Banach spaces over the field of real numbers. Modification to the complex numbers case is straightforward. For a positive measure space (Ω, Σ, μ) and $1 \leqq p \leqq \infty$ we denote by $L_p(\mu)$ the Banach space of scalar, μ-measurable functions f with $|f|^p$ integrable (with classical modification for $p = \infty$) with the usual norm.

We denote by $L_p(E) = L_p(\mu, E)$ the space of Bochner measurable E-valued functions with $\|f(\cdot)\| \in L_p(\mu)$ equipped with the norm $\|f\| = \| \ \|f(\cdot)\| \ \|_{L_p(\mu)}$.

The term "operator" means "bounded linear operator between Banach spaces". If E, F are Banach spaces, $\mathscr{L}(E, F)$ is the Banach space of operators from E into F equipped with the norm of operators.

Let E, F be Banach spaces; we say that E is finitely represented in F (abbreviation: $Ef.rF$) if for every finite dimensional subspace E_1 of E and $\varepsilon > 0$ there exists a subspace F_1 of F and an isomorphism $u: E_1 \to F_1$ with $\|u\| \|u^{-1}\| \leqq 1 + \varepsilon$. If P is a property which makes sense for Banach spaces we say that E has super-P if every space F with $Ff.rE$ has the property P.

(2) *Definitions and notations concerning Banach ideals of operators and tensor products of Banach spaces.* A standard reference in Banach ideals of operators is [8] (see also, [15] and [14]); as a reference concerning tensor products one can use [20]. If $[A, a]$ is a Banach ideal of operators we denote by $[A^*, a^*]$ the conjugate ideal and say that $[A, a]$ is perfect if $[A, a] = [A^{**}, a^{**}]$. $[A', a']$ is the adjoint ideal ($T \in A'(E, F)$ iff $T' \in A(F', E')$).

Let $[A, a]$ be a normed ideal of operators and E, F Banach spaces, a norm (called "an ideal norm") is naturally induced on the tensor product $E \otimes F$ by considering it as algebraically contained in $\mathscr{L}(E', F)$. We denote $E \otimes F$ with this norm by $E \bigotimes_a F$ and its completion by $E \hat{\bigotimes}_a F$. Let E, F be Banach spaces and $u \in E \otimes F$. Let E_1, F_1 be subspaces of E and F respectively $s.t.$ there is a representation of u as $u = \sum_{i=1}^n x_i \otimes y_i$ with $x_i \in E_i$, $y_i \in F_1$ for all i. We denote by $a(u, E_1, F_1)$ the norm of u as an element of $E_1 \bigotimes_a F_1$. If E and F are not considered as subspaces of some other spaces we denote $a(u, E, F) = a(u)$.

We say that an ideal norm a is semi-tensorial norm if for every pair of Banach spaces E, F, one which is finite dimensional, and every $u \in E \otimes F$ hold: $a(u) = \inf \{a(u, E_1, F_1)$; $E_1 \subset E$, $F_1 \subset F$, E_1 and F_1 finite dimensional and $u \in E_1 \otimes F_1\}$.

We list here a number of ideals that we shall use in the sequel.

(a) $[\mathscr{L}, \|\cdot\|]$ the ideal of all bounded operators.

(b) $[\Pi_p, \pi_p]$ $(1 \leqq p \leqq \infty)$ the ideal of p-summing operators.

(c) $[I_p, i_p]$ the ideal of p-integral operators. $U \in I_p[E, F]$ if there exists a probability space (Ω, Σ, μ) and operators $V \in \mathscr{L}(E, L_\infty(\mu))$, $W \in \mathscr{L}(L_p(\mu), F'')$ $s.t. WiV = j_F U$ where i is the formal "inclusion" map of $L_\infty(\mu)$ into $L_p(\mu)$ and j_F the canonnical inclusion of E into E''.

We define $i_p(U) = \inf\{\|V\|\|W\|; V, W, (\Omega, \Sigma, \mu)$ as in the definition}. We say that U is strongly p-integral if the preceeding factorization is for U instead of $j_F U$.

(d) $[N_p, \nu_p]$ $1 \leqq p < \infty$ the ideal of p-nuclear operators.

(e) $[\Gamma_p, \gamma_p]$ the ideal of operators factorizable through L_p. $U \in \Gamma_p(E, F)$ if there exists an $L_p(\mu)$ space and operators $A \in \mathscr{L}(E, L_p(\mu))$, $B \in \mathscr{L}(L_p(\mu), F'')$ $s.t. j_F U = BA$. We define $\gamma_p(U) = \inf \|B\|\|A\|$.

(f) (A new definition). $[M, \mu]$ the ideal of operators factorizable through a Banach lattice. $U \in M(E, F)$ iff there exists a Banach lattice L and $A \in \mathscr{L}(E, L)$, $B \in \mathscr{L}(L, F'') s.t. j_F U = BA$. $\mu(U) = \inf \|B\|\|A\|$. Using ultraproducts of Banach spaces ([1]) or the methods of [5] one can show that $[M, \mu] = [H^{**}, \eta^{**}]$ where $[H, \eta]$ is the ideal of weakly nuclear operators introduced in [7]. Therefore a Banach space E has $l.u.st$ in the sense of [6] iff E'' is isomorphic to a complemented subspace of a Banach lattice ([5]).

It is known that the ideals in (a), (b), (c) and (e) are perfect and the same is true for the ideal in (f). It is also not hard to check that all the ideal norms on tensor products induced by the above ideals are semi-tensorial.

Let E, F be Banach spaces, *the greatest tensor-norm*, π, is defined on $E \otimes F$ by $\pi(u) = \inf\{\sum_{i=1}^n \|x_i\|\|y_i\|; u = \sum_{i=1}^n x_i \otimes y_i\}$ for $u \in E \otimes F$. There is an identification $(E \hat{\otimes}_\pi F)' = \mathscr{L}(F, E')$ defined by

$$\langle u, T\rangle = \operatorname{trace} Tu = \sum_{i=1}^n \langle x_i, Ty_i\rangle$$

for

$$u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F .$$

2. Let I be an index set and $\{[A_i, a_i]\}_{i \in I}$ a family of normed ideals of operators.

DEFINITION 2.1. (a) The greatest lower bound $[\bigwedge_i A_i, \bigwedge_i a_i]$ of the family is defined by:

$$\Big(\bigwedge_i A_i\Big)(E, F) = \{T \in \mathscr{L}(E, F);\ \forall i,\ T \in A_i(E, F) \text{ and } \sup_i a_i(T) < \infty\}$$

$$\Big(\bigwedge_i a_i\Big)T = \sup_i a_i(T) \quad \text{for} \quad T \in \Big(\bigwedge_i A_i\Big)(E, F)\ .$$

(b) The least upper bound $[\bigvee_i A_i, \bigvee_i a_i]$ of the family is defined by:

$$\Big(\bigvee_i A_i\Big)(E, F) = \{T \in \mathscr{L}(E, F);\ T = \sum_{j \in J} T_j;\ J \subset I,\ J \text{ finite and for all } j \in J\ T_j \in A_j(E, F)\}$$

$$\Big(\bigvee_i a_i\Big)(T) = \inf\Big[\sum_{j \in J} a_j(T_j)\Big] \quad \text{for} \quad T \in \Big(\bigvee_i A_i\Big)(E, F)\ ,$$

the inf being taken over all finite subsets $J \subset I$ *s.t.* there is a representation $T = \sum_{j \in J} T_j$ with $T_j \in A_j(E, F)$.

PROPOSITION 2.2. (a) $[\bigwedge_i A_i, \bigwedge_i a_i]$ *and* $[\bigvee_i A_i, \bigvee_i a_i]$ *are normed ideals of operators.*

(b) *If for all* i $[A_i, a_i]$ *are Banach ideals then so is* $[\bigwedge_i A_i, \bigwedge_i a_i]$ *and if, in addition,* I *is finite, then* $[\bigvee_i a_i, \bigvee_i a_i]$ *is also a Banach ideal.*

(c) *If for all* i $[A_i, a_i]$ *are perfect then so is* $[\bigwedge_i A_i, \bigwedge_i a_i]$.

The proof is routine.

PROPOSITION 2.3. $[\bigwedge_i A_i^*, \bigwedge_i a_i^*] = [(\bigvee_i A_i)^*, (\bigvee_i a_i)]$.

Proof. Consider the following diagram, in which E, F are Banach spaces, E_1, F_1 finite dimensional Banach spaces and T, U, S, V operators.

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle V}\big\downarrow & & \big\downarrow{\scriptstyle U} \\ E_1 & \xrightarrow{S} & F_1 \end{array}$$

(a) Suppose $T \in (\bigvee_i A_i)^*(E, F)$ then

$$|\text{trace } TVSU| \leqq \Big(\bigvee_i a_i\Big)^*(T)\|V\|\|U\|\Big(\bigvee_i a_i\Big)(S)\ ,$$

hence, for all $i \in I$

$$|\text{trace } TVSU| \leqq \Big(\bigvee_i a_i\Big)^*(T)\|V\|\|U\|a_i(S)\ ,$$

therefore $\forall i \in I$ $a_i^*(T) \leqq (\bigvee_i a_i)^*(T)$ and it follows that

$$T \in \Big(\bigwedge_i A_i^*\Big)(E, F) \quad \text{and} \quad \Big(\bigwedge_i a_i^*\Big)(T) \leqq \Big(\bigwedge_i a_i\Big)^*(T)\,.$$

(b) Suppose $T \in \bigwedge_i A_i^*(E, F)$. Let $J \subset I$ be finite and $S = \sum_{j \in J} S_j$ be a representation of S *s.t.*

$$\sum_{j \in J} a_j(S_j) \leqq \Big(\bigvee_i a_i\Big)(S) + \varepsilon\,.$$

We have:

$$\begin{aligned}
|\operatorname{trace} TVSU| &\leqq \sum_{j \in J} |\operatorname{trace} TVS_jU| \\
&\leqq \sum_{j=1}^{n} a_j^*(T)\,\|V\|\,\|U\|\,a_j(S_j) \\
&\leqq \sup_i a_i^*(T)\,\|V\|\,\|U\|\Big(\sum_{j=1}^{n} a_j(S_j)\Big) \\
&\leqq \Big(\bigwedge_i a_i^*\Big)(T)\,\|V\|\,\|U\|\Big[\Big(\bigvee_i a_i\Big)(S) + \varepsilon\Big]\,,
\end{aligned}$$

therefore $T \in (\bigvee_i A_i)^*(E, F)$ and $(\bigvee_i a_i)^*(T) \leqq (\bigwedge_i a_i^*)(T)$.

COROLLARY 2.4. *If $[A_i, a_i]$ are perfect, then*

$$\Big[\Big(\bigwedge_i A_i\Big)^*, \Big(\bigwedge_i a_i\Big)^*\Big] = \Big[\Big(\bigvee_i A_i^*\Big)^{**}, \Big(\bigvee_i a_i^*\Big)^{**}\Big]\,,$$

in particular, if E and F are finite dimensional then (without assuming perfectness of $[A_i, a_i]$) for every $T \in \mathscr{L}(E, F)$ $(\bigwedge_i a_i)^*(T) = (\bigvee_i a_i^*)(T)$.

Proof. Since for all i $[A_i, a_i] = [A_i^{**}, a_i^{**}]$ we get

$$\Big(\bigwedge_i A_i\Big)^* = \Big(\bigwedge_i A_i^{**}\Big)^* = \Big[\Big(\bigvee_i A_i^*\Big)^*\Big]^* = \Big(\bigvee_i A_i^*\Big)^{**}$$

with equality of the norms. The second assertion is an obvious consequence of the first.

DEFINITION 2.5. (a) Let $[A, a]$ and $[B, b]$ be normed ideals of operators and G a fixed Banach space. We define for Banach spaces E, F:

$$\Big(\frac{A}{B}\Big)_G(E, F) = \{T \in \mathscr{L}(E, F);\ \forall U \in B(F, G) \quad UT \in A(E, G)\}\,.$$

From the closed-graph theorem it follows that for every $T \in (A/B)_G(E. F)$ there exists a $k > 0$ *s.t.* for all $U \in B(F, G) a\ (UT) \leqq$

$kb(U)$. We define $(a/b)_G(T) = \inf\{k;\ k$ as above$\}$.

(b) Let $[A, a]$ and $[B, b]$ be normed ideals of operators, E and F Banach spaces. We define

$$\frac{A}{B}(E, F) = \{T \in \mathscr{L}(E, F);\ \text{for every Banach space } G \text{ and } U \in B(F, G)\ UT \in A(E, G)\} .$$

It can be shown in a standard way that for every $T \in A/B(E, F)$ there exists a $k > 0$ *s.t.* for every Banach space G and $U \in B(F, G)$ $a(UT) \leqq kb(U)$. We define $a/b(T) = \inf\{k;\ k$ as above$\}$.

(c) Let $[A, a]$, $[B, b]$, E and F be as in (b). We define

$$\frac{A}{B}f(E, F) = \{T \in \mathscr{L}(E, F);\ \exists k > 0\ s.t.\ \text{for every Banach space } G \text{ of finite dimension and } U \in \mathscr{L}(F, G)\ a(UT) \leqq kb(U)\}$$

$$\frac{a}{b}f(T) = \inf\{k,\ k \text{ as above}\} \text{ for } T \in \frac{A}{B}f(E, F) .$$

PROPOSITION 2.6. *$[(A/B)_G, (a/b)_G]$, $[A/B, a/b]$ and $[A/Bf, a/b\,f]$ are normed ideals of operators.*

If $[A, a]$ is a Banach ideal then these ideals are Banach ideals. If $[A, a]$ is perfect then $[A/B, a/b] = [A/Bf, a/b\,f]$.

Proof. The verification of the first and third assertions is routine. We prove the second assertion for A/B.

Let $\{T_n\}_{n \in N}$ be a Cauchy sequence in $A/B\ (E, F)$. It is easy to check the following facts:

(1) There exists an operator $T \in A/B(E, F)$ *s.t.* for every Banach space G and $U \in B(F, G)$ $a(UT_n - UT) \xrightarrow[n\to\infty]{} 0$.

(2) The numerical sequence $a/b(T_n - T)$ is Cauchy, hence $a/b(T_n - T) \xrightarrow[n\to\infty]{} l \geqq 0$.

It is left to show that $l = 0$. Suppose $l > 0$. By (2) there is an integer n_0 *s.t.* for any $n \geqq n_0$ there exists a Banach space G_n and an operator $U_n \in B(F, G_n)$ with $b(U_n) \leqq 1$ *s.t.* $a(U_n(T_n - T)) > l/2$. We get for $m > n \geqq n_0$.

(3) $l/2 < a(U_n(T_n - T)) \leqq a(U_n(T_n - T_m)) + a(U_n(T_m - T))$.

Choose $n_1 > n_0$ *s.t.* for all U with $b(U) \leqq 1$ and $n, m \geqq n_1$ we have $a(U(T_n - T_m)) < l/8$ (which is possible since $\{T_n\}$ is Cauchy in $A/B(E, F)$). Fix $n > n_1$ and let $m_1 > n_1$ be *s.t.* for $m > m_1$ we have $a(U_n(T_m - T)) < l/8$ (such m_1 exists by 1).

Applying (3) to the fixed n and some $m > m_1$ we get $l/2 < l/4$ which is a contradiction that completes the proof.

PROPOSITION 2.7. *Let $[A, a]$ and $[B, b]$ be normed ideals of operators such that $[A, a]$ is perfect and b is a semi-tensorial norm. Then $[A/B, a/b]$ is perfect.*

Proof. By Proposition 2.6 it is enough to show that $[A/Bf, a/b\,f]$ is perfect. Let $T \in (A/B\,f)^{**}(E, F)$, then for every finite dimensional subspace M of E and finite codimensional subspace N of F $a/b\,f(q_N T i_M) \leqq (a/b\,f)^{**}(T)$ where $i_M\colon M \to E$ is the inclusion map and $q_N\colon N \to F/N$ the canonical surjection. Let G be a finite dimensional Banach space and $U \in B(F, G)$, since b is semi-tensorial we have:

$$\begin{aligned} b(U) &= \inf \{b(U, F^1, G);\ F^1 \text{ finite dimensional subspace of } F'\} \\ &= \inf b(U_1) \end{aligned}$$

the last infinum is taken over all operators U_1 and finite codimensional subspaces N of F such that U has a factorization of the form:

$$\begin{tikzcd} F \arrow[rr, "U"] \arrow[dr, "q_N"'] & & G \\ & F/N \arrow[ur, "U_1"'] & \end{tikzcd} \tag{1}$$

For given $\varepsilon > 0$ let N and U_1 be as in (1) with $b(U_1) \leqq b(U) + \varepsilon$. We have $a(UTi_M) = a(U_1 q_N T i_M) \leqq b(U_1) a/b\,f(q_N T i_M) \leqq (b(U)+\varepsilon)(a/b\,f)^{**}(T)$. Since ε is arbitrary and $[A, a]$ is perfect it follows that $a(UT) \leqq b(U)(a/b)^{**}(T)$, therefore $T \in A/Bf(E, F)$ and $a/b\,f(T) = (a/b\,f)^{**}(T)$.

PROPOSITION 2.8. *Let $[A, a]$ and $[B, b]$ be normed ideals of operators, E and F Banach spaces of finite dimension and $T \in \mathscr{L}(E,F)$. Then $(a/b\,f)^*(T) = \inf \sum_{i=1}^n a^*(U_i) b(V_i)$, the infinum being taken over all representations of T of the form $T = \sum_{i=1}^n U_i V_i$ with $V_i \in \mathscr{L}(E, G_i)$; $U_i \in \mathscr{L}(G_i, F)$ and G_i finite dimensional Banach spaces.*

Proof. For fixed finite dimensional G and $S \in \mathscr{L}(F, E)$ we have

$$\left(\frac{a}{b}\right)_G (S) = \sup \{a(US);\ U \in \mathscr{L}(E, G),\ b(U) \leqq 1\}\,.$$

Define the operator $\hat{S}\colon B(E, G) \to A(F, G)$

by $$\hat{S}(U) = US\,. \quad \text{Then}$$

$$\left(\frac{a}{b}\right)_G (S) = \|\hat{S}\|\,.$$

The correspondence $S \leftrightarrow \hat{S}$ enable us to identify $(A/B)_G(F, E)$ with a subspace of $\mathscr{L}(B(E, G), A(F, G))$. Therefore $(A/B)_G^*(E, F) =$

$[(A/B)_G(F, E)]'$ is a quotient space of $A^*(G, F) \otimes_\pi B(E, G)$ with the following identification: for $\phi = \sum_{i=1}^n U_i \otimes V_i \in A^*(G, F) \otimes_\pi B(E, G)$ and $S \in (A/B)_G(F, E)$ we define

$$\begin{aligned}\langle S, \phi\rangle = \langle \phi, \hat{S}\rangle &= \sum_{i=1}^n \langle U_i, \hat{S}(V_i)\rangle \\ &= \sum_{i=1}^n \langle U_i, V_i S\rangle = \sum_{i=1}^n \operatorname{trace} U_i V_i S = \operatorname{trace} TS\end{aligned}$$

where

$$T = \sum_{i=1}^n U_i V_i .$$

From the last discussion it follows that for $T \in \mathscr{L}(E, F)$

$$\left(\frac{a}{b}\right)_G^*(T) = \inf \left\{\sum_{i=1}^n a^*(U_i) b(V_i);\ T = \sum_{i=1}^n U_i V_i;\ V_i \in \mathscr{L}(E, G) U_i \in \mathscr{L}(G, F)\right\} .$$

We complete the proof by noting that

$$\left[\frac{A}{B} f, \frac{a}{b} f\right] = \left[\bigwedge_{\dim G < \infty} \left(\frac{A}{B}\right)_G, \bigwedge_{\dim G < \infty} \left(\frac{a}{b}\right)_G\right]$$

and by using Corollary 2.4 which shows that for finite dimensional E and F

$$\left[\left(\frac{A}{B} f\right)^*, \left(\frac{a}{b} f\right)^*\right] = \left[\bigvee_{\dim G < \infty} \left(\frac{A}{B}\right)_G^*, \bigvee_{\dim G < \infty} \left(\frac{a}{b}\right)_G^*\right] .$$

3.

DEFINITION 3.1. We define the ideal $[\Gamma, \gamma]$ by:

$$[\Gamma, \gamma] = \left[\frac{\Gamma_1}{\Pi_1}, \frac{\gamma_1}{\pi_1}\right] . \quad \text{Explicitly:}$$

$T \in \Gamma(E, F)$ iff for every Banach space G and $U \in \Pi_1(F, G)$ $UT \in \Gamma_1(E, G)$. For such an operator T $\gamma(T) = \sup \gamma_1(UT)$, the supremum being taken over all Banach spaces G and $U \in \Pi_1(F, G)$ with $\pi_1(U) = 1$.

DEFINITION 3.2. We say that a Banach space E has the property G. L. (Gordon-Lewis) if for every Banach space G $\Pi_1(E, G) \subset \Gamma_1(E, G)$. Of course, E has property G. L. iff the identity operator on E is in $\Gamma(E, E)$.

PROPOSITION 3.3. *A Banach space E has the property G. L. if* *and only if there exist $k > 0$ s.t. for every finite dimensional Banach space G and $U \in \mathscr{L}(E, G)$ $\gamma_1(U) \leqq k\pi_1(U)$.*

Proof. This is a result of the equality

$$\left[\frac{\Gamma_1}{\Pi_1}, \frac{\gamma_1}{\pi_1}\right] = \left[\frac{\Gamma_1}{\Pi_1}f, \frac{\gamma_1}{\pi_1}f\right]$$

which is, in turn, a consequence of Proposition 2.6 and the fact that $[\Gamma_1, \gamma_1]$ is perfect.

PROPOSITION 3.4. *Let E and F be finite dimensional Banach spaces and $T \in \mathscr{L}(E, F)$. Then* (a) $\gamma^*(T) = \inf [\sum_{i=1}^n \pi_1'(U_i)\pi_1(V_i)]$, *the infinum being taken over all representations of the form $T = \sum_{i=1}^n U_i V_i$ with $V_i \in \Pi_1(E, G_i)$, $U_i \in \Pi_1'(G_i, F)$ and G_i finite dimensional Banach spaces.*

(b) $\gamma^*(T) = \inf [\sum_{i=1}^n \|\mu_i\|\|\nu_i\|]$, *the infimum being taken over all representations of the form $T = \sum_{i=1}^n T_i$ s.t for all i there exist positive Radon measures, μ_i on the unit ball $B(E')$ of E' and ν_i on the unit ball $B(F)$ of F s.t. for all $x \in E$, $y' \in F'$ and $1 \leqq i \leqq n$ hold:*

$$|\langle T_i x, y'\rangle| \leqq \int_{B(E')} |\langle x, x'\rangle| d\mu_i(x') \int_{B(F)} |\langle y, y'\rangle| d\nu_i(y) .$$

Proof. (a) Follows from Propositions 2.8 and 3.3 combined with the fact ([10]) that $[\Gamma_1^*, \gamma_1^*] = [\Pi_1', \pi_1']$.

(b) Is a consequence of (a) and the following lemma which is proved by methods of [10].

LEMMA 3.5. (c) *Let $T \in \mathscr{L}(E, F)$ (E, F not necessarily finite dimensional) then*

$$\inf \pi_1'(U)\pi_1(V) = \inf \|\nu\|\|\mu\| \tag{1}$$

where the infinum on the left is taken over all Banach spaces G and representations $jT = UV$ with j the canonical inclusion of F into F'', $U \in \Pi_1'(G, F'')$ and $V \in \Pi_1(E, G)$. The infimum on the right is taken over all positive Radon measures μ on $B(E')$ and ν on $B(F''')$ (with the relative ω^*-topologies) s.t. for all $x \in E$, $y' \in F'$ hold

$$|\langle Tx, y'\rangle| \leqq \int_{B(E')} |\langle x, x'\rangle| d\mu(x') \int_{B(F''')} |\langle y', y''\rangle| d\nu(y'') .$$

(d) *If in* (c) *E and F are finite dimensional then the infinum on the left hand side of* (1) *can be taken over all finite dimensional Banach spaces G.*

Proof. (d) follows from (c) since π_1 and π'_1 are semi-tensorial (in fact, tensorial) norms. We prove (c).

Let $jT = UV$ be a factorization of jT with $U \in \Pi'_1(G, F'')$ and $V \in \Pi_1(E, G)$. By the Pietsch factorization theorem there exist positive Radon measures, μ on $B(E')$ and ν on $B(F'')$ *s.t.* for $x \in E$, $y' \in F'$ $\|Vx\| \leqq \int_{B(E')} |\langle x, x'\rangle| d\mu(x')$, $\|U'|_{F'}y'\| \leqq \int_{B(F'')} |\langle y', y''\rangle| d\nu(y'')$ and $\|\mu\| \leqq \pi_1(V) + \varepsilon$, $\|\nu\| \leqq \pi_1(U') + \varepsilon$. Therefore $\|\nu\|\|\mu\| \leqq (\pi'_1(U) + \varepsilon)(\pi_1(V) + \varepsilon)$ and

$$(2) \qquad |\langle Tx, y'\rangle| = |\langle Vx, U'y'\rangle| \leqq \int_{B(E')} |\langle x, x'\rangle| d\mu \int_{B(F'')} |\langle y', y''\rangle| d\nu .$$

On the other hand, suppose μ and ν are Radon measures on $B(E')$ and $B(F'')$ respectively *s.t.* (2) hold for every $x \in E$, $y' \in F'$ then we define operators:

$$U_0 \colon F' \longrightarrow L_1(\nu) ; \quad U_0(y') = \langle \cdot, y'\rangle$$

and

$$V_0 \colon E \longrightarrow L_1(\mu) ; \quad V_0(x) = \langle x, \cdot\rangle .$$

Let $H = \overline{U_0(F')}$, $G = \overline{V_0(E)}$ and let $\langle\!\langle \cdot \rangle\!\rangle$ be the bilinear form on $V_0(E) \times U_0(F')$ defined by $\langle\!\langle V_0x, U_0y'\rangle\!\rangle = \langle Tx, y'\rangle$, from (2) it follows that this form is well defined and bounded with norm $\leqq 1$, hence it defines an operator $W \in \mathscr{L}(G, H')$ with $\|W\| \leqq 1$ and $\langle\!\langle V_0x, U_0y'\rangle\!\rangle = \langle WV_0x, U_0y'\rangle$. We have then the following commutative diagram:

$$(3) \qquad \begin{tikzcd} E \arrow[r, "T"] \arrow[dr, "U_1"'] & F \arrow[r, "j"] & F'' \\ & G \arrow[r, "W"] & H' \arrow[u, "V_1'"'] \end{tikzcd}$$

where U_1 and V_1 are U_0 and V_0 considered as operators into G and H respectively. Of course $\pi_1(U_1) \leqq \|\mu\|$ and $\pi_1(V'_1) \leqq \|\nu\|$ which completes the proof of Lemma 3.5 and Proposition 3.4.

REMARK 3.6. In [7] Gordon and Lewis show that for all E, F and $T \in \mathscr{L}(E, F)$

$$(1) \qquad \mu^*(T) = \inf \|\mu\| ,$$

the infinum being taken over all positive Radon measures on $B(E') \times B(F'')$ (with the product of the ω^*-topologies) which satisfy for all x, y':

$$(2) \qquad |\langle Tx, y'\rangle| \leqq \int_{B(E')\times B(F'')} |\langle x, x'\rangle\langle y', y''\rangle| d\mu(x', y'') .$$

In fact, using compactness of the unit balls it is not hard to check that for finite dimensional E and F we can replace "$\inf \|\mu\|$" by "$\inf \sum_{i=1}^{n} \|\mu_i\| \|\nu_i\|$" in (1); μ_i, ν_i positive Radon measures on $B(E')$ and $B(F)$ respectively *s.t.* for all x, y'

$$(3) \qquad |\langle Tx, y'\rangle| \leqq \sum_{i=1}^{n} \int_{B(E')} |\langle x, x'\rangle| d\mu_i(x') \int_{B(F)} |\langle y, y'\rangle| d\nu_i(y)$$

(all the $\mu_i \otimes \nu_i$ but one may be taken as scalar multiples of $\delta(x_i') \otimes \delta(y_i)$—the products of valuations at points $x_i' \in B(E')$, $y_i \in B(F)$, the one $\mu_i \otimes \nu_i$ left may be a scalar multiple of the product of Lebesgue measures on $B(E')$ and $B(F)$). The difference between μ^* and γ^* is therefore the possibility to represent T as a sum $\sum_{i=1}^{n} T_i$ where each T_i is "majorized" by the product $\mu_i \otimes \nu_i$. It follows of course that $\mu^* \leqq \gamma^*$, hence $\mu \geqq \gamma$ and we get the result of [6]: if E'' is isomorphic to a complemented subspace of a Banach lattice then E has property G. L.

COROLLARY 3.7. $[\Gamma, \gamma] = [\Gamma', \gamma']$, *therefore E has the property G. L. if and only if E' has it.*

Proof. $[\Gamma^*, \gamma^*] = [\Gamma^{*\prime}, \gamma^{*\prime}]$; this is obvious for pairs of finite dimensional Banach spaces from (a) or (b) of Proposition 3.4 and passes over to all pairs of Banach spaces since $[\Gamma^*, \gamma^*]$ is perfect. Now perfectness of $[\Gamma, \gamma]$ gives $[\Gamma, \gamma] = [\Gamma^{**}, \gamma^{**}] = [\Gamma^{*\prime *}, \gamma^{*\prime *}] = [\Gamma^{**\prime}, \gamma^{**\prime}] = [\Gamma', \gamma']$.

The last corollary enables us to prove that a number of properties known to hold for spaces having *l.u.st.* are true also for spaces having the property G. L.

We use the next lemma of Pisier ([16] and [17]) which was originally proved for spaces E with E'' isomorphic to a complemented subspace of a Banoch lattice. However, Pisier's proof uses only the fact that such an E, and also E', has the property G. L.

LEMMA 3.8. *Let E have the property G. L.*

(a) *If E does not contain l_∞^n's uniformly, then there exist q, $2 \leqq q < \infty$ and $C > 0$ s.t.*

(1) *For any E valued operator $A\pi_q(A) \leqq C\pi_1'(A)$.*

(b) *If neither E nor E' contain l_∞^n's uniformly, then there exist q, $2 \leqq q < \infty$, p, $1 < p \leqq 2$ and $C > 0$ s.t.:*

(2) *For any E-valued operator $A\pi_q(A) \leqq C\pi_p'(A)$.*

The next theorem and its corollary is in a certain way a generalization of results of Johnson and Davis ([9] and [2]).

THEOREM 3.9. *Let E be finitely represented in a Banach space F such that F has the property G. L. and F does not contain l_∞^n-'s uniformly. Then either E contains l_1^n-'s uniformly and uniformly complementably or E does not contain l_1^n-'s uniformly.*

We need two lemmas.

LEMMA 3.10. *Let $[A, a]$ and $[B, b]$ be normed ideals of operators s.t. a is a semi-tensorial norm and $[B, b]$ is perfect and right injective (which means: if E, F, G are Banach spaces, $F \subset G$ and $T \in \mathscr{L}(E, F)$ then the b-norms of T considered as operator from E to F or from E to G are the same).*

Let F be a Banach space *s.t.* the following holds:

(1) There exists a $k > 0$ *s.t.* for every Banach space G and $T \in A(G, F)$ $b(T) \leqq ka(T)$.

Let E be a Banach space *s.t.* $Ef.r.F$ then (1) is true for E as well.

Proof. Let G be a Banach space and $T \in A(G, E)$. Let G_1 be a finite dimensional subspace of G and $T_1 = T|_{G_1}: G_1 \to E$. Then $a(T_1) \leqq a(T)$. Since a is semi-tensional and G_1 finite dimensional then $a(T_1) = \inf\{a(T_1: G_1 \to N);\ N$ a finite dimensional subspace of E with $T_1(G_1) \subset N\}$. Given $\varepsilon > 0$ there exists therefore a finite dimensional subspace $N \subset E$ with $T_1(G_1) \subset N$ *s.t.* $\bar{T}_1: G_1 \to N$ — the astriction of T_1, satisfies $a(\bar{T}_1) \leqq (1 + \varepsilon)a(T_1)$. We can find a $N_1 \subset F$ and an isomorphism $i: N \to N_1$ with $||i|| \leqq 1$; $||i^{-1}|| \leqq 1 + \varepsilon$. Let $j: N_1 \to F$ be the inclusion map from N_1 into F, then $a(ji\bar{T}_1) \leqq (1 + \varepsilon)a(T)$ and (1) gives:

$$b(ji\bar{T}_1) \leqq k(1 + \varepsilon)a(T)\,, \quad \text{injectivity of } [B, b]$$

implies now that $b(i\bar{T}_1) \leqq k(1 + \varepsilon)a(T)$. Therefore $b(\bar{T}_1) \leqq k(1 + \varepsilon)^2 a(T)$ which implies $b(T_1) \leqq k(1 + \varepsilon)^2 a(T)$. Since ε is arbitrary and $[B, b]$ perfect we conclude that $b(T) \leqq ka(T)$.

We say that a Banach space E has *property $I - K$* (respectively $I - N_r$) if for every Banach space G and strongly integral operator $T: G \to E$ T is compact (respectively — T is r-nuclear). It is known (combining results of Diestel [3] and Pisier [18]) that the property super $(I - N_1)$ is super reflexivity.

LEMMA 3.11. *The following are equivalent:*

(a) *E has the property super $(I - K)$.*

(b) *E does not contain l_1^n's uniformly.*

Proof. It is known that if E contains l_1^n-s uniformly than l_1, as well as $L_1[0, 1]$ are finitely represented in E. The formal "inclusion" map $L_\infty[0, 1] \to L_1[0, 1]$ is strongly integral, noncompact operator, therefore in this case E fails to have super $(I - K)$. Suppose, on the other hand, that E does not contain l_1^n-s uniformly but there exists an integral noncompact operator into E. The adjoint of this operator is a strongly integral noncompact operator T defined on E', hence it is a Dunford-Pettis operator (which means that it takes ω-Cauchy sequences into norm convergent sequences). Since E does not contain l_1^n-s uniformly — E' does not contain an isomorph of l_1, it follows from a result of Rosenthal [19] that every bounded sequence in E' contains a ω-Cauchy subsequence, but then T must be compact — a contradiction. Therefore E has $(I - K)$. Since "not containing l_1^n-'s uniformly" is a super-property it turns out that E has in fact super $(I - K)$.

Proof of Theorem 3.9. From Lemma 3.8 follows the existence of $c > 0$ and $2 \leqq q < \infty$ *s.t* for every Banach space G and $A: G \to F$

$$\pi_q(A) \leqq c\pi_1'(A) . \tag{1}$$

From Lemma 3.10 we deduce that (1) holds for E as well. If E does not contain l_1^n-s uniformly and uniformly complementably E' does not contain l_∞^n-s uniformly and follows as in [16] the existence of $d > 0$ and $1 < p \leqq 2$ *s.t.* for every G and $A: G \to E$ $\pi_1'(A) \leqq d\pi_p'(A)$. Therefore there exists $k > 0$ $2 \leqq q < \infty$, $1 < p \leqq 2$ *s.t* for every G and A as above

$$\pi_q(A) \leqq k\pi_p'(A) . \tag{2}$$

By Lemma 3.10 (2) is true for every Banach space which is finitely represented in E. Now, let G be a Banach space and $T: G \to E$ a strongly integral operator. Then T has a factorization

$$\begin{tikzcd} G \arrow[r, "T"] \arrow[d, "B"'] & E \\ L_\infty(\Omega, \mu) \arrow[r, "j"] & L_1(\Omega, \mu) \arrow[u, "A"'] \end{tikzcd} \tag{3}$$

with (Ω, μ) a probability space and j the formal "inclusion" map.

We look at the factorization

$$\begin{tikzcd} L_\infty(\Omega, \mu) \arrow[rr, "j"] \arrow[dr, "i_1"'] & & L_1(\Omega, \mu) \\ & L_{p'}(\Omega, \mu) \arrow[ur, "i_2"'] & \end{tikzcd} \tag{4}$$

where $1/p + 1/p' = 1$ and i_1, i_2 are the formal "inclusion" maps. Then $Ai_2 \in \pi'_p(L_{p'}(\mu), E)$ and from (2) follows $Ai_2 \in \pi_q(L_{p'}(\mu), E)$, a known result of Persson and Pietsch [14] combined with the fact that i_1B is strongly p' integral then shows that

$$T = Ai_2i_1B \in N_r(G, E) \quad \text{with} \quad \frac{1}{r} = \frac{1}{p'} + \frac{1}{q}.$$

Since the same is true for every Banach space finitely represented in E, E has super $(I - N_r)$ and of course it has super $(I - K)$. Lemma 3.11 then shows that E does not contain l_1^n-s uniformly.

REMARK. We do not know if the property super $(I - N_r)$ is in fact strictly stronger than "not containing l_1^n-'s uniformly".

COROLLARY 3.12. *Let E be a Banach space which either has the property G. L. or is finitely represented in a Banach space F s.t. F has property G. L. and does not contain l_∞^n-'s uniformly. Then for any Banach space G there exist compact nonnuclear operators from E into G and from G into E.*

Proof. From Theorem 3.9 it follows that in both cases one of the three possibilities hold: (a) E contains l_∞^n-s uniformly.

(b) E contains l_1^n-s uniformly and uniformly completably.

(c) E does not contain l_1^n-s uniformly.

In each of these cases the result follows, in (a) or (b) from results of [9] and in (c) from the result of [2].

Let E be a Banach space. We say that E has Grothendieck property ($G.$ $P.$) if $\Pi_2(\mathscr{L}_\infty, E) = \mathscr{L}(\mathscr{L}_\infty, E)$ (see [4] for discussion of this property). Maurey [12] showed that if E has cotype-2 then E has $G.$ $P.$, Pelczynski [13] shows that the inverse implication is true if E has *l.u.st.* We can generalize:

THEOREM 3.13. *Let E be a Banach space having the property G. L. Then*

(a) *E has G. P. if and only if E is of cotype-2.*

(b) *E' has G. P. and E' does not contain l_1^n-s uniformly if and only if E is of type 2.*

Proof. In both assertions only the "only-if" parts are new and will be proved.

By Corollary 3.7 we know that E' also has the G. L. property.

(a) Suppose E has $G.$ $P.$ As in [16] the fact that $\mathscr{L}(\mathscr{L}_\infty, E) = \Pi_2(\mathscr{L}_\infty, E)$ combined with the G. L. property of E' shows that there exists $c > 0$ *s.t.* Any E-valued operator A satisfies

(1) $$\pi_2(A) \leqq c\pi'_1(A) .$$

By [16] (1) is equivalent to the following condition:

(2) Let S be a subspace of an $L_1(\mu)$ space and $\omega: S \to L_2(\nu)$ a bounded operator. Then $\omega \otimes I_E$ (I_E — the identity operator of E) can be extended to a bounded operator $S \hat{\otimes}_{\Delta_1} E \to L_2(F)$ (for a subspace S of $L_p(\mu)$, Δ_p denotes the norm on $S \otimes E$ as a subspace of $L_p(\mu, E)$: of course $L_p(\mu) \hat{\otimes}_{\Delta_p} E = L_p(\mu, E)$).

We choose S to be the closed linear span in $L_1[0, 1]$ of the Rademacher functions $\{r_n\}$. ($r_n(t) = \operatorname{sign} 2^n \pi t$; $n = 0, 1, \cdots$.) It is known that S is isomorphic to l_2. Let ω be the isomorphism from S to l_2:

$$\omega(\sum b_n r_n) = (b_n)_{n \in N} .$$

From (2) it follows that

$$\omega \otimes I_E: S \hat{\otimes}_1 E \longrightarrow l_2$$

is bounded. Therefore, for $x_1, \cdots, x_n \in E$ we have:

$$\begin{aligned} \Big(\sum_{j=1}^n \|x_j\|^2\Big)^{1/2} &= \Big\|(\omega \otimes I_E)\Big(\sum_{j=1}^n r_j \otimes x_j\Big)\Big\|_{l_2(E)} \\ &\leqq \|\omega \otimes I_E\| \Big\|\sum_{j=1}^n r_j \otimes x_j\Big\|_{L_1([0,1],E)} \\ &= \|\omega \otimes I_E\| \int_0^1 \Big\|\sum_{j=1}^n r_j \otimes x_j\Big\| dt \end{aligned}$$

therefore E is of cotype 2.

(b) Let E' have G. P. and suppose E' does not contain l_1^n-'s uniformly. Then E does not contain l_∞^n-'s uniformly and Pisier's method ([16]) yields the existence of $C > 0$ and $1 < p \leqq 2$ *s.t.* Any E'-valued operator A satisfies

(3) $$\pi_2(A) \leqq C\pi'_p(A) .$$

(3) is equivalent to

(4) Let ω be a bounded operator $\omega: L_p(\mu) \to L_2(\nu)$, then $\omega \otimes I_{E'}$ is extendable to a bounded operator $\omega \otimes I_{E'}: L_p(\mu, E') \to L_2(\nu, E')$. For such a ω we get therefore that

$$(\omega \otimes I_{E'})': [L_2(\nu, E')]' \longrightarrow [L_p(\mu, E')]'$$

is bounded.

It is easy to check (identifying $L_2(\nu, E'')$ and $L_{p'}(\mu, E'')$ with subspace of $[L_2(\nu, E')]'$ and $[L_p(\mu, E')]'$) that

$$(\omega \otimes I_{E'})'(L_2(\nu, E'')) \subset L_{p'}(\mu, E'')$$

and

$$(\omega \otimes I_{E'})' = \omega' \otimes I_{E''}$$

considered as operators $L_2(\nu, E'') \to L_{p'}(\mu, E'')$.

Therefore $\omega' \otimes I_{E''}$ is well defined and bounded. Now, take $L_2(\nu) = l_2$, $L_p(\mu) = L_p[0, 1]$ and $\omega\colon L_p[0, 1] \to l_2$ defined by

$$\omega(f) = (\langle f, r_n \rangle)_{n \in N} .$$

ω is bounded and $\omega'\colon l_2 \to L_{p'}[0, 1]$ is the embedding of l_2 in $L_{p'}[0, 1]$:

$$\omega'(g) = \sum g_j r_j \quad \text{for} \quad g = (g_j)_{j \in N} \in l_2 .$$

We get for $x_1, \cdots, x_n \in E$:

$$\begin{aligned}\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\|^{p'} dt\right)^{1/p'} &= \left\|\sum_{j=1}^n r_j \otimes x_j\right\|_{L_{p'}(E'')} \\ &= \left\|(\omega' \otimes I_{E''})\left(\sum_{j=1}^n e_j \otimes x_j\right)\right\| \leqq \|\omega' \otimes I_{E''}\| \left\|\sum_{j=1}^n e_j \otimes x_j\right\|_{l_2(E'')} \\ &= \|\omega' \otimes I_{E''}\| \left(\sum_{j=1}^n \|x_j\|^2\right)^{1/2}\end{aligned}$$

(e_j being the unit vectors in l_2). Therefore E is of type 2.

Some concluding remarks. The property G. L. as it is defined is in some sense an "external" property. It is interesting to find some "internal" geometric characterization of this property. Up to now we know of no example of Banach space having the G. L. property for which E'' is not isomorphic to a complemented subspace of a Banach lattice, though Remark 3.6 hints that the existence of such example is probable (a result of Lewis [11, Cor. 4.2], together with the fact that each subspace of l_1 has G. L. constant 1, shows that the two norms are not equal).

Another course of problems may arise with respect to properties of spaces having the G. L. property, e.g., how far properties of spaces having *l.u.st* or isomorphic to complemented subspaces of Banach lattices pass over to spaces having G. L. property. Also one can ask how one can use such properties to the solution of problems concerning general Banach spaces. For example with respect to the problem of compact-nonnuclear operators arises the problem: suppose E satisfies $\mathscr{L}(E, l_2) = \Pi_1(E, l_2)$, does this imply that E can be embedded in a space having G. L. property which does not contain l_n^∞-s uniformly?

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A TREE-LIKE TSIRELSON SPACE

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An example is given of a reflexive Banach space X such that $(X \oplus X \oplus \cdots \oplus X)_{l_1^n}$, $n = 1, 2, \cdots$, are uniformly isomorphic to X. Some related examples are also given.

1. Introduction. In [4] Lindenstrauss observed that a Banach space X such that $(X \oplus X \oplus \cdots \oplus X)_{l_1^n}$ is isometric to a subspace of X for every n must contain an isometric copy of l_1. This gives a very simple proof to the fact that there exists no separable reflexive Banach space which is isometrically universal for all the separable reflexive Banach spaces. Lindenstrauss asked whether the isomorphic version of this result is true; i.e., does the fact that X contains uniformly isomorphic images of $(X \oplus X \oplus \cdots \oplus X)_{l_1^n}$, $n = 1, 2, \cdots$, imply that X contains l_1 isomorphically? An affirmative answer would give an alternative proof to the nonexistence of an isomorphically universal space in the family of all separable reflexive spaces as well as in the family of all spaces with a separable dual. (The nonexistence of these spaces was proved by W. Szlenk [8] by a completely different method.) Unfortunately the answer to Lindenstrauss' question is negative in a very strong sense.

THEOREM. *Let $1 \leqq p \leqq \infty$ and $\lambda > 1$. There exists a Banach space X with a 1-unconditional basis $\{e_i\}_{i=1}^{\infty}$ with the following properties:*

(a) *X is reflexive.*

(b) *X does not contain a subspace isomorphic to l_p (c_0 in the case $p = \infty$).*

For every $n = 1, 2, \cdots$ there exist n disjoint subsequences of the natural numbers $N_1, N_2, \cdots, N_n$ such that

(c) *$\{e_i\}_{i \in N_j}$; is isometrically equivalent to $\{e_i\}_{i=1}^{\infty}$, and*

(d) *If $x_j \in [e_i]_{i \in N_j}$; $j = 1, 2, \cdots, n$ then*

$$\lambda^{-1}\Big(\sum_{j=1}^{n} ||x_j||^p\Big)^{1/p} \leqq \Big|\Big|\sum_{j=1}^{n} x_j\Big|\Big| \leqq \lambda\Big(\sum_{j=1}^{n} ||x_j||^p\Big)^{1/p}$$
$$\Big(\lambda^{-1} \max_{1 \leqq j \leqq n} ||x_j|| \leqq \Big|\Big|\sum_{j=1}^{n} x_j\Big|\Big| \leqq \lambda \max_{1 \leqq j \leqq n} ||x_j|| \ \text{if } p = \infty\Big).$$

(e) *There exists a $K < \infty$ such that X is K-isomorphic to $(X \oplus X \oplus \cdots \oplus X)_{l_p^n}$ for every n.*

The construction uses ideas from [9] and [1] as well as the basic idea of James to construct Banach spaces on trees. The notations are standard and can be found in [5] or [6].

Proof of the theorem. We first deal with the case $p = \infty$. Let $(T, \leqq)$ be the set

$$T = \{(n, i);\, n = 0, 1, \cdots, i = 1, \cdots, 2^n\}\,.$$

With the partial order

$$(n, i) \leqq (m, j) \text{ if and only if } n \leqq m \text{ and } (i-1)2^{m-n} < j \leqq i2^{m-n}\,.$$

Let L be the linear space of all the functions on T which differ from zero only on a finite number of points of T. For $n = 0, 1, \cdots$ and $i = 1, \cdots, 2^n$ define $e_{n,i} \in L$ by

$$e_{n,i}(m, j) = \begin{cases} 1 & (n, i) = (m, j) \\ 0 & \text{otherwise}\,. \end{cases}$$

And define the operators $P_{n,i}$, $S_{n,i}$, and P_n from L to L by

$$(P_{n,i}x)(m, j) = \begin{cases} x(m, j) & (n, i) \leqq (m, j)\,, \quad x \in L \\ 0 & \text{otherwise} \end{cases}$$

$$(S_{n,i}x)(m, j) = x(m + n, (i-1)2^m + j)\,, \quad x \in L$$

and

$$P_n = \sum_{i=1}^{2^n} P_{n,i}\,.$$

Now, we define on L a sequence of norms $\|\cdot\|_n$ by induction

$$\|x\|_0 = \|x\|_{l_1} = \sum_{n,i} |x(n, i)|$$

$$\|x\|_m = \inf\left\{\|x_0\|_{m-1} + \lambda \sum_{k=1}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,i}x_k\|_{m-1}\right\}$$

where the inf is taken over all finite sequence $x_0, \cdots, x_K$ in L which satisfy

$$\sum_{k=0}^{K} x_k = x \quad \text{and} \quad P_k x_k = x_k\,, \quad k = 0, \cdots, K\,.$$

It is easy to prove by induction that for every $x \in L$ and every m

$$\|x\|_{c_0} \leqq \|x\|_m \leqq \|x\|_{m-1}\,.$$

So that we can define

$$\|x\| = \lim_{m\to\infty} \|x\|_m\,.$$

$\|\cdot\|$ is a norm. Let Y_m be the completion of L with respect to $\|\cdot\|_m$ and let Y be the completion of L with respect to $\|\cdot\|$.

LEMMA 1. (a) $\{e_{n,i}\}_{n=0,i=1}^{\infty, 2^n}$ *is a 1-unconditional basis for* Y_m *and for* Y.

(b) *If* R *is a norm one projection on* $l_1(T)$ *such that* $P_{k,i}R = RP_{k,i}$, *for all* $k = 0, 1, \cdots$ *and* $i = 1, \cdots, 2^k$, *then* R *is a norm one projection on* Y_m *and on* Y.

(c) $S_{n,j}$ *is an isometry from* $P_{n,j}Y_m$ (resp. $P_{n,j}Y$) *onto* Y_m(resp. Y) *for all* $n = 0, 1, \cdots$, $j = 1, \cdots, 2^n$.

(d) *For every* $x \in L$ *the infimum in the definition of* $\|x\|_m$ *is attained.*

(e) *For every* $x \in L$

$$\|x\| = \min\left\{\|x_0\|_{l_1} + \lambda \sum_{k=1}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,i}x_k\|; x = \sum_{k=0}^{K} x_k,\ P_k x_k = x_k\right\}.$$

Proof. (a) and (b) are proven by induction and passing to the limit. (d) is a simple consequence of (b) (for $R = I - P_n$). We prove now (e). For every $\{x_k\}_{k=0}^{K}$ such that $x = \sum_{k=0}^{K} x_k$ and $P_k x_k = x_k$, $k = 0, \cdots, K$ and for all m

$$\begin{aligned}\|x\| \leqq \|x\|_m &\leqq \|x_0\|_{m-1} + \lambda \sum_{k=1}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,,i}x_k\|_{m-1} \\ &\leqq \|x_0\|_{l_1} + \lambda \sum_{k=1}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,i}x_k\|_{m-1}.\end{aligned}$$

So, passing to the limit and using (b) to prove that the infimum is attained, we get

$$\|x\| \leqq \min\left\{\|x\|_{l_1} + \lambda \sum_{k=1}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,i}x_k\|; x = \sum_{k=0}^{K} x_k,\ P_k x_k = x_k\right\}.$$

In order to prove the other side inequality it is enough to prove that for all m and all $x \in L$

$$\|x\|_m \geqq \min\left\{\|x_0\|_{l_1} + \lambda \sum_{k=1}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,i}x_k\|; x = \sum_{k=0}^{K} x_k,\ P_k x_k = x_k\right\}.$$

We prove this by induction on m. This is obvious for $m = 0$, assume it is true for $m - 1$ and assume that

$$\|x\|_m = \|x_0\|_{m-1} + \lambda \sum_{k=1}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,i}x_k\|_{m-1}$$

where $x = \sum_{k=0}^{K} x_k$ and $P_k x_k = x_k$, $k = 0, \cdots, K$.

By the induction hypothesis

$$\|x_0\|_{m-1} \geqq \|y_0\|_{l_1} + \lambda \sum_{h=1}^{H} \max_{1 \leqq i \leqq 2^h} \|P_{h,i}y_h\|$$

for some $\{y_h\}_{h=0}^H$ such that $x_0 = \sum_{h=0}^H y_h$ and $P_h y_h = y_h$, $h = 0, \cdots, H$. We assume as we may that $H = K$, then $x = y_0 + \sum_{k=1}^K (x_k + y_k)$, $P_k(x_k + y_k) = x_k + y_k$, $k = 1, \cdots, K$ and

$$||x||_m \geqq ||y_0||_{l_1} + \lambda \sum_{k=1}^K \max_{1 \leqq i \leqq 2^k} ||P_{k,i}(x_k + y_k)|| .$$

To prove (c) it is clearly enough to show that for every x such that $P_{n,j}x = x$ and for every m

$$||x||_m = \min \left\{ ||x_n||_{m-1} + \lambda \sum_{k=n+1}^K \max_{1 \leqq i \leqq 2^k} ||P_{k,i}x_k||_{m-1} \right\}$$

where the minimum is over all the sequences $\{x_k\}_{k=n}^K$ such that $x = \sum_{k=n}^K x_k$ and $P_{n,j}P_k x_k = x_k$, $k = n, n+1, \cdots, K$.

Let x satisfy $P_{n,j}x = x$ and let $\{y_h\}_{h=0}^H$ be such that

$$||x||_m = ||y_0||_{m-1} + \lambda \sum_{h=1}^H \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h||_{m-1} ,$$

$$x = \sum_{h=0}^H y_h \quad \text{and} \quad P_h y_h = y_h , \quad h = 0, \cdots, H .$$

We can assume that $H > n$ and by (a), we can also assume that $P_{n,j}y_h = y_h$, $h = 0, \cdots, H$.

$$\begin{aligned} ||x||_m &= ||y_0||_{m-1} + \lambda \sum_{h=1}^n \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h||_{m-1} + \lambda \sum_{h=n+1}^H \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h|| \\ &= ||y_0||_{m-1} + \lambda \sum_{h=1}^n ||y_h||_{m-1} + \lambda \sum_{h=n+1}^H \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h|| . \end{aligned}$$

If $\sum_{h=1}^n ||y_h||_{m-1} > 0$ then since $\lambda > 1$

$$||x||_m > ||y_0 + y_1 + \cdots + y_n||_{m-1} + \lambda \sum_{h=n+1}^H \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h||$$

in contradiction to the fact that the minimum is attained at $y_0, \cdots, y_H$. This concludes the proof of Lemma 1.

PROPOSITION 2. (a) *For every* $n = 0, 1, \cdots$ *and* $\{y_i\}_{i=1}^{2^n}$ *such that* $P_{n,i}y_i = y_i$, $i = 1, \cdots, 2^n$,

$$\max_{1 \leqq i \leqq 2^n} ||y_i|| \leqq \left|\left| \sum_{i=1}^{2^n} y_i \right|\right| \leqq \lambda \max_{1 \leqq i \leqq 2^n} ||y_i|| .$$

(b) *Y does not contain an isomorphic image of* c_0.

Proof. (a) The left hand side follows from the 1-unconditionality of $\{e_{n,i}\}_{n=0,}^{\infty}{}_{i=1}^{2^n}$. For the right hand side put

$$x_n = \sum_{i=1}^{2^n} y_i \quad \text{and} \quad x_k = 0 \quad \text{for} \quad k \neq n$$

then, by Lemma 1.e,

$$\left\|\sum_{i=1}^{2^n} y_i\right\| \leqq \lambda \max_{1\leqq i\leqq 2^n} \|P_{n,i}x_n\| = \lambda \max_{1\leqq i\leqq 2^n} \|y_i\| .$$

(b) Assume that Y contains an isomorph of c_0. Since the unit vector basis of c_0 tends weakly to zero, we can assume that there exist a sequence $\{u_n\}_{n=1}^{\infty}$ of norm one elements in Y, an increasing sequence $\{m_n\}_{n=1}^{\infty}$ of positive integers and a constant K such that

$$(P_{m_n} - P_{m_{n+1}})u_n = u_n , \quad n = 1, 2, \cdots$$

and

$$\max_{1\leqq n<\infty} |a_n| \leqq \left\|\sum_{n=1}^{\infty} a_n u_n\right\| \leqq K \max_{1\leqq n<\infty} |a_n|$$

for every sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \to 0$ as $n \to \infty$. For every n let $1 \leqq i_n \leqq 2^{m_n}$ be such that

$$\|P_{m_n,i_n}u_n\| = \max_{1\leqq i\leqq 2^{m_n}} \|P_{m_n,i}u_n\|$$

and put

$$v_n = P_{m_n,i_n}u_n .$$

By part (a) and Lemma 1.a.

$$1 = \|u_n\| \leqq \lambda\|v_n\| \leqq \lambda\|u_n\| \leqq \lambda$$

and

$$\lambda^{-1} \max_{1\leqq n<\infty} |a_n| \leqq \left\|\sum_{n=1}^{\infty} a_n v_n\right\| \leqq \left\|\sum_{n=1}^{\infty} a_n u_n\right\| \leqq K \max_{1\leqq n<\infty} |a_n|$$

for every sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \to 0$ as $n \to \infty$. We also have $P_{m_n,i_n}v_n = v_n$ $n = 1, 2, \cdots$. By passing to a subsequence we can also assume that

$$P_{m_n,i_n}v_r = v_r \quad \text{for all} \quad r \geqq n .$$

This last property (with other m_n's) remains true for every block basis of the u_n's. Thus, by a theorem of James [3], we may assume that there exist an n, a $1 \leqq j \leqq 2^n$ and two normalized vectors w_1, w_2 in Y such that

$$(I - P_n)w_1 = w_1 , \quad P_{n,j}w_2 = w_2 \quad \text{and} \quad \|w_1 + w_2\| < \lambda - \varepsilon \quad \text{where}$$

$\varepsilon > 0$ satisfies $1 < \lambda - \varepsilon < 1 + \varepsilon/\lambda$. Let $\{x_k\}_{k=0}^{K}$ be such that $w_1 + w_2 = \sum_{k=0}^{K} x_k$, $P_k x_k = x_k$, $k = 0, \cdots, K$ and

$$(*) \qquad \|w_1 + w_2\| = \|x_0\|_{l_1} + \lambda \sum_{k=1}^{K} \max_{1\leqq i\leqq 2^k} \|P_{k,i}x_k\|$$

(such x_k's exist by Lemma 1.e). We can also assume that $K \geqq n$ and that $\operatorname{supp} x_k \subseteqq \operatorname{supp}(w_1 + w_2)$, $k = 0, \cdots, K$. We first prove that

$$(**) \qquad \left\|\sum_{k=1}^{n-1} P_n x_k\right\| \leqq \frac{\lambda - \varepsilon}{\lambda}.$$

If this were not true then, since $P_{n,j} P_n x_k = P_n x_k$ for $k = 0, \cdots, K$,

$$\begin{aligned} \lambda - \varepsilon > \|w_1 + w_2\| &\geqq \lambda \sum_{k=1}^{n-1} \max_{1 \leqq i \leqq 2^k} \|P_{k,i} P_n x_k\| \\ &= \lambda \sum_{k=1}^{n-1} \|P_n x_k\| \geqq \lambda \left\|\sum_{k=1}^{n-1} P_n x_k\right\| > \lambda - \varepsilon. \end{aligned}$$

From (**), we get that

$$(***) \qquad \left\|P_n x_0 + \sum_{k=n}^{K} x_k\right\| \geqq \frac{\varepsilon}{\lambda}.$$

Indeed,

$$\begin{aligned} \left\|P_n x_0 + \sum_{k=n}^{K} x_k\right\| &= \left\|P_n x_0 + \sum_{k=n}^{K} P_n x_k\right\| \\ &\geqq \left\|P_n\Big(\sum_{k=0}^{K} x_k\Big)\right\| - \left\|\sum_{k=1}^{n-1} P_n x_k\right\| \\ &= \|w_2\| - \left\|\sum_{k=1}^{n-1} P_n x_k\right\| \geqq 1 - \frac{\lambda - \varepsilon}{\lambda} = \frac{\varepsilon}{\lambda}. \end{aligned}$$

Now, by Lemma 1.e, the equalities

$$w_1 = \sum_{k=0}^{n-1} (I - P_n) x_k, \quad P_k(I - P_n) x_k = (I - P_n) x_k, \quad k = 0, \cdots, n-1$$

and

$$P_n x_0 + \sum_{k=n}^{K} x_k = P_n x_0 + \sum_{k=n}^{K} x_k, \quad P_k x_k = x_k, \quad k = 0, n, n+1, \cdots, K,$$

(*) and (***) we get

$$\begin{aligned} \lambda - \varepsilon > \|w_1 + w_2\| \geqq \|(I - P_n)x_0\|_{l_1} &+ \lambda \sum_{k=1}^{n-1} \max_{1 \leqq i \leqq 2^k} \|P_{k,i}(I - P_n)x_k\| \\ &+ \|P_n x_0\| + \lambda \sum_{k=n}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,i} x_k\| \\ &\geqq \|w_1\| + \left\|P_n x_0 + \sum_{k=n}^{K} x_k\right\| \geqq 1 + \frac{\varepsilon}{\lambda} \end{aligned}$$

which contradicts the choice of ε. This concludes the proof of Proposition 2.

The space Y satisfies (b), (c) and (d) of the theorem for $p = \infty$ this follows from 2.b, 1.c and 2.a, respectively it is also not hard to see that Y satisfies (e), however (a) is not satisfied, indeed, if $\{(n_k, i_k)\}_{k=1}^{\infty}$ is a totally ordered sequence in T then it is not difficult to see (using 1.e.) that $[e_{n_k,i_k}]_{k=1}^{\infty}$ is isometric to l_1, so some additional work is needed.

Proof of theorem for $p = \infty$. Define on L a new norm by

$$|||x||| = \| |x|^2 \|^{1/2} \quad x \in L$$

(for $x = \sum_{n,i} a_{n,i}e_{n,i}$ $|x|^\alpha$ is defined to be $\sum_{n,i} |a_{n,i}|^\alpha e_{n,i}$), and let X be the completion of L with respect to this norm. It is easy to check that $\{e_{n,i}\}_{n=0,i=1}^{\infty,\ 2^n}$ constitutes a 1-unconditional basis for X. Now, if $\{x_m\}_{m=1}^{M}$ is a block basis of $\{e_{n,i}\}_{n=0,i=1}^{\infty,\ 2^n}$ then

$$a \max_{1\leq m\leq M} |a_m| \leq \left\|\sum_{m=1}^{M} a_m x_m\right\| \leq b \max_{1\leq m\leq M} |a_m| \quad \text{for all} \quad a_1, \cdots, a_M$$

if and only if

$$a^{1/2} \max_{1\leq m\leq M} |a_m| \leq \left|\left|\left|\sum_{m=1}^{M} a_m |x_m|^{1/2}\right|\right|\right| \leq b^{1/2} \max_{1\leq m\leq M} |a_m| \quad \text{for all} \quad a_1, \cdots, a_M\ .$$

This proves that (b), (c) and (d) of the Theorem remain valid for X (with $\lambda^{1/2}$ instead of λ). In order to prove (a) it is enough, by James theorem [2] to prove that X does contain an isomorph of ℓ_1. This in turn is a consequence of the following simple fact: if $\{x_m\}_{m=1}^{M}$ are disjointly supported with respect to $\{e_{n,i}\}_{n=0,i=1}^{\infty,\ 2^n}$ then

$$\left|\left|\left|\sum_{m=1}^{M} x_m\right|\right|\right| \leq \left(\sum_{m=1}^{M} |||x_m|||^2\right)^{1/2}.$$

To prove (e) it is enough, in view of (c), (d) and Pelczynski's decomposition method [7], to prove that X is isomorphic to $X \oplus X$. Now, as we mentioned above for any totally ordered sequence $\{(n_k, i_k)\}_{k=1}^{\infty}$ in T $\{e_{n_k,i_k}\}_{k=1}^{\infty}$ in Y is equivalent to the unit vector basis in l_1 thus, $\{e_{n_k,i_k}\}_{k=1}^{\infty}$ in X is equivalent to the unit vector basis in l_2. So, X contains a copy of ℓ_2 and therefore is isomorphic to each of its one co-dimensional subspaces. In particular to $[e_{n,i}]_{n=1,i=1}^{\infty,\ 2^n}$ which, in turn is isomorphic to $X \oplus X$.

Proof of the theorem for $1 \leq p < \infty$. Let X and $\{e_i\}_{i=1}^{\infty}$ be the space and the basis which satisfy the theorem for $p = \infty$ and let $\{f_i\}_{i=1}^{\infty}$ be the biorthogonal basis of $\{e_i\}_{i=1}^{\infty}$ then clearly X^* and $\{f_i\}_{i=1}^{\infty}$ satisfy the theorem for $p = 1$.

For $p > 1$ define, for every eventually zero sequence $\{a_i\}_{i=1}^{\infty}$,

$$\|\{a_i\}_{i=1}^{\infty}\|_p = \left\|\sum_{i=1}^{\infty} |a_i|^p f_i\right\|^{1/p}.$$

Considerations similar to those in the proof of the $p = \infty$ case show that the completion of the space of finite sequences under $||\cdot||_p$ satisfies the theorem.

REMARK. It may be useful to know what is the dual norm to $||\cdot||$. Define on L a sequence of norms as follows

$$|x|_0 = ||x||_{c_0}$$
$$|x|_m = \max\left\{|x|_{m-1},\ \lambda^{-1} \max_{1\leq k<\infty} \sum_{i=1}^{2^n} |P_{k,i}x|_{m-1}\right\}$$

and define

$$|x| = \lim_{m\to\infty} |x|_m .$$

It can be shown that for every $x \in L$

$$|x| = \max\left\{||x||_{c_0},\ \lambda^{-1} \max_{1\leq k<\infty} \sum_{i=1}^{2^k} |P_{k,i}x|\right\}$$

and that $\{[e_{n,i}]_{n=0,i=1}^{N\ \ 2^n}, |\cdot|\}$ is the dual of $\{[e_{n,i}]_{n=0,i=1}^{N\ \ 2^n}, ||\cdot||\}$.

Once this duality is proved it can be used to simplify the proof of the theorem, in particular the proof of Proposition 2.b. We prefered, however, to give a proof which avoids the routine proof of the duality.

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FIX-FINITE HOMOTOPIES

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A well-known result by H. Hopf states that every selfmap f of a polyhedron $|K|$ can be deformed into a selfmap f' which has only a finite number of fixed points and is arbitrarily close to the given one. In addition one can locate all fixed points of f' in maximal simplexes. A map which has a finite fixed point set is here called a fix-finite map, and a homotopy $F: |K| \times I \to |K|$ is called a fix-finite homotopy if the map $f_t = F(\cdot, t)$ is fix-finite for every $t \in I$. We extend Hopf's result to homotopies, and show that two homotopic selfmaps f_0 and f_1 of a polyhedron $|K|$ which are fix-finite and have all their fixed points located in maximal simplexes can be related by a homotopy which is fix-finite and arbitrarily close to the given one. All fixed points of F can again be located in as high-dimensional simplexes as possible. Some simple properties are derived from the fact that the fix-finite homotopy is constructed in such a way that its fixed point set is a one-dimensional polyhedron in $|K| \times I$.

A. Introduction. In 1929 H. Hopf [2], Satz V, proved a well-known theorem which states that every selfmap f of a polyhedron can be deformed into a selfmap f' which is arbitrarily close to f and has only a finite number of fixed points. The construction of f' can be carried out so that all fixed points of f' are, in Hopf's terminology, "regular", i.e., they are located in maximal simplexes. We call a map which has only a finite number of fixed points a *fix-finite map*, and formulate Hopf's result accordingly.

THEOREM 1 (*Hopf*). *Let f be a selfmap of a polyhedron $|K|$. Given $\varepsilon > 0$, there exists a selfmap f' of $|K|$ such that*

(1) *f' is fix-finite,*

(2) *all fixed points of f' are contained in maximal simplexes of $|K|$,*

(3) *the distance $d(f, f') < \varepsilon$.*

We ask in this paper whether a similar result can be obtained for homotopies. We call a map $F: |K| \times I \to |K|$ (where I is the unit interval) a *fix-finite homotopy* if the map $f_t: |K| \to |K|$ defined by $f_t(x) = F(x, t)$ is a fix-finite map for every $t \in I$, and ask therefore whether two selfmaps f_0 and f_1 of a polyhedron $|K|$ which are fix-finite and homotopic can be related by a homotopy which is fix-finite and arbitrarily close to the given one. We shall show that this is possible if all fixed points of f_0 and f_1 are contained in maximal simplexes, and we shall construct the fix-finite homotopy so that its fixed points are again located as nicely as possible. They clearly cannot all be located in maximal simplexes of $|K|$, but they can be located in simplexes which are either maximal, or faces of maximal dimension. Let us make these notions precise.

We denote by $|K|$ a polyhedron which is the realization of a finite simplicial complex K, by σ an open simplex of K, by $\bar{\sigma}$ its closure, and by $\dim \sigma$ its dimension. $\sigma < \tau$ means that σ is a face of the simplex τ. The (open) *star* $\operatorname{st} \sigma$ of σ consists of all simplexes τ of $|K|$ with $\sigma < \tau$. A simplex σ is called *maximal* if $\sigma = \operatorname{st} \sigma$, and we call it a *hyperface* if $\dim \operatorname{st} \sigma = \dim \sigma + 1$. A *fixed point of a homotopy* $F\colon |K| \times I \to |K|$ is defined as a point $x \in |K|$ with $F(x, t) = x$ for some $t \in I$. If f, f' are maps and d is the metric of $|K|$, then the sup metric is given by

$$d(f, f') = \sup \{d(f(x), f'(x)) \,|\, x \in X\} .$$

We use this terminology to state our main result.

THEOREM 2. *Let F be a homotopy between two selfmaps f_0 and f_1 of a polyhedron $|K|$, let f_0 and f_1 be fix-finite, and let all their fixed points be contained in maximal simplexes. Given $\varepsilon > 0$, there exists a homotopy F' from f_0 to f_1 such that*

(1) *F' is fix-finite,*

(2) *all fixed points of F' are contained in maximal simplexes or hyperfaces of $|K|$,*

(3) *$d(F, F') < \varepsilon$.*

Special cases of Theorem 2 are known. Weier [6] constructed a fix-finite homotopy satisfying (1) and a condition related to (2) if $|K|$ is a 2-dimensional pseudomanifold satisfying a certain connectedness condition, and in [4], Satz III we constructed a fix-finite homotopy satisfying (1) and (3) if $|K|$ is an orientable and triangulable finite dimensional manifold without boundary.

The proof of Theorem 2 given below is related to Hopf's proof of Theorem 1. Hopf started with a simplicial approximation of the given map, and then carried out a succession of changes on simplexes of increasing dimension which freed the simplicial approximation of fixed points on all but maximal simplexes. The final result is a map which is again simplicial and satisfies Theorem 1. Hopf's proof is readily available in [1], pp. 117–118, where the successive changes are called "*Hopf constructions*".

In our proof of Theorem 2 a homotopy is altered successively on simplexes of increasing dimension by a "*Hopf construction for homotopies*" which is described in §B. As this construction can only be applied to simplicial homotopies, it is first necessary to approximate the given homotopy by a simplicial one. This leads to a proof of Theorem 2 in three steps. In the first, the given maps f_0 and f_1 are, with the help of the Hopf construction, approximated by fix-finite simplicial maps g_0 and g_1, and fix-finite homotopies H_i from f_i to g_i (where $i = 0, 1$) are obtained in a manner reminiscent of [4]. A homotopy between the simplicial maps g_0 and g_1 has a simplicial approximation relative to $|K| \times \{0\} \cup |K| \times \{1\}$, on which a succession of Hopf constructions for homotopies is carried out in Step 2, leading to a fix-finite homotopy G' from g_0 to g_1. Finally, in Step 3, the desired homotopy F' is obtained by constructing a homotopy from g_0 to g_1 as the composite of H_0^{-1}, F, and H_1, changing it to a homotopy G' as in Step 2, and forming the composite of H_0, G', and H_1^{-1}, where all compositions are made with suitable scale changes to ensure closeness between F and F'.

The homotopy F' is constructed in such a way that the set

$$\mathrm{Fix}\, F' = \{(x, t) \in |K| \times I \,|\, F'(x, t) = x\}$$

is a finite one-dimensional polyhedron. Some simple consequences of this fact are given in §D. One of them is the existence of an upper bound M so that the number of fixed points of f'_t is $\leqq M$ for every $t \in I$.

B. A Hopf construction for homotopies. Let G be the realization of a simplicial function $P \to K$, where P is a suitable complex with $|P| = |K| \times I$, and let τ be a given simplex of $|P|$. The Hopf construction for homotopies, which frees G of all fixed points on τ as long as $G(\tau)$ is not maximal in $|K|$, will be the basic tool in the second step of the proof of Theorem 2 and we shall embody its results in the rather technical Lemma 1 below. We write $G: |P| \to |K|$ to indicate that G is the realization of a simplicial function from P to K. The construction of K_L, the barycentric subdivision of K modulo the subcomplex L, can e.g. be found in [3], p. 49. If $L = \phi$, then it is the ordinary barycentric subdivision of K. A *refinement* of K is a complex obtained from K by means of a finite number of subdivisions modulo subcomplexes. $\mu(K)$ denotes the *mesh* of $|K|$, i.e., the maximum of the diameters of its simplexes.

LEMMA 1. *Let P be a complex with $|P| = |K| \times I$, let $G: |P| \to |K|$ be simplicial and $\pi: |P| \to |K|$ be the first projection. If τ is a simplex of $|P|$ for which $\pi(\tau)$ is contained in a simplex ρ of $|K'|$,* *where* K' *is a refinement of* K, *if* $\tau \cap \operatorname{Fix} G \neq \phi$ *where* $\operatorname{Fix} G = \{(x, t) \in |P| \mid G(x, t) = \pi(x, t)\}$, *and if* $G(\tau)$ *is not maximal in* $|K|$, *then there exists a simplicial map* $G'\colon |P_Q| \to |K|$, *with* $Q = P \backslash \operatorname{st} \tau$, *so that*

(1) $\tau \cap \operatorname{Fix} G' = \phi$,
(2) $G = G'$ *on* $|Q|$,
(3) $d(G, G') \leqq 2\mu(K)$.

Proof. Let ρ^* be a maximal simplex of K' with $\rho < \rho^*$, and σ^* be a maximal simplex of K with $\rho^* \subset \sigma^*$. Then

$$\pi(\tau) \subset \rho \subset \bar{\rho}^* \subset \bar{\sigma}^* .$$

If $\sigma = G(\tau)$, then $\pi(\tau) \cap \sigma \neq \phi$ implies $\sigma < \sigma^*$.

Define $G\colon |P_Q| \to |K|$ on the vertices of P_Q as follows: If $v \in Q$, let $G'(v) = G(v)$. If $\tau_j \in \operatorname{st} \tau \backslash \tau$ and v is the vertex of P_Q contained in τ_j, let $G'(v)$ be any vertex of σ, and if v is the vertex of P_Q contained in τ, let $G'(v)$ be any vertex of σ^* which is not a vertex of σ. (As σ is not maximal, such a vertex exists.) It can be checked that G' extends to a simplicial map $G'\colon |P'_Q| \to |K|$. The proof that G' satisfies the conditions (1), (2), and (3) closely parallels arguments in [1], p. 117–118, and is omitted.

C. The proof.

Step 1. Construction of fix-finite simplicial maps g_i which are fix-finitely homotopic to the given maps f_i.

We begin with a simple lemma.

LEMMA 2. *Let* $|K|$ *be a connected polyhedron,* $x \in |K|$, *and the carrier* σ *of* x *in* $|K|$ *maximal. Given* $\delta > 0$, *there exists a* $y \in \sigma$ *with* $d(x, y) < \delta$ *whose carrier in any refinement of* K *is maximal.*

Proof. $|K|$ is connected, therefore σ is of dimension $p > 0$. As the number of refinements of $\bar{\sigma}$ is countable, the dimension of the union A of the $(p-1)$-skeletons of all refinements is $p-1$, and $y \in \sigma \backslash A$ with $d(x, y) < \delta$ exists and satisfies the lemma.

The result of Step 1 is given as the next lemma, where

$$\operatorname{diam} H = \sup \{d(H(x, t), H(x, t')) \mid x \in |K|, t, t' \in I\}$$

denotes the diameter of a homotopy $H\colon |K| \times I \to |K|$.

LEMMA 3. *Let* $f_i\colon |K| \to |K|$, $i = 0, 1$, *be two selfmaps of a polyhedron* $|K|$ *which are fix-finite and have all their fixed points located in maximal simplexes of* $|K|$. *Given* $\varepsilon > 0$, *there exist a* *refinement K' of K, refinements K_i'' of the first barycentric subdivision of K', simplicial maps $g_i\colon |K_i''| \to |K'|$, and homotopies H_i from f_i to g_i so that*

(1) *H_i is fix-finite and has all its fixed points located in the maximal simplexes of $|K|$,*

(2) *the fixed points of g_i are located in distinct maximal simplexes of $|K_i''|$,*

(3) $\operatorname{diam} H_i < \varepsilon/4$,

(4) $\mu(K') < \varepsilon/8(n+1)$, *where* $n = \dim |K|$.

Proof. We can assume that $|K|$ is connected, otherwise the construction is made on each component.

(i) We first construct two maps $f_i'\colon |K| \to |K|$ and homotopies H_i' from f_i to f_i' such that all carriers of fixed points of f_i' are maximal in every refinement of K, all carriers of fixed points of H_i' are maximal in $|K|$, and $\operatorname{diam} H_i' < \varepsilon/2$.

Consider f_0, and let $\operatorname{Fix} f_0 = \{c_j\}$ be its fixed point set. As f_0 is uniformly continuous, we can select β with $0 < \beta < \varepsilon/16$ so that, for all $c_j \in \operatorname{Fix} f_0$, the open β-balls $U(c_j, \beta)$ are pairwise disjoint and each $U(c_j, \beta)$ is contained in the carrier of c_j in $|K|$. Now select γ with $0 < \gamma < \beta/2$ such that $d(x, f_0(x)) < \beta/2$ for all $x \in \cup\{U(c_j, \gamma) \mid c_j \in \operatorname{Fix} f_0\}$. According to Lemma 2 each $U(c_j, \gamma)$ contains a point c_j' whose carrier in all refinements of $|K|$ is maximal. If $x \in \bar{U}(c_j, \gamma)\backslash\{c_j'\}$, let y be the point in which the ray from c_j' to x intersects the boundary $\operatorname{Bd} U(c_j, \gamma)$, and z the point on the segment from c_j to y for which

$$d(c_j, z) = \frac{d(c_j, y)}{d(c_j', y)} \cdot d(c_j', x)\ .$$

To define a map f_{0j}' from $\bar{U}(c_j, \gamma)$ to $U(c_j, \beta)$, denote by $\overrightarrow{ab}$ the (free) vector from a to b in $U(c_j, \beta)$, and determine $f_{0j}'(x)$ for $x \neq c_j'$ by

$$\overrightarrow{c_j' f_{0j}'(x)} = \overrightarrow{c_j' x} + \overrightarrow{z f_0(z)}\ ;$$

also let $f_{0j}' = c_j'$.

As we have for all $x \in \bar{U}(c_j, \gamma)$

$$\begin{aligned} d(f_{0j}'(x), c_j) &\leqq d(f_{0j}'(x), x) + d(x, c_j) \\ &= d(f_0(z), z) + d(x, c_j) < \beta/2 + \gamma < \beta\ , \end{aligned}$$

this construction is well defined.

Now define $f_0'\colon |K| \to |K|$ by

$$f_0'(x) = \begin{cases} f_{0j}'(x) & \text{if } \ x \in \cup\{U(c_j, \gamma) \mid c_j \in \operatorname{Fix} f_0\}\ , \\ f_0 & \text{otherwise}\ . \end{cases}$$

f_0' is continuous, and its fixed point set is $\operatorname{Fix} f_0' = \{c_j'\}$. Hence all carriers of its fixed points are maximal in every refinement of $|K|$.

If $f'_0(x) \neq f_0(x)$, then $x \in U(c_j, \gamma)$ for some $c_j \in \operatorname{Fix} f_0$. Denote, for $0 < t \leqq 1$, by $c_j(t)$ the point which divides the segment from c_j to c'_j in the ratio $t:(1-t)$, and define $H'_{0j}(x, t)$ as the point in $U(c_j, \beta)$ which is obtained in a manner analogous to $f'_{0j}(x)$ but with the use of $c_j(t)$ instead of c'_j. Also put $H'_{0j}(x, 0) = f_0(x)$. Then a homotopy H'_0 from f_0 to f'_0 can be constructed from the H'_{0j} in the same way in which f'_0 was constructed from the f'_{0j}. If $f'_0(x) = f_0(x)$, then H'_0 is the constant homotopy, if $f'_0(x) \neq f_0(x)$, then the set $\{H'_0(x, t) \mid 0 \leqq t \leqq 1\}$ lies in some $U(c_j, \beta)$. Hence $\operatorname{diam} H'_0 < 2\beta < \varepsilon/8$. The construction of H'_0 shows that all carriers of its fixed points are maximal in K.

The map f'_1 and the homotopy H'_1 from f_1 to f'_1 are obtained analogously.

(ii) We now describe the construction of the maps g_i and the homotopies H''_i from f'_i to g_i.

Choose ρ_0 with $0 < \rho_0 < \varepsilon/32$ so that for each $c'_j \in \operatorname{Fix} f'_0$ with carrier κ_j in $|K|$ the set $\bar{U}(c'_j, 4\rho_0) \subset \kappa_j$, and so that the $\bar{U}(c'_j, 4\rho_0)$ are pairwise distinct. As f'_0 is uniformly continuous, there exists a δ_0 with $0 < \delta_0 \leqq \rho_0$ so that

$$f'_0(\bar{U}(c'_j, \delta_0)) \subset \bar{U}(c'_j, \rho_0) \quad \text{for all} \quad c'_j \in \operatorname{Fix} f'_0 .$$

Furthermore choose η_0 with $0 < \eta_0 \leqq \rho_0$ so that

$$d(x, f'_0(x)) \geqq \eta_0 \quad \text{if} \quad d(x, \operatorname{Fix} f'_0) \geqq \delta_0 .$$

Determine ρ_1, δ_1, η_1 analogously for f'_1, and select a refinement K' of K so that $\mu(K') < \min\{\delta_0, \delta_1, \eta_0/(2n+1), \eta_1/(2n+1)\}$, where n is the dimension of K.

Let ψ_0 be a simplicial approximation of f'_0 which maps a refinement of the first barycentric subdivision of K' into K', and choose g_0 as a map which is obtained from $|\psi_0|$ by a succession of Hopf constructions in the same way in which f' is obtained from $|\psi|$ in the proof of Theorem 2 on p. 118 of [1]. Then g_0 is a simplicial map $|K''_0| \to |K'|$, where K''_0 again refines the first barycentric subdivision of K'. It is fix-finite, has all its fixed points located in distinct maximal simplexes of $|K''_0|$, and $d(|\psi_0|, g_0) \leqq 2n\mu(K')$. As $d(f'_0, |\psi_0|) \leqq \mu(K')$, we have $d(f'_0, g_0) \leqq (2n+1)\mu(K') < \eta$.

Next, let us construct a homotopy H''_0 from f'_0 to g_0. If $x \notin \cup\{U(c'_j, \delta_0) \mid c'_j \in \operatorname{Fix} f'_0\}$, then it follows from [1], p. 118 that $g_0(x) = |\psi_0|(x)$. As ψ_0 is a simplicial approximation of f'_0, it is possible to define $H''_0(x, t)$ by

$$H''_0(x, t) = tf'_0(x) + (1-t)g_0(x) .$$

From $d(x, f_0'(x)) \geqq \eta$ and $d(f_0', g_0) < \eta$ follows $H_0''(x, t) \neq x$ for all $0 \leqq t \leqq 1$.

Now consider one of the sets $\bar{U}(c_j', \delta_0)$ contained in a maximal simplex κ_j of $|K|$. H_0'' has already been defined on Bd $\bar{U}(c_j', \delta_0) \times I$ such that

$$d(c_j', H_0''(x, t)) \leqq d(c_j', f_0'(x)) + d(f_0'(x), g_0(x)) \leqq 2\rho_0 .$$

Let further $H_0''(x, 0) = f_0'(x)$ and $H_0''(x, 1) = g_0(x)$ for all $x \in \bar{U}(c_j', \delta_0)$.

Then H_0'' is defined on Bd $(\bar{U}(c_j', \delta_0) \times I)$, has values in $\bar{U}(c_j', 2\rho_0)$, and its fixed point set consists of $c_j' \times \{0\}$ and finitely many points in $U(c_j', \delta_0) \times \{1\}$. To extend H_0'' over all of $\bar{U}(c_j', \delta_0) \times I$, let $\tilde{c}_j = (c_j', 1/2)$, and determine for every point $\tilde{x} = (x, t) \in (\bar{U}(c_j', \delta_0) \times I)\backslash\{c_j\}$ the point $\tilde{y} = (y, s)$ as the one in which the ray from $\tilde{c}_j$ to $\tilde{x}$ intersects Bd $(\bar{U}(c_j', \delta_0) \times I)$. Let $\tilde{d}$ denote the product metric in $|K| \times I$, and define $H_0''(x, t)$ by

$$\overrightarrow{c_j'H_0''(x, t)} = \overrightarrow{c_j'x} + \lambda\overrightarrow{yH_0''(y, s)} ,$$

where

$$\lambda = \tilde{d}(\tilde{c}_j, \tilde{x})/\tilde{d}(\tilde{c}_j, \tilde{y}) .$$

As $d(c_j', x) \leqq \delta_0$, $0 < \lambda \leqq 1$, and $d(y, H_0''(y, s)) \leqq \delta_0 + 2\rho_0 \leqq 4\rho_0$, we obtain in this way a point $H_0''(x, t) \in \bar{U}(c_j', 4\rho_0)$. Finally, let $H_0''(c_j', 1/2) = c_j'$.

In this way H_0'' is extended over $\cup\{\bar{U}(c_j', \delta_0) \times I \mid c_j' \in \operatorname{Fix} f_0'\}$, yielding a homotopy $H_0'' \colon |K| \times I \to |K|$ from f_0' to g_0 which is fix-finite and has all its fixed points located in the maximal simples κ_j of $|K|$. If $x \in \cup\{\bar{U}(c_j', \delta_0) \mid c_j' \in \operatorname{Fix} f_0'\}$, then $\sup\{H_0''(x, t), H_0''(x, t') \mid t, t' \in I\} \leqq d(f_0', g_0) < \eta$, and if $x \in \bar{U}(c_j', \delta_0)$ for some $c_j' \in \operatorname{Fix} f_0'$, then $\{H_0''(x, t) \mid t \in I\} \subset \bar{U}(c_j', 4\rho_0)$, so $\sup\{H_0''(x, t), H_0''(x, t') \mid t, t' \in I\} \leqq 8\rho_0$. Hence $\operatorname{diam} H_0'' < \varepsilon/4$. The construction of $H_1'' \colon |K| \times I \to |K|$ is analogous.

(iii) Define finally a homotopy H_i from f_i to g_i by

$$H_i(x, t) = \begin{cases} H_i'(x, 2t) & \text{for } 0 \leqq t \leqq 1/2 , \\ H_i''(x, 2t - 1) & \text{for } 1/2 \leqq t \leqq 1 . \end{cases}$$

Then $\operatorname{diam} H_i \leqq \operatorname{diam} H_i' + \operatorname{diam} H_i'' < \varepsilon/4$, and H_0 and H_1 satisfy Lemma 3.

Step 2. Construction of a fix-finite homotopy between two fix-finite simplicial maps.

The aim of Step 2 is the construction of a fix-finite homotopy between the fix-finite and simplicial maps g_i of Lemma 3. It will be achieved with the help of a succession of Hopf constructions for homotopies. For this purpose, we need to realise $|K| \times I$ as a suitable simplicial complex P. If K', K_0'' and K_1'' are the complexes obtained in Lemma 3, then we require that P is a simplicial complex with $|P| = |K| \times I$ and satisfies the following two conditions:

(P1) $K_0'' \times \{0\}$ and $K_1'' \times \{1\}$ are subcomplexes of P,

(P2) if $\tau \in |P|$ is a simplex and $\pi: |P| \to |K|$ the first projection, then $\pi(\tau) \subset \rho$, where ρ is a simplex of K'.

P can easily be obtained by starting with the complex usually associated with the polyhedron $|K'| \times I$ and then refining it modulo the complements of the simplicial neighborhoods of those simplexes in $K' \times \{0\}$ and $K' \times \{1\}$ which are subdivided in K_0'' resp. K_1''.

We state one more technical detail as a lemma.

LEMMA 4. *Let P' be a refinement of P, let $G_s: |P'| \to |K'|$ be a simplicial map, and $\tau \in |P'|$ so that $\tau \cap \mathrm{Fix}\, G_s \neq \phi$. If τ is neither maximal nor a hyperface in $|P'|$, thn $G_s(\tau)$ is not maximal in $|K'|$.*

Proof. Let $G_s(\tau) = \sigma$, where σ is a simplex of $|K'|$, and $\pi(\tau) \subset \rho$, where $\rho \in |K'|$. As $\tau \cap \mathrm{Fix}\, G_s \neq \phi$ implies $\pi(\tau) \cap \sigma \neq \phi$, we have $\rho = \sigma$, and $\dim \rho \leqq \dim \tau$. By assumption there exists a simplex $\tau^* \in |P'|$ with $\tau < \tau^*$ and $\dim \tau \leqq \dim \tau^* - 2$, therefore

$$\dim \rho + 1 \leqq \dim \tau^* - 1 \leqq \dim \pi(\tau^*) ,$$

so $\pi(\tau^*) \not\subset \rho$. But $\pi(\tau) \subset \rho$ implies $\pi(\bar{\tau}^*) \cap \rho \neq \phi$, hence ρ cannot be maximal in $|K'|$. As $\rho = \sigma$, $G_s(\tau)$ cannot be maximal either.

The next lemma contains the result of Step 2.

LEMMA 5. *Let K', K_i'' and $g_i: |K_i''| \to |K'|$ be as in Lemma 3. If g_0 and g_1 are related by a homotopy G, then there exists a homotopy G' relating them such that*

(i) *G' is fix-finite and has all its fixed points located in maximal simplexes or hyperfaces of $|K|$,*

(ii) $d(G, G') < \varepsilon/4$.

Proof. Again we can assume that $|K|$ is connected. Let P satisfy (P1) and (P2). We first select as a simplicial approximation of G a simplicial map $G_s: |P'| \to |K'|$, where P' is a refinement of P obtained by a finite number of subdivisions modulo $(K_0'' \times \{0\}) \cup (K_1'' \times \{1\})$, so that G_s satisfies $G_s = G$ on $(|K_0''| \times \{0\}) \cup (|K_1''| \times \{1\})$ and $d(G, G_s) < \mu(K')$. The existence of G_s follows from [3], p. 55.

If $\tilde{x}_0 = (x_0, t_0)$ is a vertex of $|P'|$ with $G_s(x_0, t_0) = x_0$, then x_0 is a vertex of $|K'|$ and hence not maximal. Lemma 1 allows us to make a Hopf construction which results in a simplicial map G_s': $|P''| \to |K'|$, where P'' refines P', for which $G_s'(x_0, t_0) \neq x_0$ and $G_s' = G_s$ on $|P' \backslash \mathrm{st}\{\tilde{x}_0\}|$. Hence any vertex $\tilde{x} \in |P''| \cap \mathrm{Fix}\, G_s'$ must also be a vertex of $|P' \backslash \{\tilde{x}_0\}|$. We can therefore make further Hopf constructions for all such vertices until we arrive at a simplicial map, denoted again by G_s': $|P''| \to |K'|$, where P'' refines P', which is fixed point free on all vertices of $|P''|$. As G_s is fixed point free on the vertices of $(|K_0''| \times \{0\}) \cup (|K_1''| \times \{1\})$, we have $G_s' = G_s$ on this subcomplex.

Next we carry out a succession of Hopf constructions for all one-dimensional simplexes $\tau \in |P''|$ for which $\tau \cap \mathrm{Fix}\, G_s' \neq \phi$ and $G_s'(\tau)$ is not maximal in $|K'|$, then for all two-dimensional simplexes with the same property, and so on. According to (P2) and Lemmas 1 and 4 we can continue until we arrive at a simplicial map G_s': $|P''| \to |K'|$, which equals G_s on the subpolyhedron $(|K_0''| \times \{0\}) \cup (|K_1''| \times \{1\})$ of $|P''|$ and is fixed point free on all simplexes of $|P''|$ which are neither maximal nor hyperfaces.

If τ is a hyperface of $|P''|$ for which $\tau \cap \mathrm{Fix}\, G_s' \neq \phi$, then it follows (as in [1], pp. 118–119) from the fact that G_s' is linear on $\bar{\tau}$ and that $\mathrm{Bd}\, \tau \cap \mathrm{Fix}\, G_s' = \phi$ that G_s' has at most one fixed point on τ. Now consider a maximal simplex $\tau \in |P''|$ with $\tau \cap \mathrm{Fix}\, G_s' \neq \phi$. Then $\mathrm{Bd}\, \tau \cap \mathrm{Fix}\, G_s'$ is empty or a finite set $\{\tilde{x}_j\}$. Let $\tilde{x}_j = (x_j, t_j)$, and select $\tilde{x}_0 = (x_0, t_0) \in \tau$ so that $t_0 \neq t_j$ for all t_j. For any $\tilde{x} = (x, t) \in \bar{\tau} \backslash \{\tilde{x}_0\}$, let $\tilde{y} = (y, u)$ be the point in which the ray from $\tilde{x}_0$ to $\tilde{x}$ intersects $\mathrm{Bd}\, \tau$, and modify G_s' on $\bar{\tau}$ to G' by defining $G'(x, t)$ as the point in $\bar{\sigma} = G_s'(\bar{\tau})$ with

$$\overrightarrow{x_0 G'(x, t)} = \overrightarrow{x_0 x} + \lambda \overrightarrow{y G_s'(y, u)}\ , \quad \text{where} \quad \lambda = \tilde{d}(\tilde{x}_0, \tilde{x})/\tilde{d}(\tilde{x}_0, \tilde{y})\ .$$

As $\pi(\bar{\tau}) \subset \bar{\sigma}$ and $\bar{\sigma}$ is convex, this yields a point $G'(x, t) \in \bar{\sigma}$. Also let $G'(x_0, t_0) = x_0$. Then $\bar{\tau} \cap \mathrm{Fix}\, G'$ consists of the union of the segments from $\tilde{x}_0$ to all the $\tilde{x}_j$ if $\mathrm{Bd}\, \tau \cap \mathrm{Fix}\, G' \neq \phi$, and otherwise of the point $\tilde{x}_0$ alone. If we carry out this construction on all maximal simplexes of $|P''|$ on which G_s' has fixed points, we obtain a fix-finite homotopy G': $|P''| \to |K'|$, where P'' refines P' and hence P. By construction $G'(x, 0) = g_0(x)$ and $G'(x, 1) = g_1(x)$ for all $x \in |K|$. If $\tilde{x} = (x, t) \in \mathrm{Fix}\, G'$, then $\tilde{x}$ is contained in a maximal simplex or hyperface of $|P''|$ and hence of $|P|$. It follows from (P2) that x is contained in a maximal simplex or hyperface of $|K'|$ and hence of $|K|$.

Each point $\tilde{x} \in |P|$ is moved during the succession of Hopf constructions at most n times, where again n is the dimension of $|K|$, and by a distance of at most $2\mu(K')$ on each move. During the last change of G'_s to G' it is moved by a distance of at most $\mu(K')$. So we have

$$d(G_s, G') \leqq (2n + 1)\mu(K') ,$$

and hence, according to (4) of Lemma 3,

$$d(G, G') \leqq 2(n + 1)\mu(K') < \varepsilon/4 .$$

We see that G' satisfies Lemma 5.

Step 3. Construction of a fix-finite homotopy between the given maps.

It remains to paste the constructed homotopies together in a suitable way to find a homotopy F' satisfying Theorem 2. Given $F: |K| \times I \to |K|$ as in Theorem 2 and $\varepsilon > 0$, we can choose δ with $0 < \delta < 1$ so that $d(F(x, t), F(x, t')) < \varepsilon/4$ for all $x \in |K|$ and $t, t' \in I$ with $|t - t'| < \delta$. Use the homotopies H_0, H_1 obtained in Lemma 3 and define $F'': |K| \times I \to |K|$ as a homotopy which equals $H_0H_0^{-1}FH_1H_1^{-1}$ apart from a scale change by

$$F''(x, t) = \begin{cases} H_0(x, 2t/\delta) & \text{if } 0 \leqq t \leqq \delta/2 , \\ H_0(x, 2(1 - t/\delta)) & \text{if } \delta/2 \leqq t \leqq \delta , \\ F(x, (t - \delta)/(1 - 2\delta)) & \text{if } \delta \leqq t \leqq 1 - \delta , \\ H_1(x, \delta(t + \delta - 1)/2) & \text{if } 1 - \delta \leqq t \leqq 1 - \delta/2 , \\ H_1(x, \delta(1 - t)/2) & \text{if } 1 - \delta/2 \leqq t \leqq 1 . \end{cases}$$

Then $d(F, F'') < \varepsilon/2$.

The homotopy $G: |K| \times I \to |K|$ defined by $G(x, t) = F''(x, t(1 - \delta) + \delta/2)$ for all $(x, t) \in |K| \times I$ equals $H_0^{-1}FH_1$ apart from a scale change and is hence a homotopy from g_0 to g_1. Replace it by a homotopy G' according to Lemma 5, and define $F': |K| \times I \to |K|$ by

$$F'(x, t) = \begin{cases} H_0(x, 2t/\delta) & \text{if } 0 \leqq t \leqq \delta/2 , \\ G'(x, (t - \delta/2)/(1 - \delta)) & \text{if } \delta/2 \leqq t \leqq 1 - \delta/2 , \\ H_1(x, \delta(1 - t)/2) & \text{if } 1 - \delta/2 \leqq t \leqq 1 . \end{cases}$$

It is easy to check that F' is a homotopy satisfying Theorem 2.

D. Some properties of the fix-finite homotopy. The proof of Theorem 2 allows an easy description of $\operatorname{Fix} F'$.

PROPOSITION 1. *The homotopy F' in Theorem 2 can be chosen* *so that* $\operatorname{Fix} F'$ *is a one-dimensional finite polyhedron in* $|K| \times I$ *without horizontal edges.*

Here a horizontal edge means an edge contained in a section $|K| \times \{t\}$, for some $t \in I$. Note that $\operatorname{Fix} F'$, though constructed as a polyhedron, was not constructed as a subpolyhedron of $|P|$, and its projection $\pi(\operatorname{Fix} F')$ is not a subpolyhedron of $|K|$.

As $\operatorname{Fix} F'$ has a simple structure, it has simple properties. We collect a few. The first two are immediate consequences of the homotopy and additivity axioms of the fixed point index $i(f, x)$ of the selfmap f of a polyhedron at the isolated fixed point x.

PROPOSITION 2. *Let* e *be an edge of* $\operatorname{Fix} F'$. *Then the index of* f'_t *along* e *is constant,* i.e.,

$$i(f'_t, x) = i(f'_s, y) \quad \text{if} \quad (x, t) \in e \quad \text{and} \quad (y, s) \in e .$$

PROPOSITION 3. *Let* $v = (x, t)$ *be a vertex of* $\operatorname{Fix} F'$. *Then the index of* f_t *at* x *is the sum of the indices of fixed points chosen on all edges of* $\operatorname{Fix} F'$ *either leading towards* v *or away from* v, i.e.,

$$i(f'_t, x) = \sum_k i(f'_{t_k}, x_k) ,$$

where all (x_k, t_k) *lie on edges* $e_k \in \operatorname{st} v$, *with* e_k *distinct, and the sum taken over all edges in* $\operatorname{st} v \cap \{|K| \times [0, t)\}$ *(resp. in* $\operatorname{st} v \cap \{|K| \times (t, 1]\}$*).*

Finally we note that F' is "uniformly" fix-finite.

PROPOSITION 4. *There exists a positive integer* M *so that the number of fixed points of* f'_t *is* $\leqq M$ *for all* $t \in I$.

Proof. It suffices to choose M as the number of edges in $\operatorname{Fix} F'$, as no section $|K| \times \{t\}$ can intersect the closure of an edge of $\operatorname{Fix} F'$ more than once.

E. Conclusion. For a single selfmap f of a polyhedron $|K|$ the construction of a fix-finite map which is arbitrarily close to f and has all its fixed points contained in maximal simplexes is only a first step on the road to the construction of a map homotopic to f which has a minimal number of fixed points. It is, in fact, possible to obtain a map g homotopic to f which has exactly $N(f)$ fixed points, where $N(f)$ is the Nielsen number of f, as long as $|K|$ satisfies the Shi condition, which is a somewhat stronger connectedness condition. (See [5] or [1], p. 140.) Hence a similar question arises for homotopies.

Problem. If f_0 and f_1 are two selfmaps of a polyhedron $|K|$ which satisfies the Shi condition, if f_0 and f_1 are homotopic and have each exactly $N(f_0)$ fixed points, does there exist a homotopy F from f_0 to f_1 so that, for every $t \in I$, the map $f_t = F(\cdot, t)$ has exactly $N(f_0)$ fixed points?

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A GEOMETRIC INEQUALITY WITH APPLICATIONS TO LINEAR FORMS

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Let C_N be a cube of volume one centered at the origin in $\boldsymbol{R}^N$ and let P_K be a K-dimensional subspace of $\boldsymbol{R}^N$. We prove that $C_N \cap P_K$ has K-dimensional volume greater than or equal to one. As an application of this inequality we obtain a precise version of Minkowski's linear forms theorem. We also state a conjecture which would allow our method to be generalized.

1. **Introduction.** Let $C_N = [-1/2, 1/2]^N$ be the N-dimensional cube of volume one centered at the origin in $\boldsymbol{R}^N$ and suppose that P_K is a K-dimensional linear subspace of $\boldsymbol{R}^N$. Dr. Anton Good has conjectured that the K-dimensional volume of $P_K \cap C_N$ is always greater than or equal to one. In case $K = N-1$ this has recently been proved by Hensley [6], who also obtained upper bounds for this volume. Our purpose in this paper is to prove the conjecture for arbitrary K and to give some applications to Minkowski's theorem on linear forms. In fact we prove a more general inequality for the product of spheres of various dimensions which contains the conjecture as a special case.

We write $\bar{x}$ for the column vector $\begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$ in $\boldsymbol{R}^n$ and

$$|\bar{x}| = \left(\sum_{j=1}^{n} (x_j)^2\right)^{1/2}$$

for its length. We define the sphere S_n by

$$S_n = \{\bar{x} \in \boldsymbol{R}^n : |\bar{x}| \leqq \rho_n\}$$

where $\rho_n = \pi^{-1/2}\{\Gamma(n/2+1)\}^{1/n}$. It follows that $\mu_n(S_n) = 1$ where μ_n is Lebesgue measure on $\boldsymbol{R}^n$. Also we let $\chi_U(\bar{x})$ denote the characteristic function of a subset U in $\boldsymbol{R}^n$.

Our first main result is contained in the following theorem.

THEOREM 1. *Suppose that $n_1, n_2, \cdots, n_J$ are positive integers, $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_J}$ is in $\boldsymbol{R}^N$, $N = n_1 + n_2 + \cdots + n_J$, and A is a real $N \times K$ matrix,* $\operatorname{rank}(A) = K$. *Then*

$$|\det A^T A|^{-1/2} \leqq \int_{\boldsymbol{R}^K} \chi_{Q_N}(A\bar{x})\, d\mu_K(\bar{x}) , \tag{1.1}$$

where A^T is the transpose of A.

We note that if $\operatorname{rank}(A) < K$ then each side of (1.1) is infinite. From Theorem 1 we easily deduce a lower bound for $\mu_K(Q_N \cap P_K)$.

COROLLARY. *Let Q_N be as in Theorem 1 and let P_K be a K-dimensional subspace of $\boldsymbol{R}^N$. Then $\mu_K(Q_N \cap P_K) \geqq 1$.*

Proof. Choose A in Theorem 1 so that the columns of A form an orthonormal basis for P_K in $\boldsymbol{R}^N$. Then the left hand side of (1.1) is 1 while the right hand side is $\mu_K(Q_N \cap P_K)$.

The corollary clearly contains Good's conjecture since $Q_N = C_N$ if $n_j = 1$ and $J = N$.

Next we suppose that $L_j(\bar{x})$, $j = 1, 2, \cdots, N$ are N linear forms in K variables,

$$L_j(\bar{x}) = \sum_{k=1}^{K} a_{jk} x_k \,,$$

so that $A = (a_{jk})$ is an $N \times K$ matrix. We assume that the forms L_j are real for $j = 1, 2, \cdots, r$ and that the remaining forms consist of s pairs of complex conjugate forms arranged so that $L_{r+2j-1} = \bar{L}_{r+2j}$ for $j = 1, 2, \cdots, s$. Thus $N = r + 2s$. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N$ be positive with $\varepsilon_{r+2j-1} = \varepsilon_{r+2j}$ for $j = 1, 2, \cdots, s$. We define the $N \times N$ diagonal matrix E by $E = (c_j \delta_{jk})$ where $c_j = \varepsilon_j^{-1}$ if $j = 1, 2, \cdots, r$, $c_j = (2/\pi)^{1/2} \varepsilon_j^{-1}$ if $j = r + 1, r + 2, \cdots, N$ and δ_{jk} is the Kronecker delta. Theorem 1 allows us to prove the following precise version of Minkowski's classical result on linear forms.

THEOREM 2. *Let M be a positive integer and suppose that*

$$M |\det A^* E^2 A|^{1/2} \leqq 1 \,, \tag{1.2}$$

where A^* is the complex conjugate transpose of the matrix A. Then there exist at least M distinct pairs of nonzero lattice points $\pm \bar{v}_m$, $m = 1, 2, \cdots, M$, such that

$$|L_j(\pm \bar{v}_m)| \leqq \varepsilon_j \tag{1.3}$$

for each j and each m. In particular if $|\det A^* A| > 0$ then there exists a pair of nonzero lattice points $\pm \bar{v}$ such that

$$|L_j(\pm \bar{v})| \leqq |\det A^* A|^{1/2K} \tag{1.4}$$

for $j = 1, 2, \cdots, r$, and

$$|L_j(\pm \bar{v}) \leqq \left(\frac{2}{\pi}\right)^{1/2} |\det A^* A|^{1/2K} \tag{1.5}$$

for $j = r + 1, r + 2, \cdots, N$.

Theorem 2 was first proved in the case $N \leqq K$ and $M = 1$ by Minkowski [8, p. 104]. Subsequently the extension of Minkowski's convex body theorem by van der Corput [5] allowed Theorem 2 to be proved for $N \leqq K$ and arbitrary M. Of course if $N = K$ then (1.2) becomes the more familiar condition

$$M\left(\frac{2}{\pi}\right)^s |\det A| \leqq \varepsilon_1\varepsilon_2 \cdots \varepsilon_N ,$$

and if $N < K$ then (1.2) is trivially satisfied since the left hand side is zero. The novelty in our result is that Theorem 2 now holds for $1 \leqq K < N$. Previously in the case $1 \leqq K < N$ we knew only that (1.3) held if

$$2^K M \leqq \mu_K(\{\bar{x} \in \boldsymbol{R}^K \colon |L_j(\bar{x})| \leqq \varepsilon_j,\ j = 1, 2, \cdots, N\}) . \tag{1.6}$$

We prove Theorem 2 by showing that the right hand side of (1.6) is bounded from below by $2^K |\det A^* E^2 A|^{-1/2}$. As will be clear from the proof, Theorem 2 could be generalized to include linear forms with values in $\boldsymbol{R}^n$ for various n.

In §5 we state a conjecture which would allow us to obtain a significant improvement in Theorem 1. Specifically, we deduce from this conjecture an analogue of Theorem 1 in which Q_N is replaced by an arbitrary closed, convex, symmetric subset of $\boldsymbol{R}^N$ having N-dimensional volume equal to one.

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2. Preliminary results. In this section we briefly summerize some facts about logarithmically concave measures and functions. A more detailed discription can be found in the papers of Kanter [7] and Prékopa [9].

A function $f\colon \boldsymbol{R}^n \to [0, \infty)$ is said to be *log-concave* if for every pair of vectors $\bar{x}_1, \bar{x}_2$ in $\boldsymbol{R}^n$ and every λ, $0 < \lambda < 1$, we have

$$f(\lambda\bar{x}_1 + (1 - \lambda)\bar{x}_2) \geqq (f(\bar{x}_1))^\lambda (f(\bar{x}_2))^{1-\lambda} .$$

A probability measure ν defined on the measurable subsets of $\boldsymbol{R}^n$ is *log-concave* if for every pair of open convex sets U_1 and U_2 in $\boldsymbol{R}^n$ and every λ, $0 < \lambda < 1$, we have

$$\nu(\lambda U_1 + (1 - \lambda) U_2) \geqq (\nu(U_1))^\lambda (\nu(U_2))^{1-\lambda} , \tag{2.1}$$

where $+$ on the left hand side of (2.1) indicates Minkowski addition of sets. Clearly (2.1) holds for all open convex sets U_1 and U_2 if and only if it holds for all closed convex sets U_1 and U_2. The relationship between log-concave measures and log-concave functions is contained in the following lemma.

LEMMA 3. *Let ν be a log-concave probability measure on $\boldsymbol{R}^n$ and suppose that the support of ν spans the k-dimensional subspace P_k in $\boldsymbol{R}^n$. Then there is a log-concave probability density function f defined on P_k such that $d\nu = fd\mu_k$, where μ_k is k-dimensional Lebesgue measure on P_k. Conversely for any log-concave probability density function f defined on a k-dimensional subspace P_k in $\boldsymbol{R}^n$, the probability measure defined by $d\nu = fd\mu_k$ is log-concave, where μ_k is Lebesgue measure on P_k.*

The first part of Lemma 3 is a result of Borell [2, p. 123] while the converse was proved by Prékopa [9], (see also Kanter [7, Lemma 2.1]).

Let ν_1 and ν_2 be probability measures on $\boldsymbol{R}^n$. We say that ν_2 is *more peaked* than ν_1 if

$$\nu_1(U) \leqq \nu_2(U)$$

for all closed, convex, symmetric subsets U in $\boldsymbol{R}^n$. (We recall that $U \subseteqq \boldsymbol{R}^n$ is symmetric if $U = -U$.) If f_1 and f_2 are probability density functions on $\boldsymbol{R}^n$ we say that f_2 is *more peaked* than f_1 if the measure $f_2 d\mu_n$ is more peaked than the measure $f_1 d\mu_n$. The notion of peakedness was introduced by Birnbaum [1] and Sherman [10]. A complementary relation is that of symmetric dominance in the sense of Kanter [7]. If ν_3 and ν_4 are measures on $\boldsymbol{R}^n$ then ν_3 symmetrically dominates ν_4 if

$$\nu_3(\boldsymbol{R}^n \backslash U) \geqq \nu_4(\boldsymbol{R}^n \backslash U)$$

for all closed, convex, symmetric subsets U in $\boldsymbol{R}^n$. It is clear that if ν_3 and ν_4 are both probability measures then ν_3 symmetrically dominates ν_4 if and only if ν_4 is more peaked than ν_3. For our purposes it is more convenient to work with the relation of peakedness.

If ν_1 and ν_2 are log-concave probability measures on $\boldsymbol{R}^n$ then the convolution $\nu_1 * \nu_2$ is also log-concave on $\boldsymbol{R}^n$ (Kanter [7, Lemma 2.3]). It follows that if ν_1 and ν_2 are log-concave probability measures on $\boldsymbol{R}^{n_1}$ and $\boldsymbol{R}^{n_2}$ respectively then the product measure $\nu_1 \times \nu_2$ is log-concave on $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$. Forming product measures also preserves the peakedness relation.

LEMMA 4. *Suppose that ν_1, ν_2, ν_1' and ν_2' are all log-concave probability measures such that ν_1 is mome peaked than ν_1' on $\boldsymbol{R}^{n_1}$ and*

ν_2 is more peaked than ν_2' on $\boldsymbol{R}^{n_2}$. Then $\nu_1 \times \nu_2$ is more peaked than $\nu_1' \times \nu_2'$ on $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$.

For the proof of Lemma 4 we refer to Kanter [7, Corollary 3.2] where the result is obtained for the more general class of unimodal measures.

3. Proof of Theorem 1. We begin by proving the following lemma.

LEMMA 5. *Suppose that $n_1, n_2, \cdots, n_J$ are positive integers and $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_J}$ is in $\boldsymbol{R}^N$, $N = n_1 + n_2 + \cdots + n_J$. Then $\chi_{Q_N}(\bar{x})$ is more peaked than the normal density function* $\exp\{-\pi|\bar{x}|^2\}$ *on $\boldsymbol{R}^N$.*

Proof. Since the measures $\chi_{Q_N}(\bar{x})d\mu_N(\bar{x})$ and $\exp\{-\pi|\bar{x}|^2\}d\mu_N(\bar{x})$ are both product measures which factor in $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \cdots \times \boldsymbol{R}^{n_J}$ it suffices to prove the peakedness relation in each factor space and then apply Lemma 4. Thus we need only show that for each positive integer n, $\chi_{S_n}(\bar{x})$ is more peaked than $\exp\{-\pi|\bar{x}|^2\}$ on $\boldsymbol{R}^n$. Of course it is trivial to verify that both of the density functions $\chi_{S_n}(\bar{x})$ and $\exp\{-\pi|\bar{x}|^2\}$ are log-concave on $\boldsymbol{R}^n$.

Let $\sum_{n-1} = \{\bar{x} \in \boldsymbol{R}^n : |\bar{x}| = 1\}$ so that for each $\bar{x} \neq \bar{0}$ in $\boldsymbol{R}^n$ we have the unique polar decomposition $\bar{x} = r\bar{x}'$ where $r = |\bar{x}|$ and $\bar{x}' \in \sum_{n-1}$. If U is a closed, convex, symmetric subset of $\boldsymbol{R}^n$ then it follows that

$$\int_U \exp\{-\pi|\bar{x}|^2\}d\mu_n(\bar{x}) = \int_{\Sigma_{n-1}}\int_0^\infty \chi_U(r\bar{x}')\exp\{-\pi r^2\}r^{n-1}drd\bar{x}' , \tag{3.1}$$

where $d\bar{x}'$ is the induced Lebesgue measure on $\sum_{n-1}$. Now for each fixed $\bar{x}' \in \sum_{n-1}$ we have either

$$\chi_U(r\bar{x}') \leqq \chi_{S_n}(r\bar{x}') , \quad 0 \leqq r < \infty \tag{3.2}$$

or

$$\chi_{S_n}(r\bar{x}') \leqq \chi_U(r\bar{x}') , \quad 0 \leqq r < \infty , \tag{3.3}$$

since S_n and U are convex. If (3.2) holds at $\bar{x}'$ then

$$\begin{aligned}&\int_0^\infty \chi_U(r\bar{x}')\exp\{-\pi r^2\}r^{n-1}dr \\ &\qquad \leqq \int_0^\infty \chi_U(r\bar{x}')r^{n-1}dr = \int_0^\infty \chi_U(r\bar{x}')\chi_{S_n}(r\bar{x}')r^{n-1}dr .\end{aligned} \tag{3.4}$$

If (3.3) holds at $\bar{x}'$ then

$$
\begin{aligned}
&\int_0^\infty \chi_U(r\bar{x}')\exp\{-\pi r^2\}r^{n-1}dr \\
&\qquad \leqq \int_0^\infty \exp\{-\pi r^2\}r^{n-1}dr = n^{-1}\pi^{-n/2}\Gamma\Big(\frac{n}{2}+1\Big) \\
(3.5)\qquad &\qquad = \int_0^\infty \chi_{S_n}(r\bar{x}')r^{n-1}dr \\
&\qquad = \int_0^\infty \chi_U(r\bar{x}')\chi_{S_n}(r\bar{x}')r^{n-1}dr\ .
\end{aligned}
$$

Combining (3.1), (3.4) and (3.5) we obtain

$$\int_U \exp\{-\pi|\bar{x}|^2\}d\mu_n(\bar{x}) \leqq \int_{\Sigma_{n-1}}\int_0^\infty \chi_U(r\bar{x}')\chi_{S_n}(r\bar{x}')r^{n-1}drd\bar{x}' = \int_U \chi_{S_n}(\bar{x})d\mu_n(\bar{x})\ .$$

Thus $\chi_{S_n}(\bar{x})$ is more peaked than $\exp\{-\pi|\bar{x}|^2\}$ on R^n and the lemma is proved.

We now prove Theorem 1. If $N = K$ then (1.1) is trivial so we may suppose that $K' = N - K$ is positive. Let P_K be the K-dimensional subspace of R^N spanned by the columns of A. Next let W be an $N \times N$ matrix whose first K columns are the columns of A and whose next K' columns are the columns of an $N \times K'$ matrix B. We choose the columns of B so that they form an orthonormal basis in $\boldsymbol{R}^N$ of the K'-dimensional subspace which is orthogonal to P_K. Identifying $\boldsymbol{R}^N$ with $\boldsymbol{R}^K \times \boldsymbol{R}^{K'}$ we may write each $\bar{z} \in \boldsymbol{R}^N$ as $\bar{z} = (\bar{x}/\bar{y})$ where $\bar{x} \in \boldsymbol{R}^K$ and $\bar{y} \in \boldsymbol{R}^{K'}$. For each ε, $0 < \varepsilon \leqq 1$ we define

$$H_\varepsilon = \left\{\bar{z} \in \boldsymbol{R}^N : z = \left(\frac{\bar{x}}{\bar{y}}\right),\ \max_{1\leqq j\leqq K'}|y_j| \leqq \frac{\varepsilon}{2}\right\}$$

and

$$H'_\varepsilon = \left\{\bar{y} \in \boldsymbol{R}^{K'} : \max_{1\leqq j\leqq K'}|y_j| \leqq \frac{\varepsilon}{2}\right\}\ .$$

Clearly H_ε is a closed, convex, symmetric subset of $\boldsymbol{R}^N$ and so is the image of H_ε under the nonsingular linear transformation determined by W. Thus by Lemma 5,

$$\int_{H_\varepsilon} \exp\{-\pi|W\bar{z}|^2\}d\mu_N(\bar{z}) \leqq \int_{H_\varepsilon} \chi_{Q_N}(W\bar{z})d\mu_N(\bar{z})\ . \tag{3.6}$$

Multiplying each side of (3.6) by $\{\mu_{K'}(H'_\varepsilon)\}^{-1} = \varepsilon^{-K'}$ and factoring H_ε into $\boldsymbol{R}^K \times H'_\varepsilon$ we find that

$$
\begin{aligned}
(3.7)\qquad &\varepsilon^{-K'}\int_{\boldsymbol{R}^K}\int_{H'_\varepsilon} \exp\{-\pi|A\bar{x} + B\bar{y}|^2\}d\mu_{K'}(\bar{y})d\mu_K(\bar{x}) \\
&\qquad \leqq \varepsilon^{-K'}\int_{\boldsymbol{R}^K}\int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y})d\mu_{K'}(\bar{y})d\mu_K(\bar{x})\ .
\end{aligned}
$$

By the orthogonality condition $|A\bar{x} + B\bar{y}|^2 = |A\bar{x}|^2 + |B\bar{y}|^2$ and so as $\varepsilon \to 0+$ the left hand side of (3.7) clearly converges to

$$\int_{R^K} \exp\{-\pi |A\bar{x}|^2\} d\mu_K(\bar{x}) = |\det A^T A|^{-1/2} .$$

To evaluate the corresponding limit on the right hand side of (3.7) we observe that for $0 < \varepsilon \leqq 1$ and each $\bar{x} \in \boldsymbol{R}^K$,

$$\varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) \leqq 1 .$$

Since Q_N and H'_ε are both bounded we have

$$\varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) = 0$$

for sufficiently large $|\bar{x}|$ independent of ε. Thus by dominated convergence the limit on the right of (3.7) as $\varepsilon \to 0+$ is

$$\int_{R^K} \left\{\lim_{\varepsilon \to 0+} \varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y})\right\} d\mu_K(\bar{x}) . \tag{3.8}$$

Clearly

$$\lim_{\varepsilon \to 0+} \varepsilon^{-K'} \int_{H'_\varepsilon} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) = \chi_{Q_N}(A\bar{x})$$

except possibly when $A\bar{x}$ is a boundary point of $Q_N \cap P_K$, Since this boundary has K-dimensional measure zero we see that (3.8) is equal to

$$\int_{R^K} \chi_{Q_N}(A\bar{x}) d\mu_K(\bar{x}) .$$

We have now shown that as $\varepsilon \to 0+$ on each side of (3.7) we obtain (1.1) and this proves the theorem.

4. Proof of Theorem 2. By van der Corput's extension of Minkowski's convex body theorem [5] (see also Cassels [4, Chapter III, Theorem II]) the condition (1.6) implies that there exist at least M distinct pairs $\pm\bar{v}_m$, $m = 1, 2, \cdots, M$, of nonzero lattice points such that (1.3) holds. If rank$(A) < K$ then (1.2) and (1.6) are both trivially satisfied. Thus to eatablish the first part of Theorem 2 it suffices to show that if rank$(A) = K$ then

$$2^K |\det A^* E^2 A|^{-1/2} \leqq \mu_K(\{\bar{x} \in \boldsymbol{R}^K : |L_j(\bar{x})| \leqq \varepsilon_j,\ j = 1, 2, \cdots, N\}) . \tag{4.1}$$

Let $G_j(\bar{x})$, $j = 1, 2, \cdots, N$ be linear forms defined by $G_j(\bar{x}) = L_j(\bar{x})$ for $j = 1, 2, \cdots, r$ and

$$G_{r+2j-1}(\bar{x}) = \sqrt{2}\,\mathrm{Re}\{L_{r+2j-1}(\bar{x})\}\ ,$$
$$G_{r+2j}(\bar{x}) = \sqrt{2}\,\mathrm{Im}\{L_{r+2j-1}(\bar{x})\}$$

for $j = 1, 2, \cdots, s$. We write $B = (b_{jk})$ for the corresponding real $N \times K$ matrix so that

$$G_j(\bar{x}) = \sum_{k=1}^{K} b_{jk} x_k\ .$$

Next we let $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_{r+s}}$ where $n_j = 1$ for $j = 1, 2, \cdots, r$ and $n_j = 2$ for $j = r+1, r+2, \cdots, r+s$. It follows that $|L_j(\bar{x})| \leqq \varepsilon_j$ if and only if $1/2\varepsilon_j^{-1} G_j(\bar{x}) \in S_{n_j}$, $j = 1, 2, \cdots, r$, and

$$|L_{r+2j-1}(\bar{x})| = |L_{r+2j}(\bar{x})| \leqq \varepsilon_{r+2j}$$

if and only if

$$(2\pi)^{-1/2}\varepsilon_{r+2j}^{-1}\begin{pmatrix} G_{r+2j-1}(\bar{x}) \\ G_{r+2j}(\bar{x}) \end{pmatrix} \in S_{n_{r+j}}\ ,$$

$j = 1, 2, \cdots, s$. Therefore

$$\begin{aligned}
&\mu_K(\{\bar{x} \in \boldsymbol{R}^K : |L_j(\bar{x})| \leqq \varepsilon_j,\ j = 1, 2, \cdots, N\}) \\
&\quad = \mu_K\Big(\Big\{\bar{x} \in \boldsymbol{R}^K : \frac{1}{2}EB\bar{x} \in Q_N\Big\}\Big) = \int_{\boldsymbol{R}^K} \chi_{Q_N}\Big(\frac{1}{2}EB\bar{x}\Big) d\mu_K(\bar{x}) \\
&\quad \geqq \Big|\det\Big(\frac{1}{2}EB\Big)^T\Big(\frac{1}{2}EB\Big)\Big|^{-1/2} = 2^K |\det B^T E^2 B|^{-1/2}\ .
\end{aligned}$$

An easy computation shows that $B^T E^2 B = A^* E^2 A$ and so completes the proof of (4.1).

To prove the second part of Theorem 2 we choose $\varepsilon_j = |\det A^* A|^{1/2K}$ for $j = 1, 2, \cdots, r$ and $\varepsilon_j = (2/\pi)^{1/2} |\det A^* A|^{1/2K}$ for $j = r+1, r+2, \cdots, N$. Then

$$|\det A^* E^2 A| = 1$$

and so (1.4) and (1.5) follow from the first part of the theorem.

5. Lower bounds for arbitrary convex bodies. In this section we suppose that Q_N is a closed, convex, symmetric subset of $\boldsymbol{R}^N$ with $\mu_N(Q_N) = 1$. If A is an $N \times K$ matrix, $\mathrm{rank}(A) = K$, we will be interested in the problem of finding a lower bound for

$$\int_{\boldsymbol{R}^K} \chi_{Q_N}(A\bar{x})\, d\mu_K(\bar{x})\ . \tag{5.1}$$

The method used to deduce Theorem 1 from Lemma 5 will also lead to a lower bound in this more general situation, provided that we can find a suitable normal density function on $\boldsymbol{R}^N$ which is less peaked than $\chi_{Q_N}(\bar{x})$. We succeeded in proving Lemma 5 because the special structure imposed on Q_N allowed us to appeal to Lemma 4. We now describe an alternative method which leads to a conjectured lower bound for (5.1).

We write Q for Q_N and we assume that Q is a fixed, closed, convex, symmetric subset of $\boldsymbol{R}^N$, $\mu_N(Q) = 1$. For each positive integer m let

$$\chi_Q^{(m)}(\bar{x}) = \chi_Q^* \chi_Q^* \cdots \chi_Q(\bar{x})$$

be the m-fold convolution of χ_Q. We define the dilation operator D_λ for $\lambda > 0$ and for integrable real valued functions f on $\boldsymbol{R}^N$ by

$$D_\lambda(f)(\bar{x}) = \lambda^N f(\lambda \bar{x}) \ .$$

Next we define a sequence of positive numbers λ_m, $m = 1, 2, \cdots$ by

$$(\lambda_m)^N \chi_Q^{(m)}(\bar{0}) = 1 \ .$$

With this notation we have the following

CONJECTURE 6. *For each positive integer m, $\chi_Q(\bar{x})$ is more peaked than $D_{\lambda_m}(\chi_Q^{(m)})(\bar{x})$.*

Now let Ω be the $N \times N$ covariance matrix determined by a random vector which is uniformly distributed on the convex body Q. That is $\Omega = (\omega_{rs})$ is the $N \times N$ matrix defined by

$$\omega_{rs} = \int_{R^N} y_r y_s \chi_Q(\bar{y}) d\mu_N(\bar{y}) \ ,$$

where y_r and y_s are the rth and sth co-ordinate functions of $\bar{y}$, $r = 1, 2, \cdots, N$, and $s = 1, 2, \cdots, N$. It is clear that Ω is symmetric and nonsingular since Q has a nonempty interior. By the Central Limit Theorem (Breiman [3, Theorem 11.10]) we have

$$\lim_{m\to\infty} D_{\sqrt{m}}(\chi_Q^{(m)})(\bar{x}) = (2\pi)^{-N/2}(\det \Omega)^{-1/2} \exp\left\{-\frac{1}{2}\bar{x}^T \Omega^{-1}\bar{x}\right\}$$

uniformly for $x \in \boldsymbol{R}^N$. It follows that

$$\lim_{m\to\infty} \frac{\lambda_m}{\sqrt{m}} = (2\pi)^{1/2}(\det \Omega)^{1/2N}$$

and hence

$$\lim_{m\to\infty} D_{\lambda_m}(\chi_Q^{(m)})(\bar{x}) = \exp\{-\pi(\det \Omega)^{1/N}\bar{x}^T \Omega^{-1}\bar{x}\}$$

uniformly for $x \in \boldsymbol{R}^N$. If the Conjecture 6 is true then for each positive integer m and each closed, convex, symmetric subset U of $\boldsymbol{R}^N$

$$\int_U D_{\lambda_m}(\chi_Q^{(m)})(\bar{x})d\mu_N(\bar{x}) \leqq \int_U \chi_Q(\bar{x})d\mu_N(\bar{x}) . \tag{5.2}$$

Letting $m \to \infty$ on the left hand side of (5.2) and we have proved that $\chi_Q(\bar{x})$ is more peaked than $\exp\{-\pi(\det \Omega)^{1/N}\bar{x}^T\Omega^{-1}\bar{x}\}$ on $\boldsymbol{R}^N$. By the same method used to prove Theorem 1 we obtain

THEOREM 7. *Assume that the Conjecture* 6 *holds and let A be a real $N \times K$ matrix,* $\operatorname{rank}(A) = K$. *Then*

$$(\det \Omega)^{-K/2N}|\det A^T\Omega^{-1}A|^{-1/2} \leqq \int_{\boldsymbol{R}^K} \chi_Q(A\bar{x})d\mu_K(\bar{x}) . \tag{5.3}$$

If the set Q in Theorem 7 is such that Ω is a constant multiple of the identity matrix then the left hand side of (5.3) is simply $|\det A^TA|^{-1/2}$. Just as in our proof of the corollary to Theorem 1, we deduce that in this case $\mu_K(Q \cap P_K) \geqq 1$, where P_K is a K-dimensional subspace of $\boldsymbol{R}^N$. There is also an application of Theorem 7 to linear forms. If $L_j(\bar{x})$, $j = 1, 2, \cdots, N$, are N linear forms in K-variables we could determine precise conditions under which

$$\left(\sum_{j=1}^N |L_j(\bar{v})|^p\right)^{1/p} \leqq \varepsilon$$

at a nonzero lattice point $\bar{v}$ for any $p \geqq 1$ and $\varepsilon > 0$. At present, however, these results remain hypothetical since they depend on the open problem stated in Conjecture 6.

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T AS AN $\mathscr{E}$ SUBMODULE OF G

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Let G be a mixed abelian group with torsion subgroup T. T is viewed as an $\mathscr{E}$ submodule of G, where $\mathscr{E} = \operatorname{End} G$. It is shown that T is superfluous in G if and only if, $\forall_p$, either T_p is divisible or G/T_p is not p divisible. If G is not reduced, T is essential in G if and only if T contains a $Z(p^\infty)$. Let $I(G)$ $[I(T)]$ be the $\mathscr{E}$ injective hull of G $[T]$. Then $I(G) = I(T) \oplus X$ with X torsion free divisible and T is a pure subgroup of $I(G)$. This can be used to obtain several results; for example, if $Q \not\subseteq I(T)$, TFAE: 1. $T \operatorname{ess} G$, 2. $I(G) \cong I(T)$ as abelian groups, 3. $Q \not\subseteq I(G)$. The condition $T \operatorname{ess} G$ is characterized if T is a summand or if G is algebraically compact. If T is bounded or if T is a p-group, $T^1 = (0)$ and G is reduced cotorsion, T is not essential. In fact, for bounded T there is an $\mathscr{E}$ isomorphism $I(G) \cong I(T) \oplus I(G/T)$. Some information is obtained on the p-basic subgroups of $I(T)$ as a function of those of T. A condition is given for $I(T) \supseteqq \oplus_c Q$. These last theorems specialize to $I({}_E T)$, where $E = \operatorname{End} T$.

Preliminaries. In the last fifteen years several authors have written papers concerning an abelian group G viewed as a module over $\mathscr{E}$, its ring of endomorphisms.

Let G be a mixed abelian group with maximal torsion subgroup T. In this paper we consider T as an $\mathscr{E}$ submodule of G. We determine when T is superfluous in G and then study the more difficult question of determining when T is essential in G. (If $(0) \neq T \neq G$, it is easy to prove that T is neither essential nor superfluous as a Z submodule of G.)

The latter question leads to consideration of the injective hulls $I(T)$, $I(G)$—taken with respect to $\mathscr{E}$.

Our notation, with minor exceptions, is that of [1].

1. T as a superfluous submodule of G. Henceforth, let G be a mixed abelian group, $T = t(G)$ its torsion subgroup and $\mathscr{E} = \operatorname{End} G$. To avoid stating the trivial cases of our results we always assume $(0) \neq T \neq G$. We begin by characterizing those mixed G for which ${}_{\mathscr{E}}T$ is superfluous in ${}_{\mathscr{E}}G$ ($T \ll G$). In our context $T \ll G$ if and only if whenever K is a fully invariant subgroup of G with $K + T = G$, then $K = G$.

LEMMA 1. *Let $T = \bigoplus T_p$ be a decomposition of T into its p components. Then $T \ll G$ if and only if $T_p \ll G$, $\forall p$.*

Proof. The only if part of the implication is immediate since submodules of superfluous submodules are superfluous.

Suppose $T_p \ll G$, $\forall p$, and $T \not\ll G$. Then we must have $T + K = G$ for some fully invariant $K \neq G$. Clearly, $K \not\supseteq T_p$ for some p. Let $K' = K + \sum_{q \neq p} T_q$. Since K' is fully invariant with $K' + T_p = G$, $K' = G$.

Let $t \in T_p$ and suppose that t has order $o(t) = p^l$. Write $t = x + y$ with $x \in K$, $o(y) = n$, $(n, p) = 1$. If a, $b \in Z$ with $ap^l + bn = 1$, then $t = (ap^l + bn)t = bnt = bnx \in K$. Thus, $T_p \subseteq K$, a contradiction.

THEOREM 1. *$T \ll G$ if and only if, $\forall p$, either T_p is divisible or G/T_p is not p divisible.*

We prove the contrapositive in both directions.

Proof. Suppose $\exists p$ with T_p not divisible and G/T_p p divisible. Then $T_p \not\subseteq pG$ and $G = pG + T_p$. Thus, $T_p \not\ll G$ and, by Lemma 1, $T \not\ll G$.

Conversely, suppose $T \not\ll G$. Then $\exists p$ with $T_p \not\ll G$. Let K be a proper fully invariant subgroup with $K + T_p = G$. We cannot have T_p divisible, for then $K \supseteq \mathrm{Hom}\,(G, T_p)K = T_p$. (If $x \in K$, $o(x) = \infty$, and $t \in T_p$, the map $Zx \to Zt$ extends to G.)

G/T_p is p divisible if and only if $K \subseteq pG + T_p$. Assume that G/T_p is not p divisible. Then there is an $x \in K \backslash pG + T_p$. Therefore, $\forall t \in T_p$, the p-height of $x + t$ in G, $h_p^G(x + t)$, is zero.

Thus, for every positive integer l, $\bar{x} = x + p^l G$ must have order exactly p^l in $G/p^l G$. But then, $\forall t \in T_p$, we can construct an endomorphism of G mapping $x \to \bar{x} \to t$. This implies $K \supseteq T_p$, a contradiction. The theorem follows.

2. T as an essential submodule of G-basic results.

We next consider the more difficult problem of deciding when T is essential in $G(T \operatorname{ess} G)$. We first dispose of the nonreduced case.

THEOREM 2. *Let G be a nonreduced group. Then $T \operatorname{ess} G$ if and only if T contains a $Z(p^\infty)$.*

Proof. If $T \supseteq Z(p^\infty)$ then, $\forall x \in G$ with $o(x) = \infty$, $\exists \alpha \in \mathscr{E}$ with $0 \neq \alpha(x) \in Z(p^\infty)$. This, clearly, is enough to imply $T \operatorname{ess} G$.

Conversely, suppose T contains no $Z(p^\infty)$. Then, since G is not reduced, the maximum divisible subgroup D of G is nontrivial and torsion free. Hence $T \cap D = 0$, so T is not essential in G.

From now on we assume G is reduced.

To investigate the question of when $T \operatorname{ess} G$, it is natural to consider the $\mathscr{E}$ injective hulls. Let $I(G)$ be the injective hull of the module ${}_{\mathscr{E}}G$. Since ${}_{\mathscr{E}}T \leqq {}_{\mathscr{E}}G$ we can regard $I(T)$, the injective hull of ${}_{\mathscr{E}}T$, as a maximal $\mathscr{E}$ essential extension of T in $I(G)$. If $I(T)$ is constructed in this way we have an $\mathscr{E}$ decomposition: $I(G) = I(T) \oplus X$. Clearly, $T \operatorname{ess} G$ if and only if $X = (0)$.

THEOREM 3. *Let X be as above. Then X is torsion free divisible as an abelian group.*

Proof. If $t(X)$, the torsion subgroup of X, were nonzero, then $I(T) \oplus t(X)$ would be an $\mathscr{E}$ essential extension of T in $I(G)$ properly containing $I(T)$—a contradiction. Thus, X is torsion free. Since X is an injective module, X must also be divisible.

COROLLARY. *$T \operatorname{ess} G$ if and only if $I(T)$ and $I(G)$ are isomorphic $\mathscr{E}$ modules.*

Proof. Suppose $\theta\colon I(T) \to I(G)$ is an $\mathscr{E}$ isomorphism. Then $\theta(T) \operatorname{ess} I(G)$. By Theorem 3, $\theta(T) \cap X = (0)$. Thus, $X = (0)$ and $T \operatorname{ess} G$.

The next theorem is central for our results.

THEOREM 4. *T is a pure subgroup of $I(G)$ $(T \lhd I(G))$.*

Proof. Let $D(G)$ be the Z injective hull of G and let A be the injective left $\mathscr{E}$ module $\operatorname{Hom}_Z(\mathscr{E}, D(G))$. Regard $G \subseteqq A$ via $G \cong \operatorname{Hom}_{\mathscr{E}}(\mathscr{E}, G)$ and take $I(G)$ to be a maximal $\mathscr{E}$ essential extension of G in A. It suffices to show $T \lhd A$. Let $\delta \in T$ with $p\delta = 0$. Suppose $h_p^T(\delta) = m < \infty$, but $\delta = p^{m+1}\alpha$, $\alpha \in A$.

Write $\delta = p^m\delta'$, $\delta' \in T$. Then $T = \langle\delta'\rangle \oplus T'$ ([1], Corollary 27.2). Let $\pi \in \mathscr{E}$ be projection onto $\langle\delta'\rangle$. Then $\delta(\pi) = \pi(\delta) = \delta = p^{m+1}\alpha(\pi) = \alpha(p^{m+1}\pi) = 0$—a contradiction. Thus, we have proved: $\delta \in T[p] \to h_p^T(\delta) = h_p^A(\delta)$. This shows $T \lhd A$ ([1], (h), p. 114).

COROLLARY 1. *If T is a torsion group, $E = \operatorname{End} T$, then $T \lhd I({}_E T)$.*

This is proved by putting $G = T$ in the above.

COROLLARY 2. *Suppose $T \subset G$ with $T^1 = G^1$, G/T divisible. Then $T \operatorname{ess} G$. (Here T^1 $[G^1]$ denotes the first Ulm subgroup of T $[G]$.)*

Proof. Since $T \lhd I(G)$, G/T divisible, we have $G \lhd I(G)$. If

$G^1 = T^1$ and X is as in Theorem 3, $X \cap G = (0)$, so $X = (0)$. Thus, $T \operatorname{ess} G$.

COROLLARY 3. *Let* $T \subset G$ *with* $T^1 = (0)$. *Then* $I(T)^1 = (0)$.

Proof. $I(T)^1$ is an $\mathscr{E}$ submodule of $I(T)$. Since $T^1 = (0)$ and $T \lhd I(T)$, $I(T)^1 \cap T = (0)$. Thus, $I(T)^1 = (0)$.

THEOREM 5. *Let* $T \subset G$ *with* $Q \not\subseteq I(T)$. *Then TFAE*: 1. $T \operatorname{ess} G$; 2. $I(T) \cong I(G)$ *as abelian groups*; 3. $Q \not\subseteq I(G)$. *Moreover, if* 1—3 *hold, then* $T^1 = G^1$.

Proof. The implications $1 \rightarrow 2$, $2 \rightarrow 3$ are obvious. If $Q \not\subseteq I(G)$, then the X of Theorem 3 is zero, so $T \operatorname{ess} G$.

To prove the additional statement, note that $I(T)$ is an algebraically compact group ([1], p. 178) which, by assumption, contains no Q's. Thus, there can be no elements of infinite order in $I(T)^1$. If 1—3 hold, the same is true for $I(G)^1$. Thus, in this case, $G^1 = T^1$.

COROLLARY. *Let* $T \subset G$ *with* $T^1 = (0)$. *Then conditions* 1—3 *are equivalent. Moreover, if* 1—3 *hold, then* $G^1 = (0)$.

Proof. If $T^1 = (0)$, then $I(T)^1 = (0)$, so $Q \not\subseteq I(T)$.

Theorem 5 raises the questions: When are $I(T)$ and $I(G)$ isomorphic as abelian groups? Is this sufficient for $T \operatorname{ess} G$? Here is a partial result.

THEOREM 6. *Let* $\bar{I}$ *be the* $\mathscr{E}$ *injective hull of the factor module* G/T. *Write* $I(T) = H \oplus K$, *where* H *is the maximal torsion free divisible subgroup of* $I(T)$. *Let* $r = \operatorname{rank} H$, $\bar{r} = \operatorname{rank} \bar{I}$. *If* r *is infinite and* $r \geqq \bar{r}$, *then* $I(G) \overset{+}{\cong} I(T)$.

Proof. Embed $I(G)$ into $I(T) \oplus \bar{I}$ in the standard way (via $\alpha \oplus \beta$ where α and β are the extensions to $I(G)$ of $T \subset I(T)$ and $G \rightarrow G/T \subset \bar{I}$ respectively). Then, as $\mathscr{E}$ modules, $I(G) \oplus Y \cong I(T) \oplus \bar{I}$. Since $I(G) = I(T) \oplus X$, we have:

$$(*) \qquad I(T) \oplus X \oplus Y \cong I(T) \oplus \bar{I}\,.$$

The additive group of $\bar{I}$ is torsion free divisible, since $\bar{I}$ is the injective hull of a module whose additive group is torsion free. Thus, the number of Q's on the right-hand side of $(*)$ is $r + \bar{r} = r$, so $\operatorname{rank} X \leqq r$. But then, $I(G) = I(T) \oplus X \overset{+}{\cong} I(T)$.

EXAMPLE. For each prime p, let T_p be the group generated by $\{a_i \mid i = 0, 1, 2, 3, \cdots\}$ with relations $\{pa_0 = 0,\ p^n a_n = a_0,\ n = 1, 2, 3, \cdots\}$. Let $T = \bigoplus_p T_p$ and let $G = Q \oplus T$. Then $\bar{r} = 1$ and (as we will see in Theorem 13) $r \geqq c$. Thus, $I(G) \overset{+}{\cong} I(T)$. Since T is reduced, T is not essential in G.

3. T as an essential submodule of G—some special cases. In this section we consider the essentiality of T in G in some special cases. First we consider the situation for bounded T. The following theorem shows if T is bounded, then T is never essential in G.

THEOREM 7. *Let $T \subset G$ with $nT = (0)$ and let $\bar{I} = I(G/T)$. Then:*
1. *$nI(T) = (0)$;*
2. *$I(G)$ is $\mathscr{E}$ isomorphic to $I(T) \oplus \bar{I}$.*

Proof. Let $D(G)$, $D(T)$, $D(G/T)$ be the Z injective hulls of G, T, G/T and let A, B, C be the injective left $\mathscr{E}$ modules $\mathrm{Hom}_Z(\mathscr{E}, D(M))$ where $M = G, T, G/T$, respectively. As in Theorem 4, regard $T \subseteqq G \subseteqq I(G) \subseteqq A$. Suppressing the obvious isomorphism, write $A = B \oplus C$—an $\mathscr{E}$ direct sum. Under these identifications $T = B \cap G$.

To prove (1), recall $T \lhd A$, so in this case, $T \cap nA = nT = (0)$. Thus, if $x \in I(T)$ with $nx \neq 0$, then, for some $\lambda \in \mathscr{E}$, $0 \neq \lambda(nx) \in T \cap nA$—a contradiction.

To prove (2), first note that $B \cap I(G)$ is an essential extension of $T = B \cap G$. Choose $I(T) \subseteqq I(G)$ as before—with the additional requirement $I(T) \supseteqq B \cap I(G)$.

Let $x \in I(T)$, say $x = b + c$, $b \in B$, $c \in C$. Since C is torsion free and $nx = 0$, we must have $c = 0$. Thus, $I(T) \subseteqq B$. It follows that $I(T) = B \cap I(G)$.

Let $\pi \in \mathrm{Hom}_{\mathscr{E}}(A, C)$ be projection onto C and let $\pi' = \pi|_{I(G)}$. Clearly, $\mathrm{Ker}\,\pi' = B \cap I(G) = I(T)$, so write $I(G) = I(T) \oplus Y$ with π' a monomorphism on Y.

To finish the proof of (2), we claim $\pi'(Y)$ is an $\mathscr{E}$ injective hull of G/T. To see this, first note that if G/T is embedded in C via $e\colon g + T \to$ evaluation at $g + T$, we have $e(G/T) = \pi'(G) \subseteqq \pi'(Y)$, so $\pi'(Y)$ is an injective containing $e(G/T) \cong G/T$. Furthermore, if $0 \neq \pi'(y) \in \pi'(Y)$, then $\exists \lambda \in \mathscr{E}$ with $0 \neq \lambda(y) \in G \cap Y$. Thus, $0 \neq \pi'\lambda(y) = \lambda\pi'(y) \in \pi'(G) = e(G/T)$. This proves that $e(G/T)\ \mathrm{ess}\ \pi'(Y)$. The theorem follows.

EXAMPLE. Let $T = \bigoplus_{p \in P} Z(p)$, where P is an infinite set of primes, and let $G = Z \oplus T$. Then $T\ \mathrm{ess}\ G$, so $I(G) = I(T)$ and, in view of Theorem 4, $I(T)^1 = (0)$. Moreover, it is easy to see that $\bar{I} \cong_Z Q$. Thus, if T is an unbounded group direct summand of G, we need not have the decomposition of $I(G)$ given in (2).

The following gives one characterization of $T \operatorname{ess} G$ in the splitting case.

THEOREM 8. *Let* $T = \bigoplus T_p \subset G$. *Let* $k_p = \text{l.u.b.}\{l \mid G$ *has a* $Z(p^l)$ *summand*$\}$ *and let* $H = \{x \in G \mid o(x) = \infty, h_p^G(x) \geqq k_p \forall p\}$. *Then*:

(1) *If* $H = (0)$, $T \operatorname{ess} G$;

(2) *If* $G = T \bigoplus F$ *and* $T \operatorname{ess} G$, *then* $H = (0)$.

Proof. (1) is clear. To prove (2) suppose $G = T \bigoplus F$ and $0 \neq x \in H$. Then, for some positive integer n, $0 \neq nx \in H \cap F$. Clearly, nx cannot be mapped by an endomorphism of G onto any nonzero element of a bounded T_p.

If T_p is unbounded, then G has an unbounded p-basic subgroup, so $k_p = \infty$. Thus, $h_p^G(nx) = h_p^F(nx) = \infty$. If $\lambda \in \mathscr{E}$ with $0 \neq \lambda(nx) \in T_p$, then λ restricts to a nonzero map of the subgroup $\{m/p^k(nx) \mid m, k \in Z\} \subseteqq F$ into T_p. This is impossible since T_p is reduced. Thus, nx cannot be mapped by an endomorphism of G onto a nonzero element of any T_p. The result follows.

It is easy to describe when $T \operatorname{ess} G$ for algebraically compact G.

THEOREM 9. *Let* $T = \bigoplus T_p \subset G$ *with* G (*reduced*) *algebraically compact. Write* G *as a product of* p-*adic modules*, $G = \Pi G_p$. *Then* $T \operatorname{ess} G$ *if and only if*, $\forall p$, *either* $T_p = G_p$ *or* T_p *is unbounded.*

Proof. It is immediate that $T \operatorname{ess} G$ if and only if, $\forall p$, $T_p \operatorname{ess} G_p$. If $\exists p$ with $T_p \neq G_p$ and T_p bounded, then T_p is not essential in G_p.

Conversely, by considering projections onto summands of a p-adic basis for G_p, it is easy to see that T_p unbounded implies $T_p \operatorname{ess} G_p$.

We close this section with:

THEOREM 10. *Let* $T \subset G$ *with* G (*reduced*) *cotorsion*, T *a* p-*group*, $T^1 = (0)$. *Then* T *is not essential in* G.

Proof. If T is bounded, T is not essential. If T is an unbounded p-group, $(0) \neq P\operatorname{ext}(Q/Z, T) = [\operatorname{Ext}(Q/Z, T)]^1$. Since G is reduced cotorsion, $G \cong \operatorname{Ext}(Q/Z, G) \cong \operatorname{Ext}(Q/Z, T) \bigoplus \operatorname{Ext}(Q/Z, G/T)$ ([1] H, p. 234 and Lemma 55.2). Thus $G^1 \neq (0)$, $T^1 = (0)$ and T cannot be essential in G.

4. The structure of $I(T)$. In this section we prove three theorems concerning the structure of $I(T)$. With trivial modification, each of these theorems can be rewritten to give the same result for the injective hull of a torsion group over its own endomorphism ring.

Since $I(T)$ is algebraically compact, it is natural to try to find out what its p-basic subgroups look like as a function of the p-basic subgroups of T. In the case $T^1 = (0)$, this information would characterize $I(T)$ as an abelian group. The next result shows that $I(T)$ is generally large with respect to T.

THEOREM 11. *Let B $[B']$ be a p-basic subgroup of T $[I(T)]$. Let $f =$ final rank B. If $Z(p^k)$ occurs in B, then B' contains $\bigoplus_{\gamma \in \bar{\mathscr{A}}} \langle z_\gamma \rangle$ with $|\bar{\mathscr{A}}| = 2^{2^f}$, $o(z_\gamma) \geqq p^k$, $\forall \gamma$.*

Proof. Suppose B contains a $Z(p^k)$. Write $G = \langle b \rangle \oplus Y$, $o(b) = p^k$, and let $\bigoplus_{\alpha \in A} \langle b_\alpha \rangle \subseteqq B$ with $|A| = f$, $o(b_\alpha) \geqq p^k \forall \alpha$.

Choose $\{A_\beta \mid \beta \in \mathscr{A}\}$ a collection of subsets of A such that: $|\mathscr{A}| = 2^f$, if F is any finite subset of $\mathscr{A}$ and $\beta_0 \in F$ then $[A_{\beta_0} \backslash \bigcup_{\beta \neq \beta_0, \beta \in F} A_\beta] \neq \emptyset$. (See [1[, Lemma 46.2.)

For $\beta \in \mathscr{A}$ define $\delta_\beta \in \mathrm{Hom}(\bigoplus \langle b_\alpha \rangle, \langle b \rangle)$ by $\delta_\beta(b_\alpha) = X_\beta(\alpha)b - X_\beta$ the characteristic function of A_β. Extend each δ_β to $\mathscr{E}$.

It is clear that the left ideals $\mathscr{E}\delta_\beta$ form a direct sum s in $\mathscr{E}$.

Let $\{C_\gamma \mid \gamma \in \bar{\mathscr{A}}\}$ be a family of subsets of $\mathscr{A}$ with the above independence property, $|\bar{\mathscr{A}}| = 2^{2^f}$. Consider:

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, "\lambda_\gamma"] & \mathscr{E} \arrow[dl, dashed, "\lambda'_\gamma"] \\ & I(T) & \end{tikzcd}$$

Here λ_γ is the $\mathscr{E}$ map defined by $\lambda_\gamma(\delta_\beta) = X_{C_\gamma}(\beta)b$, X_{C_γ} the characteristic function of the subset C_γ, and λ'_γ is the map obtained by injectively.

Let $z_\gamma = \lambda'_\gamma(1)$. We have $\delta_\beta(z_\gamma) = X_{C_\gamma}(\beta)b$. It is easy to see from this equation that $\{z_X \mid X \in \bar{\mathscr{A}}\}$ is a p independent set of elements of order $\geqq p^k$. This can be included as a summand of B'. The result follows.

Continuing with the same notation we have:

THEOREM 12. *If B' contains a $Z(p^k)$ so does B.*

Proof. If B' contains $Z(p^k)$ then $I(T)$ has a $Z(p^k)$ summand.

Therefore, so does Hom $(\mathscr{E}, D(T))$. ($I(T)$ can be regarded as a direct summand of Hom $(\mathscr{E}, D(T))$. Therefore, so does Hom $(\mathscr{E}, D(T)_p)$.

The pure exact sequence $0 \to t(\mathscr{E}) \to \mathscr{E} \to \mathscr{E}/t(\mathscr{E}) \to 0$ yields $0 \to [\mathscr{E}/t(\mathscr{E})]^* \to \mathscr{E}^* \to t(\mathscr{E})^* \to 0$, where $M^* = \mathrm{Hom}_Z(M, D(T)_p)$. This sequence is pure exact, so splits, since all its terms are algebraically compact. (In this proof "splits" means splits as an exact sequence of abelian groups.) Since $[\mathscr{E}/t(\mathscr{E})]^*$ is torsion free, $t(\mathscr{E})^*$ must have a $Z(p^k)$ summand.

Now $t(\mathscr{E})^* = [t(\mathscr{E})_p]^*$. Let B_0 be a basic subgroup for $t(\mathscr{E})_p$. Repeat the above procedure with $0 \to B_0 \to t(\mathscr{E})_p \to t(\mathscr{E})_p/B_0 \to 0$ to conclude that B_0^* must have a $Z(p^k)$ summand.

Since B_0 is a direct sum of cyclics, B_0 itself must have a $Z(p^k)$ summand. Thus, $\mathscr{E}$ and, therefore, Hom (G, T_p) have $Z(p^k)$ summands.

Let $\bar{B}$ be a p-basic subgroup for G. The p-pure exact sequence $0 \to \bar{B} \to G \to G/\bar{B} \to 0$ yields the p-pure exact sequence $0 \to (G/\bar{B})^\Delta \to G^\Delta \to (\bar{B})^\Delta$ where $M^\Delta = \mathrm{Hom}_Z(M, T_p)$. Since $(G/\bar{B})^\Delta \cong W \oplus \bigoplus_\gamma Q_\gamma$, where W is the p-adic completion of a direct sum of copies of the p-adic integers, this sequence also splits. It's not hard to show that $(\bar{B})^\Delta$ must have a $Z(p^k)$ summand.

Say $\bar{B} = \bar{B}_1 \oplus \bar{B}_2$, where $\bar{B}_1 = \bigoplus_\alpha Z(p^{l_\alpha})$ is a direct sum of finite p-power cyclics and $\bar{B}_2 = \bigoplus_\beta Z_\beta$ is free. Then $\bar{B}^\Delta = (\bar{B}_1)^\Delta \oplus (\bar{B}_2)^\Delta$, so one of these groups must contain a $Z(p^k)$ summand.

If $(\bar{B}_1)^\Delta \cong \prod_\alpha T_p[p^{l_\alpha}]$ has a $Z(p^k)$ summand, then $\bar{B}_1$ itself must, so T does.

If $(\bar{B}_2)^\Delta \cong \prod = \prod_\beta (T_p)_\beta$ has a $Z(p^k)$ summand, again T does. (If $\prod = \langle y \rangle \oplus Y$, $o(y) = p^k$, then $h_p^{\Pi}(p^{k-1}y) = k-1$. If $y = [y_\beta]$, $y_\beta \in (T_p)_\beta$, then, for some β_0, $h_p^{(T_p)\beta_0}(p^{k-1}y_{\beta_0}) = k-1$ and, therefore, $o(p^{k-1}y_{\beta_0}) = p$. Thus, y_{β_0} is contained in a $Z(p^k)$ summand of $(T_p)_{\beta_0}$.)

Thus, in either of the above cases, B contains a $Z(p^k)$.

In view of Theorem 5, it is of interest to discover when $Q \subseteq I(T)$. (Obviously, we must have $T^1 \neq (0)$.) We are unable to decide if $T^1 \neq (0)$ is also sufficient for $Q \subseteq I(T)$. We close the paper with a result in this direction. First, we need two lemmas.

LEMMA 2. *Let* $T = \bigoplus T_p \subset G$ *and suppose* $T_p^1 \neq (0)$ *whenever* $T_p \neq (0)$. *Then* ${}_{\mathscr{E}}T^1 \operatorname{ess} {}_{\mathscr{E}}T$.

Proof. If $t \in T \backslash T^1$, then $\Pi(t) \neq 0$, Π the projection onto $\langle a \rangle$, some $Z(p^k)$ summand of G. It is easy to construct $\theta \in \mathrm{Hom}_Z(\langle a \rangle, T_p^1)$ with $\theta\Pi(t) \neq 0$. Thus, ${}_{\mathscr{E}}T^1 \operatorname{ess} {}_{\mathscr{E}}T$.

Let $\bar{\mathscr{E}} = \mathscr{E}/t(\mathscr{E})$. Since $t(\mathscr{E})T^1 = (0)$ we can regard T^1 as an $\bar{\mathscr{E}}$ module.

LEMMA 3. *Let* $\mathscr{I}$ *be the* $\bar{\mathscr{E}}$ *injective hull of* T^1 *and let* D *be* *the maximal divisible subgroup of* $I(T)$. *Then, under the assumption of Lemma* 2, $\mathscr{I} \cong D$.

Proof. By Lemma 2, ${}_{\mathscr{E}}T^1 \operatorname{ess} {}_{\mathscr{E}}T$, so $I_{\mathscr{E}}(T^1) = I(T)$.

Now $\mathscr{I}$ is an $\mathscr{E}$ essential extension of T^1, so we can regard $\mathscr{I} \subset I_{\mathscr{E}}(T^1) = I(T)$. Since $\mathscr{I}$ is an injective module over a ring with torsion free additive group, $\mathscr{I} \subseteq D$. But D is an $\bar{\mathscr{E}}$ essential extension of T^1. Thus, $\mathscr{I} = D$.

THEOREM 13. *Let* $E = \operatorname{End} T$, $\bar{E} = E/t(E)$ *and suppose* $R: \bar{\mathscr{E}} \to \bar{E}$ *is onto, where* R *is the restriction map. Then, if* T^1 *is unbounded,* $I(T) \supseteq \bigoplus_c Q$.

Proof. Let $T_1 = \{\bigoplus T_p \mid T_p^1 \neq 0\}$, $T_2 = \{\bigoplus T_p \mid T_p^1 = (0)\}$. Clearly, T_1 and T_2 are $\mathscr{E}$ submodules and $I(T) \cong I(T_1) \bigoplus I(T_2)$. It suffices to show $I(T_1) \supseteq \bigoplus_c Q$, so, without loss of generality, assume $T = T_1$. Then Lemma 3 applies, so it is enough to construct c independent elements of infinite order in $\mathscr{I} \cong D$.

Choose $\{x_i \mid i = 1, 2, 3, \cdots\} \subseteq T^1$ with $\{o(x_i) = p_i^{s_i}\}$ unbounded. For each fixed i, choose distinct $\bigoplus_{j=1}^{\infty} \langle b_{ij} \rangle$ part of a p_i-basic subgroup of T such that $\sum_{i,j} \langle b_{ij} \rangle$ is direct and such that $o(b_{ij}) \geqq p_i^{j^2}$. (Each T_p is reduced with $T_p^1 \neq (0)$, thus has an unbounded basic.) Finally, choose $\{x_{ij}\} \subseteq T$ with $p_i^j x_{ij} = x_i$.

Now define $\delta_i \in \operatorname{Hom}_Z (\bigoplus_j \langle b_{ij} \rangle, T_{p_i})$ by $\delta_i(b_{ij}) = x_{ij}$. Each δ_i is a small homomorphism (see [1], Lemma 46.3) so each δ_i extends to an endomorphism of T_{p_i} and, thus, to an endomorphism of T. Still call this extension δ_i.

LEMMA 4. $\sum_i \bar{\mathscr{E}} \bar{\delta}_i$ *is an* $\bar{\mathscr{E}}$ *direct sum in* $\bar{E}$. *Here* $\bar{\delta}_i = \delta_i + t(E)$ *and* $\bar{E}$ *is regarded as a left* $\bar{\mathscr{E}}$ *module in the natural way.*

The proof of Lemma 4 is not difficult and is left to the reader.

Let $\{N_\alpha \mid \alpha \in A\}$ be a family of subsets of the natural numbers with $|A| = c$ such that if $F \subseteq A$ is finite and $\alpha_0 \in F$ then $[N_{\alpha_0} \backslash \bigcup_{\alpha \in F, \alpha \neq \alpha_0} N_\alpha]$ is countable.

For all $\alpha \in A$, consider the diagram of $\bar{E}$ modules:

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_i \bar{\mathscr{E}} \bar{\delta}_i \arrow[r] \arrow[d, "\lambda_\alpha"] & \bar{E} \arrow[dl, dashed, "\lambda'_\alpha"] \\ & \mathscr{I} & \end{tikzcd}$$

Here λ_α is the $\bar{\mathscr{E}}$ map defined by $\lambda_\alpha(\bar{\delta}_i) = X_{N_\alpha}(i) x_i$, X_{N_α} the characteristic function of N_α, and λ'_α the $\bar{\mathscr{E}}$ map obtained by injectivity.

Set $z_\alpha = \lambda'_\alpha(\bar{1})$, $\bar{1}$ the identity of the ring $\bar{E}$. Since $R: \bar{\mathscr{E}} \to \bar{E}$

is onto, choose $\bar{\sigma}_i \in \bar{\mathscr{E}}$ with $R(\bar{\sigma}_i) = \bar{\bar{\delta}}_i$.

Then $\bar{\sigma}_i(z_\alpha) = \lambda'_\alpha(\bar{\sigma}_i\bar{1}) = \lambda'_\alpha(\bar{\bar{\delta}}_i) = X_{N_\alpha}(i)x_i$. This equation, together with $\{o(x_i)\}$ unbounded, easily implies that $\{z_\alpha \mid \alpha \in A\}$ is an independent set of elements of infinite order. Thus, $I(T) \supseteqq \bigoplus_c Q$.

COROLLARY. *Let T be a torsion group with T^1 unbounded and $E = \mathrm{End}\, T$. Then $I_E(T) \supseteqq \bigoplus_c Q$.*

Added in proof. The proof of Theorem 13 can be modified, using a procedure similar to that of Theorem 11, to construct $\bigoplus_{2^c} Q \subseteqq I(T)$.

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THE CLASS NUMBER OF $Q(\sqrt{-p})$ MODULO 4, FOR $p \equiv 3$ (MOD 4) A PRIME

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If p is a prime congruent to 3 modulo 4, it is well-known that the class number $h(-p)$ of the imaginary quadratic field $Q(\sqrt{-p})$ is odd. In this paper we determine $h(-p)$ modulo 4.

The class number of $Q(\sqrt{-p})$ is odd, if p is a prime congruent to 3 modulo 4 (see for example [3: p. 413]. D.H. Lehmer [4: p. 9] has posed the problem of determining the Jacobi symbol $(-1/h(-p))= (-1)^{(h(-p)-1)/2}$, that is, of determining $h(-p)$ modulo 4. In this paper we evaluate $h(-p)$ modulo 4 in terms of the class number $h(p)$ and the fundamental unit $\varepsilon_p = T + U\sqrt{p}$ of the corresponding real quadratic field $Q(\sqrt{p})$. It is known that T and U are positive integers which satisfy $T \equiv 0 \pmod 2$, $U \equiv 1 \pmod 2$, $N(\varepsilon_p) = T^2 - pU^2 = +1$. We prove

THEOREM. *If $p > 3$ is a prime congruent to* 3 *modulo* 4 *then*

$$h(-p) \equiv h(p) + U + 1 \pmod 4 . \tag{1}$$

It is easily checked that (1) does not hold for $p = 3$ ($h(-3) = h(3) = U = 1$). ($p = 3$ is a special case as this is the only value of $p \equiv 3 \pmod 4$ for which the ring of integers of $Q(\sqrt{-p})$ has more than 2 units.) The method of proof is purely analytic in nature, it uses Dirichlet's class number formula (in various forms) for both real and imaginary quadratic fields and also some results from cyclotomy. It would be of interest to give a purely algebraic proof.

Proof. Let $p > 3$ be a prime congruent to 3 modulo 4 and set $\rho = \exp(2\pi i/p)$. For z a complex variable, we let

$$F_+(z) = \prod_{\substack{j=1 \\ (j/p)=+1}}^{p-1} (z - \rho^j),\ F_-(z) = \prod_{\substack{j=1 \\ (j/p)=-1}}^{p-1} (z - \rho^j) , \tag{2}$$

so that

$$F_+(z)F_-(z) = F(z) , \tag{3}$$

where $F(z)$ is the cyclotomic polynomial of index p, that is,

$$F(z) = \prod_{j=1}^{p-1} (z - \rho^j) = \frac{z^p - 1}{z - 1} = z^{p-1} + z^{p-2} + \cdots + z + 1 . \tag{4}$$

F_+ and F_- are polynomials in z of degree $(p-1)/2$ with coefficients in the ring of integers of $Q(\sqrt{-p})$ (see for example [6: p. 215]). Hence we can write

$$F_+(z) = \frac{1}{2}(Y(z) - Z(z)\sqrt{-p}) , \quad F_-(z) = \frac{1}{2}(Y(z) + Z(z)\sqrt{-p}) , \tag{5}$$

where Y and Z are polynomials with rational integral coefficients. From (3) and (5) we have

$$Y(z)^2 + pZ(z)^2 = 4F(z) . \tag{6}$$

It is also known [6: p.216] or [7: p. 209] that Y and Z have the symmetry properties expressed by

$$Y(z) = \sum_{n=0}^{(p-3)/4} a_n(z^{(p-1)/2-n} - z^n) , \quad Z(z) = \sum_{n=0}^{(p-3)/4} b_n(z^{(p-1)/2-n} + z^n) , \tag{7}$$

where the a_n and b_n are integers with

$$a_0 = 2,\ a_1 = 1,\ a_2 = (3-p)/4 , \cdots$$

and

$$b_0 = 0,\ b_1 = 1,\ b_2 = \frac{1}{2}\left(1 + \left(\frac{2}{p}\right)\right) , \cdots$$

(see [7] for further values of a_n and b_n: see [6] for a table of values of Y and Z for $p \leqq 29$).

Differentiating the expressions in (7) and (6) with respect to z, we obtain respectively

$$\begin{aligned} Y'(z) &= \sum_{n=0}^{(p-3)/4} a_n \left(\left(\frac{p-1}{2} - n\right) z^{(p-3)/2-n} - nz^{n-1}\right) , \\ Z'(z) &= \sum_{n=0}^{(p-3)/4} b_n \left(\left(\frac{p-1}{2} - n\right) z^{(p-3)/2-n} + nz^{n-1}\right) , \end{aligned} \tag{8}$$

and

$$Y(z)Y'(z) + pZ(z)Z'(z) = 2F'(z) . \tag{9}$$

Taking $z = i$ in (7) and (8) we obtain

$$\begin{aligned} Y(i) &= \begin{cases} A_3(1-i), \text{ if } p \equiv 3 \pmod 8 , \\ A_7(1+i), \text{ if } p \equiv 7 \pmod 8 , \end{cases} \\ Z(i) &= \begin{cases} -B_3(1+i), \text{ if } p \equiv 3 \pmod 8 , \\ B_7(1-i), \text{ if } p \equiv 7 \pmod 8 , \end{cases} \end{aligned} \tag{10}$$

and

$$(11)\qquad Y'(i)=\begin{cases}C_3+2D_3i, \text{ if } p\equiv 3 \pmod 8,\\ C_7+2D_7i, \text{ if } p\equiv 7 \pmod 8,\end{cases}\quad Z'(i)=\begin{cases}E_3+2F_3i, \text{ if } p\equiv 3 \pmod 8,\\ E_7+2F_7i, \text{ if } p\equiv 7 \pmod 8,\end{cases}$$

where $A_3, \cdots, F_7$ are rational integers (given in terms of the a_n and b_n). Using (10) and (11) in (6) and (9) with $z=i$, we obtain

$$(12)\qquad \begin{cases}A_3^2-pB_3^2=-2, \text{ if } p\equiv 3 \pmod 8,\\ A_7^2-pB_7^2=+2, \text{ if } p\equiv 7 \pmod 8,\end{cases}$$

and

$$(13)\qquad \begin{cases}A_3C_3+2pB_3F_3=-1,\ 2A_3D_3-pB_3E_3=p, \text{ if } p\equiv 3 \pmod 8,\\ A_7C_7+2pB_7F_7=p,\ 2A_7D_7-pB_7E_7=1, \text{ if } p\equiv 7 \pmod 8.\end{cases}$$

Clearly from (12) and (13) we see that A_3, B_3, C_3, E_3, A_7, B_7, C_7 and E_7 are all odd. Now Liouville [5: p. 415] has shown that

$$(14)\qquad Z(z)Y'(z)-Z'(z)Y(z)=\frac{2}{z-1}\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)z^{p-1-j}.$$

Taking $z=i$ in (14) we obtain

$$(15)\qquad Z(i)Y'(i)-Z'(i)Y(i)=(L+M)+i(L-M),$$

where

$$L=\sum_{j=0}^{(p-1)/2}(-1)^j\left(\frac{2j}{p}\right),\quad M=\sum_{j=0}^{(p-1)/2}(-1)^j\left(\frac{2j+1}{p}\right).$$

Applying the transformation $j\to(p-1)/2-j$ to L or M we obtain $L=M$. Also we have

$$\begin{aligned}L&=\sum_{j=1}^{(p-3)/4}\left(\frac{4j}{p}\right)-\sum_{j=0}^{(p-3)/4}\left(\frac{4j+2}{p}\right)\\ &=\sum_{j=1}^{(p-3)/4}\left(\frac{j}{p}\right)-\sum_{j=(p+1)/4}^{(p-1)/2}\left(\frac{4((p-1)/2-j)+2}{p}\right)\\ &=\sum_{j=1}^{(p-3)4}\left(\frac{j}{p}\right)+\sum_{j=(p+1)/4}^{(p-1)/2}\left(\frac{j}{p}\right)=\sum_{j=1}^{(p-1)/2}\left(\frac{j}{p}\right),\end{aligned}$$

so, by Dirichlet's class number formula (as $p\equiv 3 \pmod 4$, $p<3$) see for example [2: p. 346], we have

$$(16)\qquad L=M=\left\{2-\left(\frac{2}{p}\right)\right\}h(-p).$$

Hence from (15) and (16) we have

$$Z(i)Y'(i) - Z'(i)Y(i) = 2\left\{2 - \left(\frac{2}{p}\right)\right\}h(-p)\,. \tag{17}$$

Using (10) and (11) in (17), after equating real and imaginary parts, we obtain

$$\begin{cases} 3h(-p) = 2B_3D_3 - A_3E_3\,, & \text{if } p \equiv 3 \pmod 8\,, \\ h(-p) = 2B_7D_7 - A_7E_7\,, & \text{if } p \equiv 7 \pmod 8\,. \end{cases} \tag{18}$$

Now from (13) we have

$$\begin{cases} E_3 \equiv -2A_3B_3D_3 - B_3 \pmod 8, \text{ if } p \equiv 3 \pmod 8\,, \\ E_7 \equiv -2A_7B_7D_7 + B_7 \pmod 8, \text{ if } p \equiv 7 \pmod 8\,. \end{cases} \tag{19}$$

Using (19) in (18) we have

$$h(-p) \equiv \begin{cases} -A_3B_3 \pmod 4, \text{ if } p \equiv 3 \pmod 8\,, \\ -A_7B_7 \pmod 4, \text{ if } p \equiv 7 \pmod 8\,. \end{cases} \tag{20}$$

From (4) we have $F(i) = i$, and so taking $z = i$ in (2) and (3) we obtain

$$\begin{aligned} -i\{F_-(i)\}^2 = \frac{F_-(i)}{F_+(i)} &= \prod_{j=1}^{p-1}(1 + i\rho^j)^{-(j/p)} \\ &= \exp\left(-\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)\log(1 + i\rho^j)\right) \\ &= \exp\left(\sum_{n=1}^{\infty}\frac{(-i)^n}{n}\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)\rho^{nj}\right) \\ &= \exp\left(i\sqrt{p}\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\frac{(-i)^n}{n}\right) \\ &= \exp\left(\sqrt{p}\sum_{m=0}^{\infty}\left(\frac{2m+1}{p}\right)\frac{(-1)^m}{2m+1} + \frac{i\sqrt{p}}{2}\left(\frac{2}{p}\right)\sum_{m=1}^{\infty}\left(\frac{m}{p}\right)\frac{(-1)^m}{m}\right) \\ &= \exp\left(h(p)\log(T + U\sqrt{p}) + \frac{\pi i}{2}\left(1 - \left(\frac{2}{p}\right)\right)h(-p)\right) \\ &= (T + U\sqrt{p})^{h(p)}i^{(1-(2/p))h(-p)} \\ &= (-1)^{(p+1)/4}(T + U\sqrt{p})^{h(p)}\,, \end{aligned}$$

where we have made use of the Gauss sum

$$\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)\rho^{nj} = \left(\frac{n}{p}\right)i\sqrt{p}\,,$$

and the two results

$$\sum_{m=1}^{\infty}\left(\frac{m}{p}\right)\frac{(-1)^m}{m} = \frac{\pi}{\sqrt{p}}\left(\left(\frac{2}{p}\right) - 1\right)h(-p)$$

and

$$\sum_{m=0}^{\infty}\left(\frac{2m+1}{p}\right)\frac{(-1)^m}{2m+1} = \frac{h(p)}{\sqrt{p}}\log(T + U\sqrt{p})\ ,$$

which follow easily by standard arguments from Dirichlet's class number formula (see for example [2: p. 343]). Hence we have (using (10))

$$\begin{aligned}(T + U\sqrt{p})^{h(p)} &= (-1)^{(p-3)/4} iF_{-}(i)^2 \\ &= (-1)^{(p-3)/4} i\left\{\frac{1}{2}(Y(i) + Z(i)i\sqrt{p})\right\}^2 \\ &= \begin{cases}\dfrac{1}{2}(A_3 + B_3\sqrt{p})^2, \text{ if } p \equiv 3 \pmod 8\ , \\ \dfrac{1}{2}(A_7 + B_7\sqrt{p})^2, \text{ if } p \equiv 7 \pmod 8\ .\end{cases}\end{aligned}$$

This is essentially a result of Arndt [1].

Expanding $(T + U\sqrt{p})^{h(p)}$ by the binomial theorem and equating coefficients of $\sqrt{p}$, we have as $h(p) \equiv 1 \pmod 2$,

$$\begin{aligned}&U^{h(p)}p^{(h(p)-1)/2} + \binom{h(p)}{2}U^{h(p)-2}T^2p^{(h(p)-3)/2} + \cdots \\ &\qquad = \begin{cases}A_3B_3, \text{ if } p \equiv 3 \pmod 8\ , \\ A_7B_7, \text{ if } p \equiv 7 \pmod 8\ .\end{cases}\end{aligned}$$

As $T \equiv 0 \pmod 2$, $U \equiv 1 \pmod 2$, this gives

$$U(-1)^{(h(p)-1)/2} \equiv \begin{cases}A_3B_3 \pmod 4, \text{ if } p \equiv 3 \pmod 8\ , \\ A_7B_7 \pmod 4, \text{ if } p \equiv 7 \pmod 8\ ,\end{cases}$$

so that

$$h(p) \equiv \begin{cases}A_3B_3 - U + 1 \pmod 4, \text{ if } p \equiv 3 \pmod 8\ , \\ A_7B_7 - U + 1 \pmod 4, \text{ if } p \equiv 7 \pmod 8\ .\end{cases} \tag{21}$$

Putting (20) and (21) together, we obtain (1) as required.

From (1) we have $(-1/h(-p)) = (-1)^{(h(-p)-1)/2} = (-1)^{(h(p)+U)/2}$. In particular whenever $h(p) = 1$ (a common occurrence) we have $(-1/h(-p)) = (-1)^{(U+1)/2}$.

In [8] the author has treated, in a similar way, Lehmer's question [4: p. 10] regarding $h(-2p)$ modulo 8, when p is a prime congruent to 5 modulo 8.

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ON TOPOLOGICAL ANALOGUES OF LEFT THICK SUBSETS IN SEMIGROUPS

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We discuss the relation among various topological analogues of left thickness in semigroups and their connection with left invariant means for locally compact separately continuous semigroups. Until now, most results in this direction have been obtained for only jointly continuous semigroups. However, an important convolution formula found recently by this author made the transition to separately continuous cases possible.

1. **Introduction.** Let S be a semigroup and T a subset of S. T is called left thick if for each finite set $F \subset S$, there is some $s \in S$ such that $Fs \subset T$. In 1965, T. Mitchell obtained the following interesting results:

THEOREM 1.1 (*Mitchell* [7]). *Let S be a left amenable semigroup and T a subset of S, then T is left thick iff there is a left invariant mean m on S such that $m(\xi_T) = 1$ where ξ_T is the characteristic function of T.*

THEOREM 1.2 (*Mitchell* [7]). *If T is a left thick subsemigroup of a semigroup S, then S is left amenable iff T is left amenable.*

Since then, various attempts have been made to obtain topological analogues and extensions of these concepts and results to locally compact semigroups (with jointly continuous multiplication) with only partial success (see Day [3], [4] and Wong [10], [11]). In fact, in these attempts, a topological analogue of one or the other (but not both) of Mitchell's theorems was found. The purpose of this paper is two-fold. First, we introduce a "suitable" topological analogue of left thickness and extend *both* of Mitchell's results. Second, we shall do it in the more general setting of locally compact *separately* continuous semigroups because of an important convolution formula obtained recently by this author for such semigroups (see Wong [12] and § 2 below).

§ 2. **Notations and background.** For notations and definitions in analysis on locally compact (Hausdorff) semigroups, we shall follow [11] (to which the present paper is a sequel) except that we are now dealing with a locally compact *separately* continuous semigroup S. Although all the results cited in the references here are for jointly continuous semigroups (or compact separately continuous ones), many of them (in particular, those we are going to need here) can be carried over to general separately continuous semigroups. We shall discuss this briefly here and where appropriate, special remarks with respect to this will be made below.

As usual, let $M(S)$ be the measure algebra with convolution product and $M_0(S)$ the probability measures. Recently, this author has obtained the following convolution formula:

$$\int f d\mu*\nu = \iint f(xy)d\mu(x)d\nu(y) = \iint f(xy)d\nu(y)d\mu(x)$$

for all $f \in L_1(|\mu|*|\nu|)$, $\mu, \nu \in M(S)$. (See Wong [12].) It follows that $M_0(S)$ is a convolution semigroup (algebraically) and that

$$\text{support}\,(\mu*\nu) = [\text{support}\,\mu\cdot\text{support}\,\nu]^-$$

if $\mu, \nu \in M_0(S)$.

As a result, if S is jointly continuous, $\mu*\nu$ has compact support whenever μ, ν do. In general, this may not be the case except for example when $\mu = \delta_a$ is the Dirac measure and ν has compact support. Then $\delta_a*\nu$ has compact support ($=a\cdot$support ν).

Also, the functions $x \to f(xy)$, y fixed and $x \to \int f(xy)d\nu(y)$, though continuous, need not be in $C_0(S)$ if f is in $C_0(S)$, the continuous functions on S which vanish at infinity. Thus $M_0(S)$ need not be a topological semigroup under the weak* topology of $M(S) = C_0(S)^*$. Despite this apparent setback for separately continuous semigroups, it should however be remarked that for example the construction used in Wong [11, Lemma 3.1, p. 296] is valid for separately continuous semigroups since it requires only that $M_0(S)$ be a semigroup.

Now let T be a Borel subset of S. Consider the following conditions on T:

(TLS) For each $K \subset S$ compact, there is some $\mu \in M_0(S)$ such that $\nu*\mu(T) = 1$ for any $\nu \in M_0(S)$ with $\nu(K) = 1$. [Can assume $\mu(T) = 1$.]

(TLT) For each $\varepsilon > 0$, $K \subset S$ compact, there is some $\mu \in M_0(S)$ such that $\nu*\mu(T) > 1 - \varepsilon$ for any $\nu \in M_0(S)$ with $\nu(K) = 1$. [Can assume that μ has compact support and $\mu(T) > 1 - \varepsilon$.]

(TLL_1) For each $\varepsilon > 0$ and $\nu \in M_0(S)$ with compact support, there is some μ in $M_0(S)$ with compact support such that $\nu*\mu(T) > 1 - \varepsilon$.

(TLL) For each $\varepsilon < 0$ and $\nu \in M_0(S)$ with compact support, there is some $s \in S$ such that $\nu*\delta_s(T) > 1 - \varepsilon$ where δ_s is the Dirac measure at s.

(LT) For each $F \subset S$ finite, there is some $s \in S$ such that $Fs \subset$

T [Can assume $s \in T$.] (This is Mitchell's definition of left thickness.) and

(*) For each $\varepsilon > 0$ and $\nu \in M_0(S)$ with compact support, there is some $\mu \in M_0(S)$ with compact support such that $\mu(T) > 1 - \varepsilon$ and $\nu * \mu(T) > 1 - \varepsilon$.

T is called *topological left substantial* if T satisfies (TLS). In Wong [10], it is proved that if T is a (locally compact Borel) topological left substantial *subsemigroup* of S, then T is topological left amenable iff S is. This is a topological analogue and extension of Theorem 1.2. Also condition (TLS) remains unchanged if we require the measure μ to satisfy the additional assumption that $\mu(T) = 1$. The proof can be found in [10]. Since similar situations will frequently occur again below, we present the proof here for completeness. As in [10], if $\phi \neq K \subset S$ compact is given, choose $k \in K$ and let $K_1 = Kk \cup \{k\}$ which is also compact. There is some $\mu_1 \in M_0(S)$ such that $\nu_1 * \mu_1(T) = 1$ if $\nu_1 \in M_0(S)$ and $\nu_1(K_1) = 1$. Consider $\mu = \delta_k * \mu_1 \in M_0(S)$. $\mu(T) = 1$ since $\delta_k(K_1) = 1$. Moreover, if $\nu \in M_0(S)$ and $\nu(K) = 1$, then

$$\begin{aligned}\nu * \mu(T) = (\nu * \delta_k) * \mu_1(T) = 1 \quad \text{since} \quad & \nu * \delta_k(K_1) \\ &= \int_K \xi_{K_1}(xk) d\nu(x) = \nu(K) = 1 .\end{aligned}$$

On the other hand, T is called *topological left thick* if T satisfies (TLT). It is proved in Wong [11] that if S is uniform strong topological left amenable (hence topological left amenable), then T is topological left thick iff there is a topological left invariant mean M on $M(S)^*$ such that $M(\chi_T) = 1$ where χ_T is the characteristic functional of T in S (see [11] for more details). This is a topological analogue and extension of Theorem 1.2. Condition (TLT) remains unchanged if we require the measure μ to satisfy the additional assumptions that μ has compact support and $\mu(T) > 1 - \varepsilon$. For if $\varepsilon > o$ and $K \subset S$ compact are given, there is some $\mu \in M_0(S)$ such that $\nu * \mu(T) > 1 - \varepsilon/2$ for all $\nu \in M_0(S)$ with $\nu(K) = 1$. Since the measures in $M_0(S)$ with compact supports are norm dense in $M_0(S)$, we can choose $\mu_1 \in M_0(S)$ with compact support such that $\|\mu - \mu_1\| < \varepsilon/2$, then

$$|(\nu * \mu_1 - \nu * \mu)(T)| \leqq \|\nu * \mu_1 - \nu * \mu\| < \frac{\varepsilon}{2}$$

and $\nu * \mu_1(T) > 1 - \varepsilon$ for all $\nu \in M_0(S)$ with $\nu(K) = 1$. Next, suppose the pair (ε, K) is given and $K \neq \phi$. Choose $k \in K$ and let $K_1 = Kk \cup \{k\}$ which is compact. By the above argument, there is some $\mu_2 \in$

$M_0(S)$ with compact support such that

$$\tau * \mu_2(T) > 1 - \varepsilon \text{ for all } \tau \in M_0(S)$$

with $\tau(K_1) = 1$. Consider $\mu_3 = \delta_k * \mu_2 \in M_0(S)$, which has compact support ($= k \cdot$support μ_2), $\mu_3(T) = \delta_k * \mu_2(T) > 1 - \varepsilon$ since $\delta_k(K_1) = 1$. Moreover, if $\nu \in M_0(S)$ and $\nu(K) = 1$, then $\nu * \mu_3(T) = (\nu * \delta_k) * \mu_2(T) > 1 - \varepsilon$ since $\nu * \delta_k(K_1) = \int_K \xi_{K_1}(xk) d\nu(x) = \nu(K) = 1$.

Later, M. Day [4] improves the result in Wong [11, Theorem 4.1, p. 297] by calling T *topological left lumpy* if T satisfies (TLL) and proves that if S is topological left amenable, then T is topological left lumpy iff there is a topological left invariant mean M on $M(S)^*$ such that $M(\chi_T) = 1$. Thus for uniform strong topological left amenable semigroups (in particular, any left amenable locally compact *group*), the concepts of (TLT) and (TLL) are the same.

In general, of course (TLS) implies (TLT) which in turn implies (TLL_1). Also (TLL) and (TLL_1) are equivalent. This is due to Day [4] (under further but redundant assumption). Clearly (TLL) implies (TLL_1). Conversely, if $\nu \in M_0(S)$ has compact support and $\nu * \delta_S(T) \leqq 1 - \varepsilon$ for all s in S, then $\nu * \mu(T) = \int \nu(Ts^{-1}) d\mu(s) = \int \nu * \delta_s(T) d\mu(s) \leqq 1 - \varepsilon$ for all $\mu \in M_0(S)$. Hence (TLL_1) and (TLL) are equivalent.

Also (TLL) implies (LT). The proof is implicit in Day [4]. For given any finite $F \subset S$ with k elements, consider $\nu = 1/k \sum_{\sigma \in F} \delta_\sigma \in M_0(S)$ with compact support. By (TLL), there is some $s \in S$ such that $\nu * \delta_s(T) > 1 - 1/k$. Hence $\delta_{\sigma s}(T) = 1$ for all $\sigma \in F$ or $Fs \subset T$.

Finally, condition (*) is somewhere between topological left thickness and topological left lumpiness. Clearly (*) is formally stronger than (TLL_1). Also (TLT) implies (*) in view of the above remarks concerning the additional assumptions at the end of the condition (TLT).

This condition (*) is precisely the "suitable" condition we are looking for in order to extend both Mitchell's results.

3. Main results.

THEOREM 3.1. *Let T be a Borel subset of a locally compact semigroup S such that $M(S)^*$ has a topological left invariant mean. Then the following statements are equivalent*:

(1) *There is a topological left invariant mean M on $M(S)^*$ such that $M(\chi_T) = 1$.*

(2) *T is topological left lumpy (i.e., T satisfies (TLL) or (TLL_1)).*

(3) *T satisfies (*).*

Proof. Equivalence of (1) and (2) is due to Day [4, Theorem, p. 89]. Since the only difference between conditions (TLL_1) and (*) is that the measure μ in (*) must satisfy the additional assumption that $\mu(T) > 1 - \varepsilon$, Day's original proof in [4] can easily be adapted to show (1) implies (3). However, we shall present a modification of Day's argument to show that Theorem 3.1 remains valid if the measure μ in condition (*) is required to satisfy $\mu(T) = 1$. Suppose (1) holds and M is a topological left invariant mean such that $M(\chi_T) = 1$. Let μ_α be a net in $M_0(S)$ with compact supports such that $\mu_\alpha \to M$ weak* in $M(S)^{**}$. Then $\lim_\alpha \mu_\alpha(T) = 1$ and for each $\nu \in M_0(S)$ with compact support,

$$\nu * \mu_\alpha(T) = \chi_T(\nu * \mu_\alpha) = \mu_\alpha(\nu \odot \chi_T) \longrightarrow M(\nu \odot \chi_T) = M(\chi_T) = 1 ,$$

since M is topological left invariant.

Define $\tau_\alpha \in M^+(S)$ by

$$\int f d\tau_\alpha = \int \xi_T f d\mu_\alpha,\ f \in C_0(S) .$$

Then $\tau_\alpha(B) = \mu_\alpha(B \cap T)$ for any Borel set B in S. In particular, $\tau_\alpha(T) = \mu_\alpha(T) \to 1$. Hence we can assume $\tau_\alpha(T) \neq 0$. Let $\nu_\alpha \in M_0(S)$ be defined by $\nu_\alpha = \tau_\alpha/\tau_\alpha(T) = \tau_\alpha/\mu_\alpha(T)$. Then for any $f \in C_0(S)$, we have

$$\begin{aligned}\left|\int f d\nu_\alpha - \int f d\mu_\alpha\right| &= \left|\frac{1}{\mu_\alpha(T)}\int_T f d\mu_\alpha - \int f d\mu_\alpha\right| \\ &\leqq \left|\frac{1}{\mu_\alpha(T)}\int_T f d\mu_\alpha - \int_T f d\mu_\alpha\right| + \left|\int_{T'} f d\mu_\alpha\right| \\ &\leqq \|f\|_u \left|\frac{1}{\mu_\alpha(T)} - 1\right| + \|f\|_u \mu_\alpha(T') .\end{aligned}$$

Hence

$$\|\nu_\alpha - \mu_\alpha\| \leqq \left|\frac{1}{\mu_\alpha(T)} - 1\right| + \mu_\alpha(T') \longrightarrow 0 .$$

Let $\varepsilon > 0$ and $\nu \in M_0(S)$ with compact support be given, there is some α_0 (depending on ε and ν) such that

$$\|\nu_\alpha - \mu_\alpha\| < \frac{\varepsilon}{2}$$

and

$$\nu * \mu_\alpha(T) > 1 - \varepsilon/2 \text{ if } \alpha \geqq \alpha_0 .$$

Hence

$$\begin{aligned}|\nu*\nu_\alpha(T) - \nu*\mu_\alpha(T)|\\ &\leqq |\nu*\nu_\alpha - \nu*\mu_\alpha|(T)\\ &\leqq ||\nu_\alpha - \mu_\alpha|| < \frac{\varepsilon}{2} \text{ if } \alpha \geqq \alpha_0 .\end{aligned}$$

Consequently

$$\nu*\nu_{\alpha_0}(T) > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon .$$

The measure $\mu = \nu_{\alpha_0} \in M_0(S)$ has (compact) support $\subset K_0 \cap T^-$ where K_0 is the (compact) support of μ_{α_0} and μ satisfies the requirements

$$\mu(T) = 1 \text{ and } \nu*\mu(T) > 1 - \varepsilon .$$

This completes the proof.

REMARKS. It should be remarked that Day's result [4, Theorem, p. 89] is for jointly continuous semigroups. However, his proof (and the above adaptation) is actually valid for separately continuous semigroups (see also discussions at the beginning of § 2).

Theorem 3.1 is a topological analogue and extension of Theorem 1.1. It is also an improvement of Day's result in [4, Theorem, p. 89] (from joint continuity to separate continuity). It also shows that for *topological left amenable* semigroups conditions (TLL) and (*) are the same. It is not known in general whether condition (*) remains unchanged if we require the measure μ in (*) to satisfy $\mu(T) = 1$.

To obtain the analogue of Mitchell's second result, we need the following lemmas. From now on, unless otherwise stated explicitly, T is a locally compact Borel *subsemigroup* of S. It is known that if $\mu \in M(S)$, then the restriction $\mu_T = \mu|_T$ of μ to the Borel subsets of T is a measure in $M(T)$. In fact the correspondence $\mu \to \mu_T$ is an isometric order preserving isomorphism between the subalgebra of all measures $\mu \in M(S)$ with $|\mu|(T') = 0$ and the algebra $M(T)$. Moreover $\mu_T \in M_0(T)$ if $\mu \in M_0(S)$ and $\mu(T) = 1$ (see Wong [9] and [4] for details).

LEMMA 3.2. *Let $\mu, \nu \in M_0(S)$ with $\mu(T') \leqq \varepsilon$ and $\nu(T) = 1$. Then*
(1) $|\mu*\nu - \nu|(T') \leqq \varepsilon$
(2) $|\mu*\nu - \nu|(T) \leqq |\mu_T*\nu_T - \nu_T|(T) + \varepsilon$.
Consequently

$$||\mu*\nu - \nu|| \leqq ||\mu_T*\nu_T - \nu_T|| + 2\varepsilon .$$

Proof.
(1) Since $\mu, \nu \geqq 0$ and $\nu(T') = 0$, we have

$$\begin{aligned}
|\mu*\nu - \nu|(T') &\leqq |\mu*\nu|(T') + |\nu|(T') \\
&= \mu*\nu(T') = \iint \xi_{T'}(xy)d\mu(x)d\nu(y) \\
&= \int_T\int_T \xi_{T'}(xy)d\mu(x)d\nu(y) + \int_T\int_{T'} \xi_{T'}(xy)d\mu(x)d\nu(y) .
\end{aligned}$$

Since T is a subsemigroup, $T \cap T'y^{-1} = \phi$ if $y \in T$ and the first integral vanishes. Thus

$$|\mu*\nu - \nu|(T') \leqq \nu(T)\cdot\mu(T') \leqq \varepsilon .$$

(2) If $B \subset T$ is Borel in S then B is Borel in T and

$$\begin{aligned}
\mu*\nu(B) &= \iint \xi_B(xy)d\mu(x)d\nu(y) \\
&= \int_T\int_T \xi_B(xy)d\mu(x)d\nu(y) + \int_T\int_{T'} \xi_B(xy)d\mu(x)d\nu(y) \\
&= \mu_T*\nu_T(B) + \int_T\int_{T'} \xi_B(xy)d\mu(x)d\nu(y) ,
\end{aligned}$$

since T is a subsemigroup and $\nu(T') = 0$.

Hence if $\{B_1, B_2, \cdots, B_n\}$ is a partition of T into Borel sets in S, we have

$$|(\mu*\nu - \nu)(B_k)| \leqq |(\mu_T*\nu_T - \nu_T)(B_k)| + \int_T\int_{T'} \xi_{B_k}(xy)d\mu(x)d\nu(y)$$

and

$$\begin{aligned}
&|\mu*\nu - \nu|(T) \\
&\quad = \sup\left\{\sum_{k=1}^{n} |(\mu*\nu - \nu)(B_k)| : \{B_1, B_2, \cdots, B_n\} \text{ a Borel partition of } T \text{ in } S\right\} \\
&\quad \leqq |\mu_T*\nu_T - \nu_T|(T) + \int_T\int_{T'} \xi_T(xy)d\mu(x)d\nu(y) \\
&\quad \leqq |\mu_T*\nu_T - \nu_T|(T) + \varepsilon .
\end{aligned}$$

The last part of the lemma is now trivial.

LEMMA 3.3.

(1) *Let* ν_α *be a net in* $M_0(T)$ *such that* $||\nu*\nu_\alpha - \nu_\alpha|| \to 0$ *for each* $\nu \in M_0(T)$. *If* $0 < \varepsilon < 1$ *and* τ *is a measure in* $M^+(T)$ *such that* $1 - \varepsilon < \tau(T) = ||\tau|| \leqq 1$, *then there is some* α_0 *(depending on* τ *and* ε*) such that*

$$||\tau*\nu_\alpha - \nu_\alpha|| \leqq 2\varepsilon \text{ if } \alpha \geqq \alpha_0 .$$

(2) *Let ν_α be a net in $M_0(T)$ such that for each $F \subset T$ compact, $\|\nu*\nu_\alpha - \nu_\alpha\| \to 0$ uniformly for all $\nu \in M_0(T)$ with $\nu(F) = 1$. Let $0 < \varepsilon < 1$ and F a compact subset of T be given. Then there is some α_0 (depending on ε and F) such that for any $\tau \in M^+(T)$ with $\tau(T\backslash F) = 0$ and $1 - \varepsilon < \tau(T) = \|\tau\| \leqq 1$, we have*

$$\|\tau*\nu_\alpha - \nu_\alpha\| \leqq 2\varepsilon \text{ for } \alpha \geqq \alpha_0 .$$

Proof.

(1) Let $c = \|\tau\| \neq 0$ and write $\tau = c\nu$ with $\nu \in M_0(T)$. Then $0 \leqq 1 - c < \varepsilon$ and

$$\begin{aligned}\|\tau*\nu_\alpha - \nu_\alpha\| &= \|c(\nu*\nu_\alpha - \nu_\alpha) + c\nu_\alpha - \nu_\alpha\| \\ &\leqq \|\nu*\nu_\alpha - \nu_\alpha\| + |c - 1| \\ &\leqq 2\varepsilon \text{ if } \alpha \geqq \alpha_0 .\end{aligned}$$

(2) Let $0 < \varepsilon < 1$ and $F \subset T$ compact be given. There is some α_0 (depending on ε and F) such that

$$\|\nu*\nu_\alpha - \nu_\alpha\| < \varepsilon \text{ if } \alpha \geqq \alpha_0 \text{ and } \nu \in M_0(T)$$

such that $\nu(T\backslash F) = 0$. Let $\tau \in M^+(T)$ with $\tau(T\backslash F) = 0$ and $1 - \varepsilon < \tau(T) = \|\tau\| \leqq 1$. Write $\tau = c\nu$ where $c = \|\tau\| \neq 0$ and $\nu \in M_0(T)$. Then as before $0 \leqq 1 - c < \varepsilon$ and $\nu(T\backslash F) = 0$, and

$$\begin{aligned}\|\tau*\nu_\alpha - \nu_\alpha\| &\leqq \|\nu*\nu_\alpha - \nu_\alpha\| + |c - 1| \\ &\leqq 2\varepsilon \text{ if } \alpha \geqq \alpha_0 .\end{aligned}$$

THEOREM 3.4. *Let S be a locally compact semigroup and T a locally compact Borel subsemigroup of S satisfying condition (*) of § 2, then S is topological left amenable iff T is topological left amenable.*

Proof. Assume that S is topological left amenable (i.e., $M(S)^*$ has a topological left invariant mean). Since T satisfies (*), by Theorem 3.1, there is a topological left invariant mean M on $M(S)^*$ such that $M(\chi_T) = 1$. Therefore $M(T)^*$ also has a topological left invariant mean by a topological analogue (separately continuous version) of Day's well-known criterion for amenability of (discrete) subsemigroups (Day [1] and Wong [14, Theorem 4.1]).

Conversely, suppose $M(T)^*$ has a topological left invariant mean. Then there is a net ν_α in $M_0(T)$ such that $\|\tau*\nu_\alpha - \nu_\alpha\| \to 0$ for each $\tau \in M_0(T)$. Let μ_α be the unique measure in $M_0(T)$ with $\mu_\alpha(T') = 0$ and $\mu_{\alpha|T} = \nu_\alpha$. Suppose now $\nu \in M_0(S)$ has compact support. We claim that $\|\nu*\mu_\alpha - \mu_\alpha\| \to 0$. By (*), given $0 < \varepsilon < 1$, there is some $\mu \in M_0(S)$ with compact support such that

$$\mu(T) > 1 - \varepsilon \text{ and } \nu*\mu(T) > 1 - \varepsilon$$

(i.e., $\mu(T') < \varepsilon$ and $\nu*\mu(T') < \varepsilon$).

Now apply Lemma 3.3 (1) to the measures $\tau = \mu_T$ and $(\nu*\mu)_T$, there is some α_0 such that if $\alpha \geqq \alpha_0$

$$\|\mu_T*\nu_\alpha - \nu_\alpha\| < 2\varepsilon$$

and

$$\|(\nu*\mu)_T*\nu_\alpha - \nu_\alpha\| < 2\varepsilon .$$

By Lemma 3.2, if $\alpha \geqq \alpha_0$

$$\begin{aligned}\|\mu*\mu_\alpha - \mu_\alpha\| &\leqq \|\mu_T*\mu_{\alpha|T} - \mu_{\alpha|T}\| + 2\varepsilon \\ &\leqq \|\mu_T*\nu_\alpha - \nu_\alpha\| + 2\varepsilon \leqq 4\varepsilon\end{aligned}$$

and similarly

$$\begin{aligned}\|(\nu*\mu)*\mu_\alpha - \mu_\alpha\| &\leqq \|(\nu*\mu)_T*\mu_{\alpha|T} - \mu_{\alpha|T}\| \\ &\leqq \|(\nu*\mu)_T*\nu_\alpha - \nu_\alpha\| + 2\varepsilon \\ &\leqq 4\varepsilon .\end{aligned}$$

Consequently,

$$\begin{aligned}&\|\nu*\mu_\alpha - \mu_\alpha\| \\ &\quad\leqq \|\nu*\mu_\alpha - \nu*(\mu*\mu_\alpha)\| + \|(\nu*\mu)*\mu_\alpha - \mu_\alpha\| \\ &\quad\leqq \|\mu*\mu_\alpha - \mu_\alpha\| + \|(\nu*\mu)*\mu_\alpha - \mu_\alpha\| \\ &\quad\leqq 8\varepsilon \text{ if } \alpha \geqq \alpha_0 .\end{aligned}$$

Therefore S is topological left amenable and this completes the proof.

REMARKS. Theorem 3.4 is an extension of Wong [10, Theorem 3.2, p. 233].

4. **Uniform strong topological left amenability.** It is quite natural to ask whether Mitchell's second result has also an analogue for uniform strong topological left amenability. To answer this in affirmative, we need the following concept of left lumpiness first introduced by Day [4] for a Borel subset T in S (not necessarily a subsemigroup):

(LL) For each $K \subset S$ compact, there is some $s \in S$ such that $Ks \subset T$.

Like Mitchell's concept of left thickness, there is no loss of generality here in assuming that $s \in T$. Thus we have the following string of implications

$$(LI) \Longrightarrow (LL) \Longrightarrow (TLS) \Longrightarrow (TLT) \Longrightarrow (*) \Longrightarrow (TLL) \Longrightarrow (LT)$$

with (LI) which stands for left ideal being the strongest and Mitchell's (LT) the weakest of all these conditions.

THEOREM 4.1. *Let T be a locally compact Borel subsemigroup of a locally compact semigroup S. Consider the following statements:*

(a) *S is uniform strong topological left amenable*

(b) *T is uniform strong topological left amenable.*

If T satisfies (*), *then* (a) *implies* (b). *If T is left lumpy, then* (a) *and* (b) *are equivalent.*

Proof. Suppose T satisfies (*) and S is uniform strong topological left amenable. Then by Theorem 3.1, there is a topological left invariant mean M such that $M(\chi_T) = 1$. By [11, Lemma 3.1, p. 296, (separately continuous version, same proof)], we can assume that there is a net $\mu_\alpha \in M_0(S)$ such that for each compact set $K \subset S$, $||\mu*\mu_\alpha - \mu_\alpha|| \to 0$ uniformly for $\mu \in M_0(S)$ with $\mu(K) = 1$ and that $\mu_\alpha \to M$ weak* in $M(S)^{**}$. Define τ_α and ν_α as in the proof of Lemma 3.1 above and let $\theta_\alpha \in M(T)$ be defined by

$$\int g d\theta_\alpha = \int g' d\nu_\alpha , \quad g \in C_0(T)$$

where $g'(s) = g(s)$ if $s \in T$ and $g'(s) = 0$ if $s \notin T$. Then $\theta_\alpha = \nu_{\alpha|T} \in M_0(T)$. (See Wong [9] and [14, Lemma 3.1] (separately continuous versions).) Now let $F \subset T$ be compact and $\nu \in M_0(T)$ with $\nu(F) = 1$. Then there is a unique $\mu \in M_0(S)$ with $\mu(T') = 0$ and $\mu|_T = \nu$. Clearly $\mu(F) = 1$. Since $\mu(T') = 0$, $\nu_\alpha(T') = 0$, we have

$$\begin{aligned} ||\nu*\theta_\alpha - \theta_\alpha|| &= ||\mu_T*\nu_{\alpha|T} - \nu_{\alpha|T}|| = ||(\mu*\nu_\alpha - \nu_\alpha)|_T|| \\ &= ||\mu*\nu_\alpha - \nu_\alpha|| \\ &\leqq ||\mu*\nu_\alpha - \mu*\mu_\alpha|| + ||\mu*\mu_\alpha - \mu_\alpha|| + ||\mu_\alpha - \nu_\alpha|| \, . \end{aligned}$$

Now $||\mu_\alpha - \nu_\alpha|| \to 0$ and $||\mu|| = 1$, this last sum tends to zero uniformly for $\nu \in M_0(T)$ with $\nu(F) = 1$. Hence (a) implies (b).

If T satisfies (LL) which is stronger than (*), then (a) certainly implies (b). Conversely, suppose T is uniform strong topological left amenable. Let $\nu_\alpha \in M_0(T)$ be such that for any $F \subset T$ compact, $||\nu*\nu_\alpha - \nu_\alpha|| \to 0$ uniformly for $\nu \in M_0(T)$ with $\nu(F) = 1$. Let μ_α be the unique measure in $M_0(S)$ such that $\mu_{\alpha|T} = \nu_\alpha$, and $\mu_\alpha(T') = 0$. We claim that the net μ_α converges strongly to topological left invariance uniformly on compacta in S. Let $K \subset S$ be compact. By (LL), there is some $a \in T$ such that $Ka \subset T$. Then $F = Ka \cup \{a\}$ is a compact subset of T. Given $\varepsilon > 0$, there is some α_0 depending on (ε, F) such that

$$||\delta_t*\nu_\alpha - \nu_\alpha|| < \varepsilon \text{ if } \alpha \geqq \alpha_0 \text{ and } t \in Ka \cup \{a\} \, .$$

Therefore for any $\alpha \geqq \alpha_0$, $k \in K$, we have $ka \in Ka$ and

$$\begin{aligned}
&||\delta_k*\mu_\alpha - \mu_\alpha|| \\
&\quad \leqq ||\delta_k*\mu_\alpha - \delta_k*\delta_a*\mu_\alpha|| + ||\delta_{ka}*\mu_\alpha - \mu_\alpha|| \\
&\quad \leqq ||\delta_a*\mu_\alpha - \mu_\alpha|| + ||\delta_{ka}*\mu_\alpha - \mu_\alpha|| \\
&\quad = ||\delta_a*\nu_\alpha - \nu_\alpha|| + ||\delta_{ka}*\nu_\alpha - \nu_\alpha|| < 2\varepsilon .
\end{aligned}$$

This implies that S is uniform strong topological left amenable (See Day [4, (1sau) $\Leftrightarrow$ (W) pp. 88-89].) and the proof is complete.

If the semigroup S is jointly continuous, then the result can be partially strengthened.

THEOREM 4.2. *Let T be a closed topological left thick subsemigroup of a jointly continuous semigroup S. Then T is uniform strong topological left amenable iff S is.*

Proof. Since T is closed, T is necessarily locally compact Borel. Sufficiency is clear by Theorem 4.1. On the other hand, if T is uniform strong topological left amenable, there is a net $\nu_\alpha \in M_0(T)$ such that for each compact $F \subset T$, $||\nu*\nu_\alpha - \nu_\alpha|| \to 0$ uniformly for all $\nu \in M_0(T)$ with $\nu(F) = 1$. Again let $\mu_\alpha \in M_0(S)$ satisfy $\mu_\alpha(T') = 0$ and $\mu_{\alpha|T} = \nu_\alpha$. We claim that μ_α converges strongly to topological left invariance uniformly on compacta. Let $K \subset S$ be compact and $0 < \varepsilon < 1$. By (TLT), there is some $\mu_1 \in M_0(S)$ with compact support $K_1 \subset S$ such that $\mu_1(T) > 1 - \varepsilon$ and $\mu*\mu_1(T) > 1 - \varepsilon$ for all $\mu \in M_0(S)$ with $\mu(K) = 1$. Since T is closed and S is jointly continuous, both $F_1 = K_1 \cap T$ and $F_2 = K_2 \cap T$ where $K_2 = KK_1$ are compact subsets of T. So is $F = F_1 \cup F_2$. By Lemma 3.3 (2), there is some α_0, depending on (ε, F) such that for any $\tau \in M^+(T)$ with $\tau(T\backslash E) = 0$ and $1 - \varepsilon < \tau(T) = ||\tau|| \leqq 1$, we have

$$||\tau*\nu_\alpha - \nu_\alpha|| < 2\varepsilon \text{ if } \alpha \geqq \alpha_0 .$$

Now apply this to the measures $\tau = \mu_{1|T}$ and $(\mu*\mu_1)_T$ where $\mu \in M_0(S)$, $\mu(K) = 1$. We have

$$1 \geqq \mu_{1|T}(T) = \mu_1(T) > 1 - \varepsilon$$

and

$$\mu_{1|T}(T\backslash F) = \mu_1(T\backslash F) \leqq \mu_1(T\backslash F_1) = \mu_1(T \cap K_1') \leqq \mu_1(K_1') = 0 ,$$

since support $\mu_1 = K_1$.

Similarly,

$$1 \geqq (\mu*\mu_1)_T(T) = \mu*\mu_1(T) > 1 - \varepsilon$$

and

$$\begin{aligned}(\mu*\mu_1)_T(T\backslash F) &= \mu*\mu_1(T\backslash F) \leqq \mu*\mu_1(T\backslash F_2)\\ &= \mu*\mu_1(T\cap K_2') \leqq \mu*\mu_1(K_2')\\ &= 0\ ,\end{aligned}$$

since

$$\mu*\mu_1(K_2) = \iint \xi_{KK_1}(xy)d\mu(x)d\mu_1(y) = 1\ .$$

Hence by Lemma 3.3 (2),

$$\|\mu_{1|T}*\nu_\alpha - \nu_\alpha\| < 2\varepsilon$$

and

$$\|(\mu*\mu_1)_T*\nu_\alpha - \nu_\alpha\| < 2\varepsilon \text{ for all } \alpha \geqq \alpha_0,\ \mu\in M_0(S) \text{ with } \mu(K) = 1\ .$$

Consequently, for all $\alpha \geqq \alpha_0$, $\mu \in M_0(S)$, $\mu(K) = 1$, we have

$$\begin{aligned}&\|\mu*\mu_\alpha - \mu_\alpha\|\\ &\quad = \|\mu*\mu_\alpha - \mu*(\mu_1*\mu_\alpha)\| + \|(\mu*\mu_1)*\mu_\alpha - \mu_\alpha\|\\ &\quad \leqq \|\mu_1*\mu_\alpha - \mu_\alpha\| + \|(\mu*\mu_1)*\mu_\alpha - \mu_\alpha\|\\ &\quad \leqq \|\mu_{1|T}*\nu_\alpha - \nu_\alpha\| + \|(\mu*\mu_1)_T*\nu_\alpha - \nu_\alpha\| + 4\varepsilon \leqq 8\varepsilon\ ,\end{aligned}$$

by Lemma 3.2 and above. This completes the proof.

5. Pointwise strong left amenability. As mentioned in Day [4], an analogue of Theorem 1.1 is still needed for left amenable locally compact semigroups which characterizes those subsets on which some left invariant mean can be concentrated. He also remarked without proof that to obtain a left invariant mean which concentrates on a Borel subset T, under the assumption that T is left thick, would require something like left amenability of S regarded as a discrete semigroup which is not a common property of left amenable locally compact semigroups.

In this section, we shall first show that if S is left amenable as a discrete semigroup, then S is left amenable as a locally compact semigroup and then supply a proof of Day's remark, using an elegant application of the fixed point property for left amenable discrete semigroups. Also we shall obtain an analogue of Mitchell's second result (Theorem 1.2).

THEOREM 5.1. *Let S be a locally compact semigroup which is left amenable as a discrete semigroup, then S is left amenable. In this case, if T is a Borel subset of S, then the following statements are equivalent*:

(1) *There is a left invariant mean* M *on* $M(S)^*$ *such that* $M(\chi_T) = 1$.

(2) T *is left thick.*

(3) *There is a left invariant mean* m *on* $m(S)$ *such that* $m(\xi_T) = 1$.

Proof. Suppose S is left amenable as a discrete semigroup. Let $\varphi: BM(S) \rightarrow M(S)^*$ be the natural embedding of the bounded Borel measurable functions $BM(S)$ into $M(S)^*$ defined by $\varphi(f)(\mu) = \int f d\mu$, $\mu \in M(S)$. It is known that φ is an order preserving isometric isomorphism (into) which commutes with left translations and $\varphi(1) = 1$. Let m be a left invariant mean on $m(S)$ and n its restriction to $BM(S)$. Then n is left invariant on $BM(S)$. Let K be the set of means N on $M(S)^*$ which extends n. (In other words $\varphi^*(N) = n$.) By Hahn-Banach Extension Theorem, $K \neq \phi$. (A mean M on $M(S)^*$ can be defined equivalently as $M(1) = ||M|| = 1$.) K is a compact convex subset of the separated locally convex space $M(S)^{**}$ with the weak* topology. Moreover, if $a \in S$ and $N \in K$, then $l_a^* N \in K$ where $l_a; M(S)^* \rightarrow M(S)^*$ is the left translation operator in $M(S)^*$ defined by $l_a F = \delta_a \odot F$. Therefore the map $(s, N) \rightarrow l_s^* N$ is an action of S as continuous affine maps in K. By Day's Fixed Point Theorem (Day [2, Theorem 1] or Mitchell [7, Theorem 5]), this action has a fixed point N which is a left invariant mean on $M(S)^*$ (extending n). By Day [4, Theorem, p. 91], (1) implies (2) which is equivalent to (3) by Mitchell [7, Theorem 7, p. 257]. It remains to show that (3) implies (1). This however follows from the above arguments since we can assume in the definition of K, the mean n to satisfy $n(\xi_T) = 1$, then any fixed point N has the property that $N(\chi_T) = 1$ because $\varphi(\xi_T) = \chi_T$. This completes the proof.

REMARKS. Theorem 5.1 is an analogue of a result in Wong [11, Theorem 5.2, p. 301] for locally compact groups.

THEOREM 5.2. *Let* T *be a locally compact Borel subsemigroup of a locally compact semigroup* S. *If* T *satisfies* (TLL), *then* T *is left amenable iff* S *is.*

Proof. Suppose S is left amenable and T satisfies (TLL). Then there is a left invariant mean M on $M(S)^*$ such that $M(\chi_T) = 1$ by Day [4, Theorem, p. 91]. Hence $M(T)^*$ also has a left invariant mean (Wong [14, Theorem 4.2, separately continuous version]).

Conversely, suppose T is left amenable, and ν_α is a net in $M_0(T)$ such that $||\delta_t * \nu_\alpha - \nu_\alpha|| \to 0$ for each $t \in T$. Let $\mu_\alpha \in M_0(S)$ be such that $\mu_\alpha(T') = 0$ and $\mu_{\alpha|T} = \nu_\alpha$. Since T satisfies (TLL), T is left thick. For $s \in S$, there is some $t \in T$ such that $st \in T$. Consequently

$$\begin{aligned}
&||\delta_s * \mu_\alpha - \mu_\alpha|| \\
&\quad \leqq ||\delta_s * \delta_t * \mu_\alpha - \delta_s * \mu_\alpha|| + ||\delta_{st} * \mu_\alpha - \mu_\alpha|| \\
&\quad \leqq ||\delta_t * \mu_\alpha - \mu_\alpha|| + ||\delta_{st} * \mu_\alpha - \mu_\alpha|| \\
&\quad = ||\delta_t * \nu_\alpha - \nu_\alpha|| + ||\delta_{st} * \nu_\alpha - \nu_\alpha|| \to 0 \ .
\end{aligned}$$

Hence S is left amenable.

6. Some examples.

(1) Let $S = R$ be the real numbers under addition and usual topology. Then S is a locally compact abelian group. S is amenable in every sense we have considered. Let T be either $[0, \infty)$ or $(0, \infty)$, then T is a locally compact Borel subsemigroup of S which is clearly left lumpy in S. Therefore by Theorem 4.1, T is uniform strong topological left amenable.

(2) Let S be a compact semigroup with identity. Suppose $CB(S)$, the continuous bounded functions on S has a left invariant mean. By DeLeeuw and Glicksberg [5, Lemma 2.8, p. 70], S has a unique minimal right ideal, the kernel $K(S)$ of S which is a disjoint union of minimal left ideals of S that are compact topological groups. Let T be any one of these. Then T is left lumpy. Being a compact group, T is uniform strong topological left amenable. By Theorem 4.1, so is S. On the other hand if $M(S)^*$ has a left invariant mean, so does $CB(S)$ by restriction. It follows that for compact semigroups with identity, uniform strong topological left amenability, the existence of topological left invariant mean or left invariant mean on $M(S)^*$ or $CB(S)$ are all equivalent. [Note that the restriction of the natural embedding $\varphi: BM(S) \to M(S)^*$ to $CB(S)$ commutes (besides with left translations) also with left convolutions: $\varphi(\mu \odot f) = \mu \odot \varphi(f)$ if $f \in CB(S)$ and $\mu \in M(S)$, while if $f \in BM(S)$ and $\mu \in M(S)$, $\mu \odot f$ need not be in $BM(S)$ but is in $GL(S)$, the generalized functions on S (See Wong [13] for details).] In fact, we can show that any left invariant mean m on $CB(S)$ is always topological left invariant. For with notations as above, let ν be the normalised left Haar measure in T. Again by [5, Lemma 2.8, p. 70], $m(f) = \int f_{|T} d\nu$, $f \in CB(S)$. Let $\mu \in M_0(S)$ be such that $\mu(T') = 0$ and $\mu_{|T} = \nu$. By Wong [9, Lemma 3.3, p. 129], $\delta_a * \mu = \mu$ for all $a \in T$. Since T is a left ideal in S, $\delta_s * \mu = \mu$ for all $s \in S$. It follows

that $\tau * \mu = \mu$ for all $\tau \in M_0(S)$. Moreover $m(\tau \odot f) = \int (\tau \odot f)_T d\nu = \int \tau \odot f d\mu = \int f d\tau * \mu = \int f d\mu = \int f_{|T} d\nu = m(f)$ for any $\tau \in M_0(S)$ and m is topological left invariant. [Recall that $\tau \odot f(x) = \int f(yx) d\tau(y)$ for $f \in CB(S)$, $\tau \in M(S)$.]

Addendum. After the submission of the present paper, we have been informed by M.M. Day that in general the measure μ in condition $(*)$ can be chosen such that $\mu(T) = 1$ and that as a consequence, topological left lumpiness is equivalent to condition $(*)$. This latter result was also communicated to us independently by H. Junghenn.

M. M. Day also claims that if a Borel subset T is topological left substantial, then T^- is left lumpy and as a consequence, these two concepts coincide for *closed* sets.

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