

# Pacific Journal of Mathematics

**NORM ATTAINING OPERATORS ON LEBESGUE SPACES**

ANZELM IWANIK

# NORM ATTAINING OPERATORS ON LEBESGUE SPACES

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**Let  $X$  and  $Y$  be Lebesgue spaces (AL-spaces). Then the norm attaining operators mapping  $X$  to  $Y$  are dense in the space of all linear bounded operators from  $X$  to  $Y$ .**

For any two real Banach spaces  $X$  and  $Y$  by  $B(X, Y)$  we denote the Banach space of all bounded linear operators from  $X$  to  $Y$ . In [7] Uhl proved that for any strictly convex Banach space  $Y$  the norm attaining operators are (norm) dense in  $B(L^1[0, 1], Y)$  if and only if  $Y$  has the Radon-Nikodym property. The question of whether the norm attaining operators are dense in  $B(L^1[0, 1], L^1[0, 1])$  has remained unsolved (cf. [7], p. 299). Here we answer this question in the affirmative. In fact we prove a slightly more general result.

First we introduce some notations. Let  $I$  stand for the unit interval. For any function  $\mu$  defined on the product algebra in  $I \times I$  by  $\mu^i (i = 1, 2)$  we denote the corresponding marginal functions defined on the Borel subsets of  $I$ :

$$\begin{aligned}\mu^1(A) &= \mu(A \times I), \\ \mu^2(B) &= \mu(I \times B).\end{aligned}$$

The vector lattice of all finite signed Borel measures on  $I \times I$  will be denoted by  $M$ . Given any two finite positive Borel measures  $m_1, m_2$  on  $I$  we write  $M(m_1, m_2)$  for the set of all measures  $\mu$  in  $M$  such that  $|\mu|^i$  is absolutely continuous with respect to  $m_i (i = 1, 2)$  and

$$\frac{d|\mu|^1}{dm_1} \in L^\infty(m_1).$$

The measures  $m_1$  and  $m_2$  will be fixed throughout the rest of the paper.

Let us recall that  $B(L^1(m_1), L^1(m_2))$  is a Banach lattice under its canonical order (see [5], IV Theorem 1.5 (ii)).

The forthcoming theorem establishes an isomorphism between  $M(m_1, m_2)$  and  $B(L^1(m_1), L^1(m_2))$ , and extends a corresponding result of J. R. Brown on doubly stochastic operators ([1], p. 18). As was kindly indicated by the referee, our Theorem 1 is also related

to N. J. Kalton's representation of the endomorphisms from  $L^p$  to  $L^p$  for  $0 < p \leq 1$  ([3], Theorem 3.1).

By  $\langle \cdot, \cdot \rangle$  we denote the canonical bilinear form on  $L^\infty(m_2)^* \times L^\infty(m_2)$ .

**THEOREM 1.** *The space  $M(m_1, m_2)$  is a vector lattice ideal in  $M$  and to each  $\mu \in M(m_1, m_2)$  there corresponds a unique operator  $T_\mu \in B(L^1(m_1), L^1(m_2))$  such that*

$$\langle T_\mu f, h \rangle = \int f(x)h(y)d\mu(x, y)$$

for all  $f \in L^1(m_1)$  and  $h \in L^\infty(m_2)$ . Moreover, the mapping  $\mu \rightarrow T_\mu$  is a vector lattice isomorphism of  $M(m_1, m_2)$  onto  $B(L^1(m_1), L^1(m_2))$  and

$$\|T_\mu\| = \left\| \frac{d|\mu|^1}{dm_1} \right\|_\infty$$

for every  $\mu \in M(m_1, m_2)$ .

*Proof.* First we note that  $M(m_1, m_2)$  is a vector subspace of  $M$ . Since  $\nu \in M(m_1, m_2)$  whenever  $0 \leq \nu \in M$  and  $\nu \leq \mu \in M(m_1, m_2)$ , we observe that  $M(m_1, m_2)$  is a lattice ideal (and clearly a sublattice) in  $M$ . If  $\mu \in M(m_1, m_2)$  then it is easy to see that the bilinear form

$$[f, h] = \int f(x)h(y)d\mu(x, y)$$

is well-defined and continuous on  $L^1(m_1) \times L^\infty(m_2)$ . Therefore there exists a unique operator  $T_\mu \in B(L^1(m_1), L^\infty(m_2)^*)$  such that

$$[f, h] = \langle T_\mu f, h \rangle$$

(see e.g., [5], IV § 2). Clearly the mapping  $\mu \rightarrow T_\mu$  is one-to-one and  $\mu \geq 0$  if and only if  $T_\mu$  is a positive operator in the Banach lattice sense. Moreover, for an arbitrary  $\nu \geq 0$  in  $M(m_1, m_2)$  and for any  $h \in L^\infty(m_2)$  we have  $\langle T_\nu 1, h \rangle = \int h d\nu^2$ , so

$$T_\nu 1 = \frac{d\nu^2}{dm_2} \in L^1(m_2),$$

whence  $T_\nu f \in L^1(m_2)$  for any  $f \in L^\infty(m_1)$ . Consequently,  $T_\nu \in B(L^1(m_1), L^1(m_2))$  by continuity. Since every  $\mu \in M(m_1, m_2)$  is a difference of two positive measures in  $M(m_1, m_2)$  and  $\mu \rightarrow T_\mu$  is a linear map, we have  $T_\mu \in B(L^1(m_1), L^1(m_2))$  for all  $\mu \in M(m_1, m_2)$ .

We now show that  $\mu \rightarrow T_\mu$  is an "onto" mapping. Since  $B(L^1(m_1), L^1(m_2))$  is a Banach lattice, it suffices to prove that every

positive operator  $T \in B(L^1(m_1), L^1(m_2))$  is of the form  $T_\mu$ . Given any such  $T$  we define a set function

$$\mu(A \times B) = \langle T\chi_A, \chi_B \rangle$$

on all Borel rectangles in  $I \times I$ . Evidently  $\mu$  extends uniquely to a finitely additive positive measure (denoted also by  $\mu$ ) on the product algebra. The marginal measures  $\mu^1(A) = \int_A T^*1 dm_1$  and  $\mu^2(B) = \int_B T1 dm_2$  are finite, positive, and countably additive, so they are compact by the classical result of Ulam. Since  $\mu$  is a nondirect product of  $\mu^1$  and  $\mu^2$ , it is countably additive by Theorem 1 (i) in [4]. The unique extension of  $\mu$  to a finite positive (countably additive) Borel measure on  $I \times I$  is again denoted by  $\mu$ . By a standard approximation argument,

$$\int f(x)h(y)d\mu(x, y) = \langle Tf, h \rangle$$

for all  $f \in L^1(m_1)$  and  $h \in L^\infty(m_2)$ . Therefore  $T = T_\mu$ . Finally, we note that for every  $\mu \in M(m_1, m_2)$

$$\begin{aligned} \|T_\mu\| &= \|T_{|\mu^1|}\| = \sup \|T_{|\mu^1|}f\|_1 = \sup \langle T_{|\mu^1|}f, 1 \rangle \\ &= \sup \int f(x)d|\mu^1|(x) = \sup \int f(x) \frac{d|\mu^1|}{dm_1}(x) dm_1(x) \\ &= \left\| \frac{d|\mu^1|}{dm_1} \right\|_\infty, \end{aligned}$$

where the suprema are taken over all nonnegative functions  $f \in L^1(m_1)$  with  $\|f\|_1 \leq 1$ .

**COROLLARY 1.** *Let  $\nu \in M(m_1, m_2)$ . If there exists a function  $g \in L^\infty(m_2)$  with  $|g| = 1$  such that the Radon-Nikodym derivative of the marginal measure  $(g(y)d\nu(x, y))^1$  with respect to  $m_1$  equals*

$$\left\| \frac{d|\nu^1|}{dm_1} \right\|_\infty$$

on a set  $B$  of positive  $m_1$  measure, then the operator  $T_\nu$  attains its norm on the unit ball in  $L^1(m_1)$ .

*Proof.* We put  $d\lambda(x, y) = g(y)d\nu(x, y)$ . Then

$$\begin{aligned} \langle T_\nu(\chi_B/m_1(B)), g \rangle &= \frac{1}{m_1(B)} \int \chi_B(x) d\lambda(x, y) \\ &= \frac{1}{m_1(B)} \int_B \frac{d\lambda^1}{dm_1} dm_1 = \left\| \frac{d|\nu^1|}{dm_1} \right\|_\infty, \end{aligned}$$

implying  $\|T_\nu(\chi_B/m_1(B))\|_1 = \|T_\nu\|$  by Theorem 1.

The algebra of sets generated by all dyadic-rational rectangles in  $I \times I$  will be denoted by  $\mathcal{A}$ . The  $\sigma$ -algebra generated by  $\mathcal{A}$  coincides with the Borel algebra in  $I \times I$ .

**THEOREM 2.** *The norm attaining operators are dense in  $B(L^1(m_1), L^1(m_2))$ .*

*Proof.* Let  $T \in B(L^1(m_1), L^1(m_2))$ . By Theorem 1 we have  $T = T_\mu$  for some measure  $\mu$  in  $M(m_1, m_2)$ . Without any loss of generality we may assume

$$\left\| \frac{d|\mu|^1}{dm_1} \right\|_\infty = 1.$$

Given  $0 < \varepsilon < 1$ , the set

$$D = \left\{ x \in I : \frac{d|\mu|^1}{dm_1}(x) > 1 - \frac{\varepsilon}{4} \right\}$$

is of positive  $m_1$  measure, say,  $m_1(D) = \delta > 0$ . Now let  $P$ ,  $(I \times I) - P$  be the Hahn decomposition for  $\mu$  with  $\mu^+$  concentrated on  $P$  (see [2], § 29 Theorem A). Since  $P$  is a Borel set, there exists  $\tilde{P} \in \mathcal{A}$  such that  $|\mu|(P \Delta \tilde{P}) < \delta\varepsilon/4$  ([2], § 13 Theorem D). We define a new measure  $\tilde{\mu}$  by

$$d\tilde{\mu} = \chi_{\tilde{P}}d\mu^+ - \chi_{(I \times I) - \tilde{P}}d\mu^-.$$

Evidently  $\tilde{P}$ ,  $(I \times I) - \tilde{P}$  is the Hahn decomposition for  $\tilde{\mu}$  and  $d|\mu - \tilde{\mu}| = \chi_{P \Delta \tilde{P}}d|\mu|$ . Since  $|\mu - \tilde{\mu}|(I \times I) < \delta\varepsilon/4$ , the Radon-Nikodym derivative of  $|\mu - \tilde{\mu}|^1$  with respect to  $m_1$  must be less than  $\varepsilon/4$  on some set  $C \subset D$  of positive  $m_1$  measure. As  $\tilde{P} \in \mathcal{A}$ , there exists a natural number  $n$  such that  $\tilde{P}$  is a union of finitely many squares corresponding to the dyadic partition of  $I$  into  $2^n$  subintervals of equal length. Let  $I_0$  be any such open subinterval intersecting  $C$  on a set  $B = C \cap I_0$  of positive  $m_1$  measure. We let

$$d\nu(x, y) = \chi_B(x) \left( \frac{d|\mu|^1}{dm_1} \right)^{-1}(x) d\tilde{\mu}(x, y) + \chi_{I-B}(x) d\mu(x, y).$$

Note first that

$$\begin{aligned} d|\nu - \mu| &= \chi_B(x) \left( \frac{d|\mu|^1}{dm_1} \right)^{-1}(x) |d(\tilde{\mu} - \mu)(x, y)| \\ &+ \left( 1 - \frac{d|\mu|^1}{dm_1}(x) \right) d\mu(x, y) \left| \leq 2\chi_C(x) d|\tilde{\mu} - \mu|(x, y) + \frac{\varepsilon}{2} d|\mu|(x, y) \right|. \end{aligned}$$

Therefore

$$\frac{d|\nu - \mu|^1}{dm_1} < 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon ,$$

whence  $\|T_\nu - T_\mu\| = \|T_{\nu-\mu}\| \leq \varepsilon$ . Moreover,

$$\frac{d|\mu|^1}{dm_1} = 1 \text{ on } B \text{ and } \leq 1 \text{ elsewhere.}$$

The set  $(I_0 \times I) \cap \tilde{P}$  is a finite union of squares of the form  $I_0 \times I_k (k = 1, \dots, m)$ , where each  $I_k$  is an element of the dyadic partition of  $I$  into  $2^n$  subintervals of equal length. Therefore  $(B \times I) \cap \tilde{P}$  is the finite union of the Borel rectangles  $B \times I_k$ . We define a function  $g \in L^\infty(m_2)$  as follows

$$g(y) = \begin{cases} 1 & \text{if } y \in \cup I_k , \\ -1 & \text{otherwise.} \end{cases}$$

Clearly the Radon-Nikodym derivative of the marginal measure  $(g(y)d\nu(x, y))^1$  coincides with

$$\frac{d|\nu|^1}{dm_1} = 1$$

on  $B$ . Therefore, by Corollary 1,  $T_\nu$  attains its norm and the proof is completed.

By the known representation theorems for Lebesgue spaces (see e.g., [5], II 8.5 Corollary and [2], §41 Theorem C, or [6], 26.4.9 Exercise (C)), every separable Lebesgue space (i.e., separable AL-space in terms of [5]) is Banach lattice isomorphic with  $L^1(m)$  for some finite positive Borel measure  $m$  on  $I$ . Therefore we obtain the following corollary to our result:

**COROLLARY 2.** *Let  $X$  and  $Y$  be separable Lebesgue spaces. Then the norm attaining operators are dense in  $B(X, Y)$ .*

After the paper was accepted for publication, the last corollary has been generalized to arbitrary (nonseparable) Lebesgue spaces as a result of the author's conversations with Professors J. Bourgain and H. P. Lotz. The proof is outlined below:

Theorem 1 remains true if we replace  $(I, m_i)$  by  $(J_i, m_i)$  with  $J_i$  compact Hausdorff and  $m_i$  a finite regular (compact) positive measure on the Borel  $\sigma$ -algebra  $\mathcal{B}_i$ , and with  $M$  being the space of all finite signed measures on the product  $\sigma$ -algebra  $\mathcal{B}_1 \times \mathcal{B}_2$ . Indeed, the marginal measures  $\int_A T^*1 dm_1, \int_B T1 dm_2$  are compact since the measures  $m_i$  are regular, and so Theorem 1 (i) of [4] is still

applicable. The rest of the proof remains unchanged.

Theorem 2 is valid for the general spaces  $L^1(J_i, m_i)$  with essentially the same proof as before,  $\mathcal{A}$  being replaced now by the algebra of all finite unions of Borel rectangles in  $J_1 \times J_2$ .

Now if  $X_1, X_2$  are arbitrary Lebesgue spaces then every  $T \in B(X_1, X_2)$  can be approximated by norm attaining operators. Indeed, let  $(x_n)$  be a sequence in  $X_1$  such that  $\|x_n\| \leq 1$  and  $\lim \|Tx_n\| = \|T\|$ . The Banach lattice ideal  $Y_1$  spanned by  $(x_n)$  is a Lebesgue subspace with a weak order unit. Also the image  $TY_1$  is contained in a Lebesgue subspace  $Y_2 \subset X_2$  with a weak order unit. By the Kakutani representation theorem there exist compact spaces  $J_i$  with finite regular positive measures  $m_i$  such that  $Y_i = L^1(J_i, m_i)$ . By the above, the restriction  $T_1$  of  $T$  to  $Y_1$  can be approximated within a given  $\varepsilon > 0$  by a norm attaining operator  $T_0 \in B(Y_1, Y_2)$  satisfying  $\|T_0\| = \|T\|$ . If  $P$  denotes the canonical band projection of  $X_1$  onto  $Y_1$  then it is easy to see that  $T_0P + T(I - P)$  has norm  $\|T_0\|$ , is norm attaining, and approximates  $T$  within  $\varepsilon$ .

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