SUPERHARMONIC INTERPOLATION IN SUBSPACES OF $C_c(X)$

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Let $E$ be a closed subset of the compact Hausdorff $X$ and let $A$ be a closed separating subspace of $C_c(X)$. Let $\rho$ be a dominator (strictly positive, l.s.c.) defined on $X \times T$, $T$ the unit circle in $C$. Conditions, formulated in terms of boundary measures, are discussed for approximate and exact solutions to the problem of finding $\rho$-dominated extensions in $A$ of functions $g \in (A|_E)^-$ satisfying $\text{re} \, g(x) \leq \rho(x, t)$ on $E \times T$. Various interpolation theorems of Rudin-Carleson type for superharmonic dominators are incorporated into this framework.

We do not assume that $A$ contains the constant functions. We denote $M(X) = C(X)^*$, the space of regular Borel measures on $X$.

We consider $N = M(E)$ as situated in $M(X)$ as the range of the projection $\pi_1 \mu = \mu|_E$ and denote the complementary projection $\pi_2 \mu = \mu|_{X\setminus E}$. Thus $(A|_E)^\perp$ is identified with the subspace $A^\perp \cap N$ in $M(X)$.

We call $\mu \in M(X)$ a boundary measure if $|\mu|$ is maximal with respect to the Choquet ordering as a measure of $X$ (embedded by evaluation) in the $w^*$ compact unit ball $A^*_\text{e}$. If $1 \in A$ then this is the same as $|\mu|$ being maximal on the state space $S_A$, as $X \subset S_A$, a $w^*$ closed face of $A^*_\text{e}$.

For brevity we denote the boundary measures by $\partial A M(X)$, or $\partial M(X)$, if $A$ is understood, and in general, adopt the convention of writing $\partial A S$ for $S \cap \partial A M(X)$. Thus, $\partial A A^\perp$ refers to the boundary measures annihilating $A$. The space $A^*$ is the quotient space $M(X)/A^\perp$ and images under the quotient map are denoted $\hat{\mu}$ for $\mu \in M(X)$. A subset $S \subset M(X)$ is called $A$-stable if $\hat{S} = (\partial A S)^\wedge$.

We call $E$ an interpolation set if $A|_E$ is closed in $C(E)$. Gamelin [8] shows that $E$ is an interpolation set if and only if there is a $k; 0 \leq k < \infty$, such that for each $m \in A^\perp$,

$$||\pi_1 m + A^\perp \cap N|| \leq k ||\pi_2 m||.$$  (1)

The best value of $k$ is called the extension constant, $e(A, E)$.

In [10] Roth introduces a general framework for interpolation problems by means of a dominator, $\rho$, defined as a strictly positive l.s.c. extended real-valued function on $X \times T$ ($T$ the unit circle in $C$). We let

$$U = \{f \in C(X); \text{re} \, f(x)/\rho(x, t) \leq 1 \text{ for all } (x, t) \in X \times T\}$$
and write
\[ \|f\|_\rho = \sup\{re\,tf(x)/\rho(x, t) : (x, t) \in X \times T\} \]
for the Minkowski functional of $U$. Thus $\|f\|_\rho \leq 1$ if and only if $re\,tf(x) \leq \rho(x, t), (x, t) \in X \times T$. Then $\|\mu\|_\rho, \mu \in M(X)$, refers to the polar functional given by
\[ \|\mu\|_\rho = \sup\{re(f, \mu) : f \in U\} . \]

Since $\rho$ is l.s.c and positive there is a constant $c$ such that $\|f\|_\rho \leq c\|f\|$ (the uniform norm corresponding to $\rho \equiv 1$) and if $\rho$ is bounded above the two are equivalent.

We say $E$ is an approximate $\rho$-interpolation set for $A$ if $E$ is an interpolation set and for each $g \in (A|_\rho)^-$ and $\varepsilon > 0$ there is an $f \in A$ such that $f|_E = g$ and $\|f\|_\rho < \|g\|_\rho + \varepsilon$. We say $E$ is an exact $\rho$-interpolation set if $f$ can be chosen with $\|f\|_\rho = \|g\|_\rho$. It is shown in [5] that for bounded $\rho$, $E$ is an approximate $\rho$-interpolation set for $A$ if and only if for each $m \in A^\perp$,
\begin{equation}
\|\pi_1 m + A^\perp \cap N\|_\rho \leq -\|\pi_2 m\|_\rho .
\end{equation}

If, in addition, the image $\hat{U}$ of $U^\circ$ under the quotient map is decomposable by $\hat{N}$ then $E$ is an exact $\rho$-interpolation set. If there is an $s, 0 \leq s < 1$, such that for each $m \in A^\perp$,
\begin{equation}
\|\pi_1 m + A^\perp \cap N\|_\rho \leq s \|\pi_2 m\|_\rho ,
\end{equation}
then the above holds and $E$ is $\rho$-exact for $A$. Gamelin’s results [8] can be phrased as follows: Let $G$ be a compact set in $X \setminus E$ and let
\[ \rho(G, k)(x, t) = \begin{cases} 1 & \text{for } (x, t) \in E \times T \\ k & \text{for } (x, t) \in G \times T \\ 1 \lor k & \text{otherwise.} \end{cases} \]

Then $E$ is an approximate $\rho(G, k)$-interpolation set for all such $G$ if and only if (1) holds and if, in addition, $e(A, E) < 1$ then $E$ is an exact $\rho$-interpolation set for any continuous $T$-invariant $\rho$ such that $\rho > e(A, E)$ on $X \times T$. This was obtained in abstract form using polar techniques by Ando [3].

In [6] Briem shows that if $E$ is a subset of the Choquet boundary, $\partial_\Lambda X$, then $E$ is an interpolation set if and only if (1) holds only for $m \in \partial_\Lambda A^\perp$. Further, if $X$ is metrizable then (1) holds for $\partial_\Lambda A^\perp$ if and only if $E$ is an approximate $\rho(G, k)$-interpolation set for each compact $G \subset \partial_\Lambda X \setminus E$. The $A$-stability of the unit ball $M(A)(X)$ (Hustad’s theorem [9]) and of $N = M(E)$ (since $E \subset \partial_\Lambda X$) are
essential here. If (1) holds for $\overline{e}(A, E) < 1$ (again, $\overline{e}$ is the smallest $k$ such that (1) holds for all $m \in \partial A^\perp$) then $E$ is $\rho(G, k)$ exact for any $G \subset \partial A \setminus E$ and $k > \overline{e}$.

If (1) holds for all $m \in \partial A^\perp$ with $k = 0$ this can be expressed as

$$m \in \partial A^\perp \implies \pi_m \in A.$$  

The set $E$ is called an $M$-set if $M(E)$ is $A$-stable and (4) holds. Roth [10] shows that if $E$ is an $M$-set and $\rho$ is a bounded $A$-superharmonic (if $1 \in A$ this means $\rho(x, t) \geq \int \rho(\cdot, t) d\mu$ for any $\mu \in M^+_a(X)$ and $\hat{\mu} = x \in X \subset A^*_a$) dominator then $E$ is an exact $\rho$-interpolation set for $A$. This generalizes the Alfsen-Hirschberg theorem [2] which deals with $T$-invariant $\rho$ and $E \subset \partial A X$.

In this note we consolidate these results by showing that for $E$ an interpolation set with $M(E)$ $A$-stable and $\rho$ $A$-superharmonic then $E$ is an approximate $\rho$-interpolation set if and only if (2) holds for $m \in \partial A^\perp$. If in addition $\hat{U}$ is decomposable by $\hat{N}$ in $A^*$ then the interpolation is exact. This is the case if $\rho$ is bounded and (3) holds for $m \in \partial A^\perp$. (If $\rho$ is bounded and (2) or (3) holds then $E$ is already an interpolation set.) We give a measure theoretic condition for the decomposability of $\hat{U}$ and show by means of simple examples of $A(K)$ spaces that exactness of interpolation can be deduced in this way even though equality holds in (2) which, of course, precludes the use of (3).

1. Hustad-Roth stability theorems. Let $A$ be a closed separating subspace of $C(X)$. Define $\Phi: C(X) \to C(X \times T)$ by $\Phi f(x, t) = tf(x)$. By separating we shall mean that the range of $\Phi|_A$ separates the points of $X \times T$. This assumption can be avoided, as is shown in Fuhr-Phelps [7], but at the expense of additional technicalities. If $\nu \in M(X \times T)$ then the Hustad map is given by

$$\mu = \Phi^* \nu \in M(X); \mu(f) = \int_{X \times T} t f(x) d\nu(x, t).$$

If $\Phi = \Phi|_A$ has range $B \subset C(X \times T)$ and $\nu$ is a maximal probability measure on $X \times T \subset B^*_a$ representing $\tilde{L} \in B^*_a$ then Hustad’s theorem says $\mu = \Phi^* \nu$ belongs to $\partial_d M(X)$, with $\hat{\mu} = L = \phi^* \tilde{L}$. We combine this with the following observations concerning $T$-invariant $A$-superharmonic dominators to obtain a general stability theorem due to Roth [11].

Thus let $\rho$ be a strictly positive l.s.c. extended real-valued function on $X$ such that for each $x \in X$ and $\mu \in M^+_a(X)$ with $\hat{\mu} = x \in A^*$, we have $\rho(x) \geq \int x \rho d\mu$, that is, $\rho$ is $A$-superharmonic. If $U = \{ f \in C(X): \text{ref}_{/\rho} 1 \}$ then $U^\circ$ is a $w^*$ compact convex subset of the
positive cone \( M^+(X) \), and we let \( \hat{U} \) be the quotient image in \( A^* \). Take \( \hat{R}^+ \) to be the one-point-compactification of \( R^+ \) and

\[
X_0 = \{(x, s) \in X \times \hat{R}^+: \rho(x) \leq s \leq +\infty\},
\]

\[
Y_0 = \{(x, \rho(x)) \in X_0: \rho(x) < \infty\},
\]

\[
Y_\infty = \{(x, \rho(x)) \in X_0: \rho(x) = +\infty\}.
\]

Since \( \rho \) is l.s.c., \( Y_0 \cup Y_\infty \) and \( Y_\infty \) are both \( G_\delta \) subsets of \( X_0 \) so that \( Y_0 \) is a Borel set. Define

\[
\psi: C(X) \to C(X_0); \psi f(x, s) = f(x)/s,
\]

and let \( \theta = \psi |_A \) with (not necessarily closed) range \( B \subset C(X_0) \). Since \( \rho \) is strictly positive \( \psi \) is bounded and \( \theta^* \) is one-to-one from \( B^* \) into \( A^* \). Let

\[
\phi_0: X_0 \to B_{1^*}
\]

be the evaluation map and let \( \hat{V} = w^* - \overline{co\phi_0(X_0)} \).

**Proposition 1.1.** Let \( \rho \) be a \( T \)-invariant \( A \)-superharmonic dominator on \( X \) as above.

1. \( \phi_0 \) is one-to-one on \( X_0((X \times \{\infty\}), X \times \{\infty\} = \phi_0^{-1}(0) \), and \( \theta^* \hat{V} = \hat{U} \).

2. If \( \nu \) is a maximal probability measure on \( \hat{V} \) then \( \nu[\phi_0(Y_0) \cup \{0\}] = 1 \) and \( \nu \) may be identified with the measure on \( Y_0 \) given by \( \nu \circ \phi_0 \).

3. If \( \nu \) is as in (2) and \( \mu = \psi^* \nu \) then for any bounded Borel function \( h \) on \( X \)

\[
\int_X h d\mu = \int_{Y_0} (h(x)/\rho(x))d\nu(x, \rho(x)).
\]

In particular, \( \mu \in U^0 \).

4. Let \( \mu_0 \in M_i^+(X) \) with \( \tilde{\mu}_0 = x_0 \in X \subset A_i^* \) and define \( \tilde{\mu}_0 \in M(X_0) \) by

\[
\tilde{\mu}_0(F) = (1/\rho(x_0))\int_X F(x, \rho(x))\rho(x)d\mu_0(x).
\]

Then for any bounded Borel function \( h \) on \( X \)

\[
\int_{X_0} (h(x)/s)d\mu_0(x, s) = (1/\rho(x_0))\int_X h d\mu_0.
\]

In particular \( \mu_0 \geq 0, \mu_0(X_0) = \tilde{\mu}_0(Y_0) \leq 1, \) and \( \mu_0 \) represents \( (x_0, \rho(x_0)) \in \hat{V} \).

5. If \( \nu \) is maximal on \( \hat{V} \) then \( \mu = \psi^* \nu \) is maximal on \( K = coX \subset A^* \).
Proof. (1) The separation theorem shows \( \hat{U} = w^*co\{x/s: (x, s) \in X_0\} \). Now

\[
\theta^* \circ \phi_0(x, s) = x/s \in A^*
\]

so the rest of (1) follows from the fact that \( A \) separates points in \( X \). For (2) let \( p = 1 - \chi_{[0]} \) on \( \hat{V} \) and note that the lower envelope \( \bar{\rho} \) is the Minkowski functional of \( \hat{V} \). Since \( \nu \) is maximal,

\[
1 = \nu[\{x: p(x) = \bar{\rho}(x)\}] = \nu[\{x: \bar{\rho}(x) = 1 \text{ or } 0\}].
\]

Now \( \lambda \geq 1 \) implies \( \phi_0(x, \lambda s) = (1/\lambda)\phi_0(x, s) \), so that

\[
\nu[\phi_0(Y_0) \cup \{0\}] = 1.
\]

If \( f \in C(X) \) then \( \psi^*\nu(f) = \int_{x_0} (f(x)/s)d\nu(x, s) = \int_{x_0} (f(x)/\rho(x))d\nu(x, \rho(x)) \)
and so (3) holds.

(4): If \( F \in C(X_0) \) and \( 0 \leq F \leq 1 \) then

\[
0 \leq \bar{\mu}_0(F) \leq (1/\rho(x_0))\int_X \rho d\mu_0 \leq 1.
\]

Thus \( \bar{\mu}_0 \geq 0, \bar{\mu}_0(X_0) \leq 1 \) and \( \mu_0[\{x: \rho(x) = + \infty\}] = 0. \) For \( F = \psi h, \)

\[
\bar{\mu}_0(F) = \int_{x_0} (h(x)/s)d\bar{\mu}_0(x, s) = (1/\rho(x_0))\int_X h d\mu_0.
\]

(5): Let \( f \) be a continuous convex function of \( K \) and denote the upper envelope of \( f \) by \( \hat{f}(K) \), where [1, I. 3.6]

\[
\hat{f}(K)(x_0) = \sup\{\mu(f): \mu \in M^+_1(X) \text{ and } \hat{\mu} = x_0 \in A^*\}.
\]

If \( g = \psi(f | x) \) then \( g \in C(X_0) \) with \( g = 0 \) on \( X \times \{\infty\} \). If \( \bar{\mu}_0 = x_0 \) and \( \bar{\mu}_0 \) is as in (4) then \( \bar{\mu}_0 \) represents \( (x_0, \rho(x_0)) \in \hat{V} \) and the upper envelope, \( \hat{g}(\hat{V}) \), satisfies

\[
\hat{g}(\hat{V})(x_0, \rho(x_0)) = \sup\{\mu_0(g): \hat{\mu}_0 = x_0 \} = (1/\rho(x_0))\hat{f}(K)(x_0)
\]
by part (4). Thus, using part (3), and [1, I. 4.5],

\[
\int_X [\hat{f}(K) - f]d\mu = \int_{x_0} [\hat{f}(K) - f]/\rho d\nu \leq \int_{y_0} [\hat{g}(\hat{V}) - g]d\nu = 0
\]
since \( \nu \) is maximal. Hence, \( \mu \) is maximal on \( K \).

We now consider the case where \( \rho \) is defined on \( X \times T \). We say such a \( \rho \) is \( A \)-superharmonic if for each \( (x, t) \in X \times T \) and \( \mu \in M(X \times T)_1^+ \) with
\[ \int_{X \times T} sf(y) d\mu(y, s) = tf(x) \quad \text{for all } f \in A \]

we have \( \rho(x, t) \geq \int_{X \times T} \rho d\mu. \)

**Theorem 1.2 (Hustad-Roth).** If \( \rho \) is an \( A \)-superharmonic dominator then \( U^0 \) is \( A \)-stable.

**Proof.** Let \( \Phi : C(X) \rightarrow C(X \times T); \Phi f(x, t) = tf(x) \) and let

\[ U^1 = \{ F \in C(X \times T) : \text{re} F(x, t)/\rho(x, t) \leq 1 \} \]

and \( \phi = \Phi|_A \) with range \( B \).

Let \( \Psi : C(X \times T) \rightarrow C(X_0) ; \Psi F(x, t, s) = F(x, t)/s \), where \( X_0 \) is the closed epigraph of \( \rho \) in \((X \times T) \times \mathbb{R}^+\). Now \( \Phi U \subset U^1 \) and \( \phi(A \cap U) = B \cap U^1 \). Given \( L \in \hat{U} \), let \( \tilde{L} \in (U^1)^{\circ} \subset B^* \) and \( L' \in \hat{V} \) (as in Proposition 1.1) with \( \theta^* L' = \tilde{L} \) and \( \phi^* \tilde{L} = L \). We have \( B^* \subset \rho^* \subseteq (X \times T) \) and the hypothesis says \( \rho \) on \( X \times T \) is \( B \)-superharmonic. Hence the results of Proposition 1.1 apply. Thus if \( \nu' \) is maximal on \( \hat{V} \) representing \( L' \) then 1.1 (3) and (5) show \( \nu = \Psi^* \nu' \) is maximal on \( B^* \) representing \( \tilde{L} \in (U^1)^{\circ} \). Then \( \mu = \phi^* \nu \in U^0 \) and \( \hat{\mu} = L \in \hat{U} \). Furthermore, Hustad's theorem shows \( \mu \) is a boundary measure.

If \( 1 \in A \) then the condition for \( A \)-superharmonicity is somewhat simpler.

**Proposition 1.3.** If \( 1 \in A \) then \( \rho \) is \( A \)-superharmonic if and only if for each \( \mu \in M_1^+(X) \) with \( \hat{\mu} = x \),

\[ \rho(x, t) \geq \int_X \rho(\cdot, t) d\mu \]

**Proof.** If \( \rho \) is \( A \)-superharmonic and \( \mu \in M_1^+(X) \) with \( \hat{\mu} = x \) we can embed \( X \) as \( X \times \{t\} \subset X \times T \) so that the measure \( \mu \) satisfies

\[ \int_{X \times T} sf(y) d\mu = tf(x) \]

and hence

\[ \rho(x, t) \geq \int_{X \times \{t\}} \rho(x, t) d\mu = \int_X \rho(\cdot, t) d\mu. \]

Conversely, if \( \mu \in M_1^+(X \times T) \) and represents \( tx \) then, since \( 1 \in A \), we have \( t^* \circ \rho X = tS_d(S_d \text{ the state space of } A) \) is a face of \( A^*_1 \). Hence \( \text{supp} \mu \subset X \times \{t\} \) and the measure \( \mu_1(f) = \int_{X \times T} f(x) d\mu \) represents \( x \) so that
\[ \rho(x, t) \geq \int_X \rho(\cdot, t) d\mu = \int_{X \times T} \rho d\mu. \]

2. Dominated interpolation. If \( E \) is a compact subset of \( X \) we let
\[ M = \{ f \in C(X) : f|_E = 0 \} \]
and denote \( M \cap A \) by \( E^+ \). It is well known that \( E \) is an interpolation set for \( A \) if and only if \( A + M \) is closed in \( C(X) \) and this in turn is equivalent to \( \hat{N} \) being \( w^* \) (or norm) closed in \( A^* \). The following characterization of approximate \( \rho \)-interpolation sets follows from results in \([5; 4.2]\). We denote \( N = M(E) \subset M(X) \).

**Theorem 2.1.** Let \( \rho \) be a (strictly positive l.s.c) dominator on \( X \) such that either \( \rho \) is bounded or \( E \) is an interpolation set. The following are equivalent:

(i) \( E \) is an approximate \( \rho \)-interpolation set for \( A \),

(ii) \( A + M \) is closed in \( C(X) \) and
\[ (A + M) \cap (U + M) = (A \cap U + M)^-, \]

(iii) \( \hat{U} \cap \hat{N} = (U^o \cap N)^- \),

(iv) \( ||\mu + A^1 \cap N||_\rho = ||\mu + A^1||_\rho \) for all \( \mu \in N \),

(v) \( ||\pi^1 m + A^1 \cap N||_\rho \leq ||-\pi^1 m||_\rho \) for all \( m \in A^1 \).

For \( x \in A^* \) we write \( ||x||_\rho \) for the Minkowski functional of \( \hat{U} \) so that if \( \hat{\mu} = x \)
\[ ||x||_\rho = ||\mu + A^1||_\rho. \]
The set \( U^o \) is split, that is, \( ||\mu||_\rho = ||\pi^1 \mu||_\rho + ||\pi^2 \mu||_\rho \) \([10, 5]\).

**Proposition 2.2.** Let \( N \) and \( U^o \) be \( A \)-stable sets in \( M(X) \). Then for \( \mu \in \partial_A M(X) \),

1. \( ||\mu + A^1||_\rho = ||\mu + \partial A^1||_\rho = ||\hat{\mu}||_\rho, \)
2. \( ||\mu + N + A^1||_\rho = ||\pi^1 \mu + \pi^1 \partial A^1||_\rho (\pi^1 \mu = \mu|_{X \times T}), \)
3. If \( ||\mu||_\rho = ||\hat{\mu}||_\rho \) then
\[ ||\pi^i \mu||_\rho = ||(\pi^i \mu)^\circ||_\rho \quad (i = 1, 2). \]

**Proof.** If \( \mu \in \partial M(X) \) and \( ||\hat{\mu}||_\rho \leq r \) then \( \mu = rv + m \) with \( v \in U^o \) and \( m \in A^1 \). The stability of \( U^o \) shows we can assume \( v \in \delta U^o \), so that \( m \in \delta A^1 \). Then (1) follows. If \( \mu = rv + \eta + \zeta \) with \( v \in \delta U^o, \eta \in \delta N, \zeta \in A^1 \), then \( \zeta \in \delta A^1 \) and \( \pi^1 \mu = rv + \pi^1 \zeta \in \tau \pi^1 U^o + \pi^1 \partial A^1 \). Conversely, if \( \pi^1 \mu = rv + \pi^1 \zeta, v \in \delta U^o, \zeta \in \delta A^1 \) then
\[ \mu = rv + (\pi_i\mu - \pi_i\zeta) + \zeta \in rU^0 + \partial N + \partial A^\perp. \]

For (3), we have
\[
||\pi_i\mu||_\rho \geq ||(\pi_i\mu)^T||_\rho = ||\pi_i\mu + A^\perp||_\rho = ||\mu - \pi_i\mu + A^\perp||_\rho,
\]
\[
\geq ||\mu||_\rho - ||\pi_i\mu + A^\perp||_\rho \geq ||\mu||_\rho - ||\pi_i\mu||_\rho = ||\pi_i\mu||_\rho.
\]

Since we do not assume \(1 \in A\), we take the Choquet boundary, \(\partial_A X\), to be \(X \cap \text{ext}A^*\). There are two main instances where the \(A\)-stability of \(N\) can be deduced.

**Proposition 2.3.** Let \(E\) be a closed subset of \(X\) such that either
(a) \(E \subset \partial_A X\) or
(b) \(E = F \cap X, F \text{ a w}^* \text{ closed face of } A^*\).

Then \(N\) is \(A\)-stable.

**Proof.** In the case (a) each probability measure on \(E\) is maximal and so the result follows since \(\overline{co}E\) spans \(\hat{N}\). In case (b) each maximal probability measure \(\mu\) with \(\hat{\mu} \in \overline{co}E\) has its support on \((\text{ext } F)^\perp \subset E\).

**Theorem 2.4.** Let \(E\) be a closed subset of \(X\) such that either
(a) \(E \subset \partial_A X\), or
(b) \(E = F \cap X, F \text{ a closed face of } A^*\).

Let \(\rho\) be an \(A\)-superharmonic dominator such that either \(\rho\) is bounded or \(E\) is an interpolation set. Then the following are equivalent:

(i) \(E\) is an approximate \(\rho\)-interpolation set,
(ii) \(||\mu + A^\perp \cap N||_\rho = ||\mu + \partial A^\perp||_\rho\) for all \(\mu \in \partial N\),
(iii) \(||\pi_i\mu + A^\perp \cap N||_\rho \leq ||-\pi_i\mu||_\rho\) for all \(m \in \partial A^\perp\).

**Proof.** The hypotheses imply that \(U^0\) and \(N\) are \(A\)-stable and so 2.2. (1) shows for \(\mu \in \partial M\),
\[
||\mu + A^\perp||_\rho = ||\mu + \partial A^\perp||_\rho.
\]
Thus (i) \(\rightarrow\) (ii) \(\rightarrow\) (iii) follows from 2.1. If (ii) holds and \(x \in \hat{U} \cap \hat{N}\) then choose \(\mu \in \partial N\) with \(\hat{\mu} = x\) and \(\mu \in U^0 + A^\perp\). Then
\[
||\mu + A^\perp \cap N||_\rho = ||\mu + \partial A^\perp||_\rho = ||\mu + A^\perp||_\rho \leq 1
\]
so that \(\mu = \nu + m; \nu \in U^0, m \in A^\perp \cap N\). Hence \(\nu \in N\) and \(\hat{\mu} = x = \hat{\nu} \in (U^0 \cap N)^\perp\). Thus 2.1 (iii) holds and hence (i) is shown.

The exactness of \(\rho\)-interpolation is characterized by the sum
$A \cap U + E^\perp(E^\perp)$ the ideal of functions in $C(X)$ vanishing on $E'$ being closed in $A$, a condition which is implied by the decomposability of $\hat{U}$ by $\hat{N}$ in $A^* [5;\text{Theorem 3.2}]$. If $E$ is an interpolation set (so that $\hat{N}$ if $w^*$ closed in $A^*$) then $\hat{U}$ is said to be decomposable by $\hat{N}$ if there is an $\alpha \geq 1$ such that each $x \in \hat{U}$ is a convex combination of elements $y, z$ with $y \in \hat{U} \cap \hat{N}, z \in \hat{U}$ and $\|z\| \leq \alpha\|z + \hat{N}\|$ (dual uniform norm).

The condition for decomposability, and hence exact interpolation, can be formulated in terms of representing measures in $M(X)$. We illustrate this for boundary measures in the case where $\rho$ is super-harmonic.

**Theorem 2.5.** Let $E$ be a closed subset of $X$ and $A$ a closed separating subspace such that either

(a) $E \subset \partial_\rho X$, or

(b) $E = F \cap X$, $F$ a closed face of $A^*$,

and let $\rho$ be an $A$-superharmonic dominator such that either $\rho$ is bounded or $E$ is an interpolation set.

If for each $x \in \hat{U}$ there is a $\mu \in \partial_\rho U^0$ with $\mu = x$ and

$$\|\pi_x \mu + \partial A^\perp\| \leq \alpha\|\pi_x \mu + \pi_\rho \partial A^\perp\|$$

($\alpha$ a constant independent of $\mu$) then $E$ is an exact $\rho$-interpolation set.

**Proof.** Given $x \in \hat{U}$ choose a boundary measure $\mu$ satisfying $\hat{\mu} = x$, $\|\hat{\mu}\|_\rho = \|\mu\|_\rho$ and $\|\pi_x \mu + \partial A^\perp\| \leq \alpha\|\pi_x \mu + \pi_\rho \partial A^\perp\|$. Now $\|\mu\|_\rho = \|\pi_x \mu\|_\rho + \|\pi_\rho \mu\|_\rho$ shows that $\mu$ is a convex combination of $\mu_i \in U^0 \cap N$ and $\mu_i \in U^0$, scalar multiples of $\pi_i \mu, \pi_\rho \mu$ respectively. Thus, $\|\pi_x \mu + \partial A^\perp\| \leq \alpha\|\mu_x + \pi_\rho \partial A^\perp\|$ and $x$ is a convex combination of $y \in (U^0 \cap N)^\perp$ and $z \in \hat{U}$ with (using 2.2 (1) and (2))

$$\|z\| = \|\mu_x + \partial A^\perp\| \leq \alpha\|\mu_x + \pi_\rho \partial A^\perp\| = \alpha\|\mu + N + A^\perp\|$$

$$= \alpha\|z + \hat{N}\|.$$ 

This shows that $(U^0 \cap N)^\perp = \hat{U} \cap \hat{N}$ and that $\hat{U}$ is decomposable by $\hat{N}$. Therefore $E$ is an exact $\rho$-interpolation set.

If $E$ is an $M$-set then $\pi_\rho \partial A^\perp \subset \partial A^\perp$ so that

$$\|\pi_x \mu + \pi_\rho \partial A^\perp\| \geq \|\pi_x \mu + \partial A^\perp\|$$

and the condition of 2.5 is automatically satisfied (for $A$-stable $U^0$). More generally, if $U^0$ and $N$ are $A$-stable and, for some $s < 1$

$$\|\pi_m + A^\perp \cap N\|_\rho \leq s\|\pi_\rho m\|_\rho$$

for all $m \in \partial A^\perp$. 

then a computation based on [5; 4.8] shows the condition of Theorem 2.5 holds, so that $E$ is an exact $\rho$-interpolation set.

**Corollary 2.6.** If $E$ is an $M$-set for the closed separating subspace $A \subset C(X)$ then $E$ is an exact $\rho$-interpolation set for $A$ for any $A$-superharmonic dominator $\rho$.

**Proof.** If $E$ is an $M$-set then $\tilde{N}$ is the range of a projection in $A^*$ so that $E$ is an interpolation set for $A$. The conclusion then follows from 2.5.

3. Examples. We illustrate the results of §2 with various choices of $\rho$. First, let $X$ be a compact metric space with $E$ a closed subset and $M(E)$ $A$-stable for the closed separating subspace $A \subset C(X)$. Let $G$ be the collection of compact subsets $G \subset \partial A \setminus E$ and let $\rho = \rho(G, k)$ be the dominator mentioned in the introduction. Then (for $k < \infty$)

\begin{equation}
||\pi_\rho m + A^\perp \cap N|| \leq k ||\pi_\rho m|| \text{ for all } m \in \partial A^\perp
\end{equation}

if and only if $E$ is an approximate $\rho(G, k)$-interpolation set for all $G \in \mathcal{G}$. To see this we note that since $G \subset \partial X, U^o$ is $A$-stable so that the second property holds if and only if

\begin{equation}
||\pi_\rho m + A^\perp \cap N||_\rho \leq ||-\pi_\rho m||_\rho \text{ for all } m \in \partial A^\perp, G \subset \mathcal{G}.
\end{equation}

It follows easily from [5; 4.1] that if $Y = X \setminus (E \cup G)$ then

$$
||\mu||_\rho = ||\mu||_\rho + k ||\mu||_\rho + (1 \wedge k) ||\mu||_\rho
$$

so that

$$
||\pi_\rho m + A^\perp \cap N|| = ||\pi_\rho m + A^\perp \cap N||_\rho
$$

and, since for boundary measures $\mu$, the metrizability of $X$ gives

$$
||\mu||(X \setminus E) = ||\mu||(\partial X \setminus E) = \sup\{||\mu||(G): G \in \mathcal{G}\},
$$

we have

$$
k ||\pi_\rho m|| = \sup\{||\pi_\rho m||_\rho: \rho = \rho(G, k), G \in \mathcal{G}\}.
$$

The equivalence of (1) and (2) is now immediate. If (1) holds for $k_0 < 1$ and $k_0 < k \leq 1$ then for $\rho = \rho(G, k)$

\begin{align*}
||\pi_\rho m + A^\perp \cup N||_\rho &= ||\pi_\rho m + A^\perp \cap N|| \leq k_0 (||m||_\rho + ||m||_\rho) \\
&\leq (k_0/k)(k ||m||_\rho + ||m||_\rho) = (k_0/k) ||\pi_\rho m||_\rho.
\end{align*}
so that $E$ is an exact $\rho(G, k)$-interpolation set for $A$.

The study of sufficient conditions for the $A$-convex hull of $E$ to be a generalized peak set (we now assume $1 \in A$) has been shown [4] to be related to an ordering on $C_c(X)$ and $M(X)$ induced by choosing $P$ to be a closed proper convex cone with nonempty interior in $C$. Let $\alpha, \beta$ be the generators (of modulus one) of the dual cone $P^* = \{z : reaz \geq 0 \text{ for all } a \in P\}$. We denote by $e$ the element of $P$ such that $ree\gamma = 1 \ (\gamma = \alpha, \beta)$. If $f \in C_c(X)$ we say $f \geq 0(P)$ if $f(X) \subset P$ and $\mu \geq 0(P^*)$ means $\mu(B) \in P^*$ for all Borel sets $B \subset X$. Then the function $e \equiv e$ becomes an order unit for $C(X)$ in which the order unit norm $||\cdot||_e$ (equivalent to the uniform norm) is given by

$$
\rho(x, t) = \begin{cases} 1 & \text{for } t = \pm \gamma \\ 1/\epsilon & \text{for } t \neq \pm \gamma \end{cases}, \quad \gamma = \alpha, \beta
$$

where $\epsilon$ is a constant such that

$$
cz \leq |re\alpha| \lor |re\beta|.
$$

This provides an example of a $\rho$ which is not $T$-invariant.

Let $\rho^+$ and $\rho^-$ be strictly positive l.s.c. functions on $X$ and take

$$
\rho(x, t) = \begin{cases} \rho^+(x) & \text{on } X \times \{1\} \\ \rho^-(x) & \text{on } X \times \{-1\} \\ + \infty & \text{otherwise.}
\end{cases}
$$

Then $U = \{f \in C(X) : -\rho^- \leq ref \leq \rho^+\}$. If $\mu \in U^0$ and $f$ is real then $\lambda f \in U$ for all real $\lambda$ so that

$$
1 \geq \lambda e\mu(\lambda f) = -\lambda \text{im} \mu(f)
$$

and hence $\text{im} \mu(f) = 0$. Thus $\mu$ is a real measure and $U^0 \subset reM(X)$.

If $A_0$ is a real subspace of $C(X)$ then we can apply the results of §2 to the self-adjoint space $A_0 + iA_0 = A$. Then $||f||_\rho = ||ref||_\rho$ and $m \in A^\perp$ if and only if $m = m_1 + im_2$ with $m_1, m_2$ real measures in $A^\perp$. Also $m$ is a boundary measure if and only if $m_1, m_2$ are boundary. Hence $E$ is an approximate (exact) $\rho$-interpolation set for $A$ if and only if it is for $A_0 = reA$, and the measure conditions of §2 need only involve real measures in $M(X)$. If $X$ is a compact convex subset of a locally convex space and $A_0 = A(X)$ (real affine continuous functions) then $\rho$ is $A$-superharmonic if and only if $\rho^+ = (\rho^+)^*$ and $\rho^- = (\rho^-)^*$, that is, if and only if $\rho^+$ and $\rho^-$ are concave on $X$.

Let $X$ be a square in $R^2$ with vertices denoted $\{1, 2, 3, 4\}$ with
\( E = \{1, 2\} \) diagonally opposite and \( A_0 = A(X), \rho^+, \rho^- = 1 \). Then \( \partial A^\perp \) is a one-dimensional subspace of the four-dimensional space \( \partial M(X) \) spanned by the point-masses \( \{\delta_j\}_{j=1}^4 \). A generator for \( \partial A^\perp \) is \( m = \delta_1 + \delta_2 - \delta_3 - \delta_4 \). Clearly \( A^\perp \cap N = \{0\} \) since \( coE \) is a simplex and so

\[
\|\pi_1 m + A^\perp \cap N\| = \|\pi_1 m\| = \|\pi_2 m\|.
\]

This shows \( E \) is an approximate \( \rho \)-interpolation set for \( A(X) \). Obviously \( E \) is in fact an exact interpolation set, but this cannot be concluded from a condition such as (3) in the introduction. Nevertheless, the condition of 2.5 holds, since if

\[
\mu = \Sigma \lambda_i \delta_i,
\]

then

\[
\|\mu\| = \Sigma |\lambda_i|
\]

and

\[
\|\pi_2 \mu + \pi_2 \partial A^\perp\| = \inf \{|\lambda_3 - \lambda| + |\lambda_4 - \lambda|: \lambda \in R\} = |\lambda_4 - \lambda_3|.
\]

If \( \lambda_3 \) and \( \lambda_4 \) are opposite in sign then

\[
\|\pi_2 \mu + \partial A^\perp\| \leq \|\pi_2 \mu\| = |\lambda_3| + |\lambda_4| = |\lambda_4 - \lambda_3| = \|\pi_2 \mu + \pi_2 \partial A^\perp\|.
\]

If, say \( 0 \leq \lambda_3 \leq \lambda_4 \), consider \( \nu = \mu + \lambda_3 m \). Then \( \hat{\nu} = \hat{\mu} \) and

\[
\|\nu\| = \Sigma |\lambda_i - \lambda_3| \leq (|\lambda_1| + |\lambda_2| + 2|\lambda_3|) + |\lambda_4| - |\lambda_3| = \|\mu\|
\]

and

\[
\|\pi_2 \nu + \partial A^\perp\| \leq \|\pi_2 \nu\| = |\lambda_4 - \lambda_3| = \|\pi_2 \mu + \pi_2 \partial A^\perp\|.
\]

We conclude with an example of an approximate interpolation set which is not exact. Let \( X \) be the unit ball of the sequence space \( l^1(w^* \text{ topology}) \) and let \( \rho = 1 \). Then take \( A = c_0 \), the pre-dual of \( l^1 \), so that \( \|a\|_\rho = \|a\|_\infty = \sup \{|a_n|\} \). Let \( E \) be the singleton \( \{x^n\}, x^n = 1/2^n, \ n = 1, 2, \cdots \). If \( (a, x^n) = 1 \) then \( \sum_{n=1}^\infty a_n/2^n = 1 \) so that some \( a_n \) must be greater than one. Clearly we can find such an \( a \) with \( \|a\| \leq 1 + \varepsilon \) for any \( \varepsilon > 0 \). Thus \( E \) is an approximate, but not exact, \( \rho \)-interpolation set.

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