CONTINUOUSLY VARYING PEAKING FUNCTIONS

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Let $X$ be a compact metric space, $A \subseteq C(X)$ a closed subalgebra. Let $\mathcal{P} \subseteq X$ be the set of peak points for $A$. It is shown that there is a continuous function $\Phi: \mathcal{P} \to A$ such that $\Phi(x)$ peaks at $x$ for all $x \in \mathcal{P}$.

0. Let $X$ be a compact Hausdorff space, $C(X)$ the continuous functions on $X$ under the uniform norm, and $A$ a closed subspace of $C(X)$ containing $1$. Let $\mathcal{P}$ be the set of peak points for $A$. Clearly if $X$ has more than one point and $x \in \mathcal{P}$ then there are infinitely many functions in $A$ which peak at $x$. Can one construct a function

$$\Phi: \mathcal{P} \to A$$

so that $\Phi(x)$ peaks at $x$ and $\Phi$ has some regularity properties?

In [4], using the von Neumann selection principle, it was shown that for $X = \mathcal{D} \subset \subset C^*$ with smooth boundary, $A = A(\mathcal{D})$ (the analytic functions on $\mathcal{D}$ which extend continuously to $\overline{\mathcal{D}}$), one can choose $\Phi$ to be measurable. The same argument is valid under much more general circumstances.

In the present note we prove that, for quite general $X$ and for $A$ an algebra, $\Phi$ can be chosen to be continuous. This generalizes results in [1, Theorem 3.1] and [2, Proposition 4].

1. Throughout the discussion, $X$ will be a fixed compact metric space with metric $d$. We let $C(X)$ denote the continuous, complex-valued functions on $X$ with the uniform norm and $A \subseteq C(X)$ will be a closed complex linear subspace. If $x \in X$, $r > 0$, then $B(x, r) = \{t \in X : d(x, t) < r\}$.

**DEFINITION.** A point $x \in X$ is said to be a peak point for $A$ if there is an $f \in A$ with $f(x) = 1$ and, for all $y \in X \sim \{x\}$, $|f(y)| < 1$. The function $f$ is said to peak at $x$.

We let $\mathcal{P}(A)$ denote the set of peak points for $A$.

**THEOREM.** Let $X$ be a compact metric space, $A \subseteq C(X)$ a closed subalgebra (with or without $1$). Then there is a continuous map

$$\Phi: \mathcal{P}(A) \to A$$

such that $\Phi(x)$ peaks at $x$ for each $x \in \mathcal{P}(A)$. 

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The remainder of the paper is devoted to the proof of the theorem. We proceed via a sequence of lemmas. The plan of the proof is as follows.

For each $k \in \{1, 2, \cdots\}$ we will construct a continuous function
\[
\Phi_k : \mathcal{P}(A) \longrightarrow A
\]
such that for each $x \in \mathcal{P}(A)$ we have
(i) $||\Phi_k(x)|| = 1$;
(ii) $[\Phi_k(x)](x) = 1$;
(iii) if $t \in X \sim B(x, 1/k)$ then $[\Phi_k(x)](t) \leq 1 - 1/(k + 2)$.

Once the $\{\Phi_k\}$ are constructed, the proof is immediate. For let $\Phi = \sum_{i=1}^{\infty} 2^{-i}\Phi_i$. Then $\Phi$ is continuous and for each $x \in \mathcal{P}(A)$ we have $\Phi(x) \in A$ and $[\Phi(x)](x) = 1$. Moreover, if $t \neq x$ and $k > 1/d(x, t)$ then
\[
||\Phi(x)(t)|| \leq \sum_{i=k}^{\infty} 2^{-i} ||\Phi_i(x)(t)|| + 2^{-k}[\Phi_k(x)](t) || \\
\leq 1 - 2^{-k} + 2^{-k}(1 - 1/(k + 2)) < 1 .
\]

So $\Phi(x)$ peaks at $x$. Thus it remains to construct the $\Phi_k$.

**Lemma 1.** Let $x_0 \in \mathcal{P}(A)$. Let $p$ be a strictly positive continuous function on $X$ with $p(x_0) = 1$. Then there is an $f \in A$ with $f(x_0) = 1$ and $|f(x)| \leq p(x)$ for all $x \in X$.

**Proof.** This is a special case of Theorem 12.5 of Gamelin [3], p. 58.

**Corollary 2.** With hypotheses as in Lemma 1, there is a $g \in A$ such that $g(x_0) = 1$, $|g(x)| < p(x)$ for all $x \in X \sim \{x_0\}$.

**Proof.** Immediate.

**Lemma 3.** Let $x_0 \in \mathcal{P}(A)$. Let $\psi \in A$ peak at $x_0$. There is a map
\[
\Psi : \mathcal{P}(A) \cap \{|\psi(x)| > 1/2\} \longrightarrow A
\]
so that
(i) $\Psi(x)$ peaks at $x$ for each $x \in \mathcal{P}(A) \cap \{|\psi(x)| > 1/2\}$,
(ii) $\Psi(x_0) = \psi$,
(iii) $\Psi$ is continuous at $x_0$.

**Proof.** For each $x \in \mathcal{P}(A) \sim \{x_0\}$ choose, by Corollary 2, a function $\varphi_x \in A$ such that $\varphi_x(x) = 1$ and
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( * ) \[ |\varphi(x)| < \min \left\{ \frac{(2 - |\psi(x)| - |\psi(t)|)}{2(1 - |\psi(x)|)}, 1 \right\} \]
for all \( t \in X \sim \{x\} \).

Now for each \( x \in \mathcal{P}(A) \cap \{|\psi(x)| > 1/2\} \) we define

\[
\Psi(x) = \begin{cases} 
(2(1 - |\psi(x)|)|\varphi_x + \text{sgn} \psi(x))\varphi|/[2 - |\psi(x)|] & \text{if } x \neq x_0, \ |\psi(x)| > 1/2 , \\
\psi & \text{if } x = x_0 .
\end{cases}
\]

Here \( \text{sgn} \ z = z/|z|, \) any \( z \in \mathbb{C} \sim \{0\} \).

Clearly if \( x \neq x_0 \) and \( x \) is sufficiently close to \( x_0 \) then \( |\psi(x)| > 1/2 \) and we have

\[
||\Psi(x) - \psi|| \leq ||\Psi(x) - \text{sgn} \psi(x) \cdot \psi|| + ||\text{sgn} \psi(x) \cdot \psi - \psi||
\]
\[
\leq \|(2(1 - |\psi(x)|)|\varphi_x + \text{sgn} \psi(x) \cdot \psi|/[2 - |\psi(x)|] - \text{sgn} \psi(x) \cdot \psi||
\]
\[
+ \|\psi(1 - \text{sgn} \psi(x))||
\]
\[
\leq \|(2(1 - |\psi(x)|)|\varphi_x - \text{sgn} \psi(x) \cdot \psi||
\]
\[
+ (1 - |\psi(x)|)||\text{sgn} \psi(x) \cdot \psi||/[2 - |\psi(x)|] + |1 - \text{sgn} \psi(x)|
\]
\[
\to 0 \text{ as } x \to x_0 .
\]

It remains to verify that \( \Psi(x) \) peaks at \( x \) when \( |\psi(x)| > 1/2 \). For such \( x \), we have \( [\Psi(x)](x) = 1 \). Further, if \( t \neq x \) then by ( * ) we have

\[
2(1 - |\psi(x)|)|\varphi_x(t)| < 2 - |\psi(x)| - |\psi(t)|
\]
or

\[
|2(1 - |\psi(x)|)|\varphi_x(t)| + |\psi(t)| < 2 - |\psi(x)|
\]

whence

\[
|2(1 - |\psi(x)|)|\varphi_x(t) + \text{sgn} \psi(x)\psi(t)| < 2 - |\psi(x)|
\]
or

\[
||[\Psi(x)](t)|| < 1 .
\]

**LEMMA 4.** Fix a positive integer \( k \). There is a sequence \( \{\Phi_i\}_{i=1}^k \) of functions,

\[
\Phi_i: \mathcal{P}(A) \longrightarrow A
\]
satisfying, for each \( z \in \mathcal{P}(A) \) and every \( j, \)

(i) \( ||\Phi_i(x)|| = 1; \)
(ii) \( [\Phi_i(x)](x) = 1; \)
(iii) \( \limsup_{\mathcal{P}(A) \to y \to x} ||\Phi_i(x) - \Phi_i(y)|| \leq 4^{-j} \cdot (1/k); \)
for every $t \in X \sim B(x, (1 - 2^{-j}) \cdot (1/k))$,

$$||\Phi^j(t)|| \leq (1 - 2/(k + 2)) + \sum_{i=1}^{j} 2^{-i} \cdot (1/(k + 2)) ;$$

(v) $||\Phi^j(x) - \Phi^{j-1}(x)|| \leq 2^{-j} \cdot (1/k)$, $j \geq 2.$

**Proof.** This lemma is the heart of the matter. We construct the $\Phi^j$ inductively on $j$. First consider $j = 1$. For each $x \in \mathcal{P}(A)$ construct, by Lemma 1, a function $\varphi_x \in A$ which satisfies $\varphi_x(x) = 1$ and

$$|\varphi_x(t)| \leq \min \{1 - 8kd(x, t)/(k + 2), 1 - 2/(k + 2)\} .$$

Using $\psi = \varphi_x$, construct a function

$$\Psi^j: \mathcal{P}(A) \cap \{|\psi(x)| > 1/2\} \rightarrow A$$

satisfying the conclusions of Lemma 3. Choose $r_x^1$, $0 < r_x^1 < 1/4k$ so that $t \in B(x, r_x^1)$ implies that $|\varphi_x(t)| > 1/2$ and

$$||\Psi^j(x) - \Psi^j(t)|| < 4^{-2} \cdot (1/(k + 2)) .$$

Now observe that if $y \in B(x, r_x^1)$ and $t \in B(y, 1/2k)$ then

$$d(x, t) \geq d(y, t) - d(y, x) \geq 1/4k .$$

Therefore for such $y, t$ we have

$$||\Psi^j(y)|| \leq ||\Psi^j(x)|| + ||\Psi^j(x) - \Psi^j(y)||$$

$$\leq |\varphi_x(t)| + 4^{-2} \cdot (1/(k + 2))$$

$$\leq (1 - 2/(k + 2)) + 2^{-1} \cdot (1/(k + 2)) .$$

Now since $\mathcal{P}(A)$ is a metric space, it is paracompact ([5], p. 160, Cor. 35). Hence there is a locally finite refinement $\mathcal{U}^1 = \{U^1_\omega\}_{\omega \in \Omega}$ of the covering $\{B(x, r_x^1)\}_{x \in \mathcal{P}(A)}$ of $\mathcal{P}(A)$. Let $x_\omega$, $\omega \in \Omega$, be chosen so that $U^1_\omega \subseteq B(x_\omega, r_x^1)$. Let $B^1_\omega$ denote $B(x_\omega, r_x^1)$. We may assume that $\mathcal{U}^1_\omega \subseteq B^1_\omega$. Let $\{\chi^1_{\omega}\}$ be a continuous partition of unity subordinate to $\mathcal{U}^1$ and define

$$\Phi^1_j = \sum_{x \in B^1_\omega} \chi^1_{\omega} \Psi^1_\omega .$$

Then conclusions (i) and (ii) are immediate. Conclusion (iv) follows from (**). Conclusion (v) is vacuous for $j = 1$. It remains to verify (iii).

Fix $x \in \mathcal{P}(A)$. Then there is a neighborhood $W$ of $x$ and $\{\omega_1, \cdots, \omega_m\} \subseteq \Omega_1$ so that $W \cap \text{supp} \chi^1_{\omega} \neq 0$ only if $\omega \in \{\omega_1, \cdots, \omega_m\}$. Of course $m$ may depend on $x$. Letting $x_i$ denote $x_{\omega_i}$, $i = 1, \cdots, m$, we have that
\[
\limsup \| \Phi_k(x) - \Phi_k(y) \| \leq \sum_{t=1}^{m} \limsup_{(A) \ni \Phi_j(x) - \Phi_j(y) \| \\
+ \sum_{t=1}^{m} \chi_{\omega_j}(x) \limsup_{(A) \ni \Phi_j(x) - \Phi_j(y) \| \\
\leq 0 + \sum_{t=1}^{m} \chi_{\omega_j}(x) \limsup_{(A) \ni \Phi_j(x) - \Phi_j(y) \| \\
+ \sum_{t=1}^{m} \chi_{\omega_j}(x) \limsup_{(A) \ni \Phi_j(x) - \Phi_j(y) \| \\
\leq 2 \cdot 4^{-i}(k + 2) \leq 4^{-i} \cdot (1/k).
\]

Now suppose that \( \Phi_i, \ldots, \Phi_k \) have been constructed so that (i)-(v) are satisfied. Let \( x \in \mathcal{P}(A) \). Using \( \psi = \Phi_i(x) \), we construct a function

\[
\Psi^{i+1}_x: \mathcal{P}(A) \cap \{ |\psi(x) | > 1/2 \} \rightarrow A
\]
satisfying the conclusions of Lemma 3. Choose \( r^{i+1}_j, 0 < r^{i+1}_j < 2^{-i-1} \cdot (1/k) \) so that \( t \in B(x, r^{i+1}_j) \) implies that \( |[\Phi_i(x)](t) | > 1/2 \) and both

\[
|\Psi^{i+1}_x(x) - \Psi^{i+1}_x(t) | \leq 4^{-i-2} \cdot (1/(k + 2))
\]

(***), and

\[
|\Phi_i(x) - \Phi_i(t) | \leq (4/3) \cdot 4^{-i} \cdot (1/k).
\]

If now \( y \in B(x, r^{i+1}_j), t \in B(y, (1 - 2^{-i-1}) \cdot (1/k)) \) then

\[
d(x, t) \geq d(y, t) - d(y, x) \geq (1 - 2^{-i})/(1/k).
\]

Hence for such \( y, t \) we have

\[
|\Psi^{i+1}_x(y)(t) | \leq |\Psi^{i+1}_x(x)(t) | + |\Psi^{i+1}_x(x)(t) - \Psi^{i+1}_x(y)(t) | \\
\leq |\Phi_i(x)(t) | + 4^{-i-1} \cdot (1/(k + 2)) \\
\leq (1 - 2/(k + 2)) + \sum_{t=1}^{i+1} 2^{-i} \cdot (1/(k + 2)) + 2^{-i-1} \cdot (1/(k + 2)) \\
= (1 - 2/(k + 2)) + \sum_{t=1}^{i+1} 2^{-i} \cdot (1/(k + 2)).
\]

Choose a locally finite refinement \( \mathcal{W}^{i+1}_j = \{ U^{i+1}_w \}_{w \in \Omega^{i+1}_j} \) of the covering \( \{ B(x, r^{i+1}_j) \}_{x \in \mathcal{A}} \) of \( \mathcal{P}(A) \). Let \( \{ \chi_{\omega_j} \}_{\omega_j \in \Omega^{i+1}_j} \) be chosen so that \( U^{i+1}_w \subseteq B(x, r^{i+1}_j) \equiv B^{i+1}_w \), each \( \omega \in \Omega^{i+1}_j \). We may assume that \( \tilde{U}^{i+1}_w \subseteq B^{i+1}_w \). Let \( \{ \chi^{i+1}_w \} \) be a continuous partition of unity subordinate to \( \mathcal{W}^{i+1}_j \). Define

\[
\Phi^{i+1}_x = \sum_{w \in \Omega^{i+1}_j} \chi^{i+1}_w U^{i+1}_w.
\]

It follows as in the case \( j = 1 \) that (i), (ii), (iii), and (iv) hold. To verify (v) fix \( x \in \mathcal{P}(A) \). Let \( \omega_1, \ldots, \omega_m \) satisfy the property that
\( \chi_\omega(x) \neq 0 \) iff \( \omega \in \{\omega_i, \cdots, \omega_m\} \). Let \( x_i \) denote \( x_{\omega_i}, i = 1, \cdots, m \). Then

\[
\|\Phi_k^{i+1}(x) - \Phi_k^i(x)\| \leq \left| \sum_{i=1}^m \chi^{i+1}_\omega(x]\left[\mathcal{W}^{i+1}_\omega(x) - \mathcal{W}^{i+1}_\omega(x_i)\right]\right| \\
+ \left| \sum_{i=1}^m \chi^{i+1}_\omega(x_\omega)\left[\Phi^{i+1}_\omega(x) - \Phi^i_\omega(x_i)\right]\right| \\
+ \left| \sum_{i=1}^m \chi^{i+1}_\omega(x_\omega)\left[\Phi^i_\omega(x_i) - \Phi^i_\omega(x)\right]\right| \\
\leq 4^{-j-2}(1/k + 2) + 0 + (4/3)4^{-j-1}(1/k) \leq 2^{-j}(1/k).
\]

The induction is complete.

**Lemma 5.** For \( k \in \{1, 2, \cdots\} \) there exist functions

\[ \Phi_k: \mathcal{P}(A) \longrightarrow A \]

such that

- (i) \( \|\Phi_k(x)\| = 1 \) for all \( x \in \mathcal{P}(A) \),
- (ii) \( \{\Phi_k(x)\}(x) = 1 \),
- (iii) \( \Phi_k \) is continuous;
- (iv) \( \|\Phi_k(x)\|(t) \leq 1 - 1/(k + 2) \) for all \( x \in \mathcal{P}(A), t \in X \sim B(x, 1/k) \).

**Proof.** Let \( \Phi_k^i \) be as in Lemma 4 and define \( \Phi_k = \lim_{j \rightarrow \infty} \Phi_k^i \). That the limit exists follows from (v) of Lemma 4. The conclusions (i)-(iv) of the present lemma now follow from the corresponding parts of Lemma 4.

By the discussion preceding Lemma 1, the proof of the theorem is complete.

**Remark.** Our proof yields something more general. Indeed, instead of assuming \( X \) to be metric, one need only assume that the relative topology on \( \mathcal{P} \) has a \( \sigma \)-locally finite base. By [5], p. 128, this is equivalent to assuming that \( \mathcal{P} \) is metric, hence paracompact, and the proof goes through as before.

The referee has kindly observed that given our Lemma 3, one can use Theorem 3.1" of [6] to prove that if \( X \) is compact Hausdorff and \( A \) is separable then the theorem holds. This is a weaker result than the one outlined in the preceding paragraph. Moreover, the proof using [6] is not essentially shorter than the elementary one presented here, and the construction of \( \Phi \) as the uniform limit of discontinuous functions has intrinsic interest.

**Remark.** It would be interesting to know whether, in the presence of differentiable structure in \( X \) and \( A \), the peaking functions may be chosen to vary differentiably.
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