POINTWISE COMPACTNESS AND MEASURABILITY

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Among other results it is proved that if \((X, \mathcal{A}, \mu)\) is a probability space, \(E\) a Hausdorff locally convex space such that \((E', \sigma(E', E))\) contains an increasing sequence of absolutely convex compact sets with dense union, and \(f: X \to E\) weakly measurable with \(f(X) \subset K\), a weakly compact convex subset of \(E\), then \(f\) is weakly equivalent to \(g: X \to E\) with \(g(X)\) contained in a separable subset of \(K\).

In [8] and [9] some remarkable results are obtained for the pointwise compact subsets of measurable real-valued functions and some interesting applications to strongly measurable Banach space-valued functions are established. In this paper we continue those ideas a little further. We first give a somewhat different proof of ([9], Theorem 1) and then apply it to give a generalization of classical Phillip's theorem ([5]). Also some result about equicontinuous subsets of \(C(X)\), the space of all continuous real-valued functions on \((X, \tau_\rho)\) (\(\tau_\rho\) is the lifting topology, [10], p. 59; in [8] this topology is denoted by \(T_\rho\)) are obtained.

All locally convex spaces are taken over reals and notations of [6] are used. For a topological space \(Y\), \(C(Y)\) (resp. \(C_b(Y)\)) will denote the set of all (resp. all bounded) real-valued continuous functions of \(Y\). \(N\) will denote the set of natural numbers.

In this paper \((X, \mathcal{A}, \mu)\) is a complete probability measure space. Let \(\mathcal{L}\) be the set of all real-valued \(\mathcal{A}\)-measurable functions on \(X\), \(\mathcal{L}^e\), the essentially bounded elements of \(\mathcal{L}\), and \(M^\infty\), the bounded elements of \(\mathcal{L}\). We fix a lifting, [10], \(\rho: \mathcal{L}^\infty \to M^\infty\) and on \(X\) we always take the lifting topology \(\tau_\rho\) ([10], p. 59). For \(f \in \mathcal{L}\), \(g \in \mathcal{L}\), we write \(f = g\) if \(f(x) = g(x), \forall x \in X\), and \(f \equiv g\) if \(f(x) = g(x), \text{ a.e. } \mu\). For a Hausdorff locally convex space \(E\), a function \(f: X \to E\) is said to be weakly measurable if \(h \circ f\) is \(\mathcal{A}\)-measurable, \(\forall h \in E'\), the topological dual of \(E\). Two weakly measurable functions \(f_i: X \to E\), \(i = 1, 2\), are said to be weakly equivalent if \(h \circ f_1 = h \circ f_2\), \(\forall h \in E'\). The space \(\mathcal{L}_1\) and norms \(\| \cdot \|_1\) and \(\| \cdot \|_\infty\) have the usual meanings. We shall call a topological space, countably compact if every sequence in it has a cluster point, and sequentially compact if every sequence has a convergent subsequence.

We start with a different proof of the following result of [9].
THEOREM 1 ([9], Theorem 1). Let $H$ be a subset of $L$ such that for any $h_1 \in H$, $h_2 \in H$, $h_1 \neq h_2$ implies $h_1 \neq h_2$. Then, with the pointwise topology on $H$, the following are equivalent:

(i) $H$ is sequentially compact;
(ii) $H$ is compact and metrizable.

If $H$ is convex, then each of (i) and (ii) is also equivalent to:

(iii) $H$ is compact;
(iv) $H$ is countably compact.

Proof. By ([6], Theorem 11.2, p. 187) each of (i), (ii), (iii), (iv) implies that $H$ is relatively compact in $R^k$, with product topology. Thus each of these conditions implies that $H$ is pointwise bounded. Denote by $\varphi$ the homeomorphism, $[0, \infty] \to [0, 1], x \to x/(1 + x)$. For any $\alpha \in I$, the directed net of all finite subsets of $H$, let $h_\alpha = \sup \{|h| : h \in \alpha\}$, and $p_\alpha = \rho(\varphi \circ h_\alpha)$. \{p_\alpha\} is a monotone bounded net in $C_0(X)$, which is boundedly complete. Let sup $p_\alpha = p \in C_0(X)$. This means there is an increasing sequence $\{\alpha(n)\} \subset I$ such that $p = \sup p_{\alpha(n)}$ (this follows from the fact that $\mu(p) = \sup \mu(p_\alpha)$). Since $p_\alpha \equiv \varphi \circ h_\alpha$, we get $p_\alpha^{-1}(1)$ is $\mu$-null, $\forall \alpha$. From this it follows that $K = p^{-1}(1)$ is $\mu$-null. Thus $q = (\varphi^{-1} \circ p)_{X/K}$ is a measurable function such that $|h| \leq q$ a.e. $[\mu], \forall h \in H$.

(i) $\iff$ (ii) is simple ([8], Prop. 1, p. 197), the metric $d$ of (ii) being defined by $d(f, g) = ||(f - g)/(1 + q)||$. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are trivial. Now we come to the proof of (iv) $\Rightarrow$ (i). Take a sequence \{f'\}_n \subset H. Since 1/(1 + q)H is relatively weakly compact in $(L_1, ||\cdot||_1)$ there exists a subsequence \{f'_n\} of \{f'\} and an $f_0 \in L_1$ such that 1/(1 + q)f'_n $\to$ $f_0$ weakly. Thus there exists a sequence \{g_n\} in the convex hull of \{f_n : 1 \leq n < \infty\} (note \{g_n\} $\subset$ H) such that 1/(1 + q)g_n $\to$ $f_0$ a.e. $[\mu]$ (because a convergent sequence in $(L_1, ||\cdot||_1)$ has a subsequence converging a.e. $[\mu]$). Taking $f$ to be a cluster point of \{g_n\} in $H$, we get 1/(1 + q)f $\equiv$ $f_0(\mu)$. We claim $f'_n \to f$ in $H$. If $f'_n \to f$ there exists an $x \in X$, an $\varepsilon > 0$, and a subsequence \{f''_n\} of \{f'_n\} such that one of the two following conditions are satisfied:

(i) $f''_n(x) > f(x) + \varepsilon, \forall n$;
(ii) $f''_n(x) < f(x) - \varepsilon, \forall n$.

Since 1/(1 + q)f''_n $\to$ 1/(1 + q)f weakly, proceeding as before we get a sequence \{g''_n\} in the convex hull of \{f''_n : 1 \leq n < \infty\} such that 1/(1 + q)g''_n $\to$ 1/(1 + q)f a.e. $[\mu]$. If $f''$ is a cluster point of \{g''_n\} in $H$ we get $f'' \equiv f(\mu)$ but because of (i) or (ii), $f''(x) \neq f(x)$, a contradiction. This proves that $H$ is sequentially compact.

This result is also proved in [11] by a different method.
By a classical theorem of Phillips [5], if \( f: X \to E, E \) being a Banach space, is weakly measurable and \( f(X) \) is relatively weakly compact in \( E \), then \( f \) is weakly equivalent to a strongly measurable function ([8], Theorem 3, p. 200). What one really needs to do is to find a weakly equivalent function \( g \) such that \( g(X) \) is separable.

The next theorem is a generalization of Phillips’ theorem.

**Theorem 2.** Let \((E, \mathcal{J})\) be a Hausdorff locally convex space such that there exists an increasing sequence \( \{A_n\} \) of absolutely convex compact subsets of \((E', \sigma(E', E))\) whose union is dense in \((E', \sigma(E', E))\). Suppose \( f: X \to E \) is weakly measurable and \( f(X) \subset K \), for some weakly compact convex subset of \( E \). Then there exists a weakly measurable function \( g: X \to E, g = f(w) \) and \( g(X) \subset K_0 \), a separable closed convex subset of \( K \).

**Proof.** Since \((E, \sigma(E, E'))\) can be considered as a subspace of \( R^{E'} \), with product topology, \( f \) can be considered as \( f: X \to R^{E'} \). For each \( h \in E' \), define \( g(h) = \rho(h \circ f) \) and let \( g: X \to R^{E'}, (g)_h = g(h), \forall h \in E' \).

\( g \) is evidently continuous. If \( g(x_0) \in K \) for some \( x_0 \in X \), there exists, by separation theorem ([6], p. 65), an \( h \in E' \) such that \( h \circ g(x_0) > \sup (K) \). This is a contradiction since \( h \circ f \leq \sup h(K) \) implies \( \rho(h \circ f) \leq \sup h(K) \). Evidently \( g = f(w) \). Fix \( n \in N \). By Theorem 1, \( B_n = \{h \circ g; h \in A_n\} \), with the topology of pointwise convergence on \( X \), is a compact metric space. We metrize \( E \) by the seminorms \( p_n \), \( p_n(x) = \sup \{|h(x)|; h \in A_n\} \). We denote this metric topology by \( \mathcal{J}_n \).

For each \( n \), \( E_n = (C(B_n), \|\cdot\|) \) is a separable Banach space (here \( \|\cdot\| \) is sup norm), and so \( F = \prod_{n=1}^{\infty} E_n \) is a separable Frechet space. Let \( X_0 \) be the quotient space obtained from \( X \) by the equivalent relation, \( x = y \iff g(x) = g(y) \). Each \( x \in X_0 \) gives rise to \( x \in C(B_n), x(t) = t(x) \) for each \( t \in B_n \), for every \( n \). Thus \( X_0 \) can be embedded in \( F \), \( x_0 \to (x_0, x_0, \ldots) \in F \). Taking, on \( X_0 \), the topology induced by \( F \), we easily verify that \( g: X_0 \to (E, \mathcal{J}_0) \) is continuous and so \( (g(X), \mathcal{J}_0) \) is separable. Let \( K_0 = \) the closed convex hull, in \((E, \mathcal{J})\), of a countable dense subset of \((g(X), \mathcal{J}_0)\). If \( g(X) \not\subset K_0 \), by separation theorem, there exists an \( h \in E' \) and \( x_0 \in X \) such that \( h \circ g(x_0) > \sup h(K_0) \). Since \((E, \mathcal{J}_0) = \bigcup_{n=1}^{\infty} A_n, g \circ g(x_0) \leq \sup q(K_0), \forall q \in \bigcup_{n=1}^{\infty} A_n \). Now there exists a net \( \{h_\alpha\} \subset \bigcup_{n=1}^{\infty} A_n \) such that \( h_\alpha \to h \) uniformly on each compact convex subset of \((E, \sigma(E, E'))\). From this it follows \( h \circ g(x_0) \leq \sup h(K_0) \), a contradiction. This proves the result.

**Remark 3.** If \( E \) is metrizable then \((E', \sigma(E', E))\) contains a sequence of compact absolutely convex sets whose union is \( E' \). If \( Y \) is a completely regular Hausdorff space containing a \( \sigma \)-compact dense set and \( E = C_\beta(Y) \) with strict topology \( \beta_\beta, \beta_\beta \), then it is...
proved in ([3], Theorem 3) that \((E', \sigma(E, E'))\) has an increasing sequence of absolutely convex compact sets with dense union — here \(E\) is not metrizable.

**Remark 4.** The function \(g: X \rightarrow (E, \sigma(E, E'))\), obtained in this theorem, is measurable in the sense of ([2], Def. 4, p. 89).

The next theorem, in some sense, is a generalization of ([9], Theorem 3).

**Theorem 5.** Let \(E\) be a Hausdorff locally convex space such that there exist, in \((E', \sigma(E', E))\), an increasing sequence \(\{A_n\}\) of absolutely convex compact sets whose union is \(E'\). Suppose \(g: X \rightarrow E\) is weakly measurable such that \(g \circ f \neq 0\) implies \(g \circ f \neq 0\), for every \(f \in E'\). Then \(g(X)\) is contained in a separable subspace of \(E\).

**Proof.** In the notations of Theorem 2, \(B_n = \{h \circ g: h \in A_n\}\) are compact and metrizable, with the topology of pointwise convergence, and \(\mathcal{T}_0\) is the metric topology, on \(E_0\) of uniform convergence on \(A_n\). Proceeding exactly as in Theorem 2, we prove that \(g(X)\) is a separable subset of \((E, \mathcal{T}_0)\). Let \(F = (E, \mathcal{T}_0)'\) and \(E_0 =\) the closed separable subspace, in \((E, \mathcal{T})\), generated by a countable dense subset of \((g(X), \mathcal{T}_0)\). If \(g(x_0) \in E_0\) for some \(x_0 \in X\) there exists, by separation theorem, an \(h \in E'\) such that \(h \circ g(x_0) > 0\) and \(h \equiv 0\) on \(E_0\). Since \(E' = \bigcup_{n=1}^{\infty} A_n \subset F\), \(h \circ g(x_0) \leq \sup (h \circ g(X)) \leq \sup h(E_0) = 0\), a contradiction. This proves the result.

In the next theorem we do not assume \(H\) to be uniformly bounded ([8], Theorem 4, p. 203).

**Theorem 6.** Let \(H\) be a pointwise bounded subset of \(C(X)\). If \(H\) is equicontinuous then, with the topology of pointwise convergence on \(X\), its closure in \(C(X)\) is compact and metrizable. Conversely if \(H\) is sequentially compact then there is a \(\mu\)-null set \(A\) such that \(H\) is equicontinuous at each point of the open set \(X\setminus A\) of \((X, \tau_p)\).

**Proof.** If \(H\) is equicontinuous then its pointwise closed convex hull \(H_{\text{cv}}\) in \(R^f\), lies in \(C(X)\) and is compact and convex, and so the result follows from Theorem 1.

Conversely suppose \(H\) is sequentially compact. Then, by Theorem 1, \(H\) is compact and metrizable. By the generalized Egoroff's theorem ([4], p. 198) there exists a \(\mathcal{U}\)-partition of \(X = \bigcup_{i=0}^{\infty} X_i\), with \(\mu(X_0) = 0\) and \(\mu(X_i) > 0, \forall i \geq 1\) such that \(H|_{X_i}\) is compact in the topology of uniform convergence on \(X_i, \forall i \geq 1\).
$Y_i = X_i \cap \rho(X_i), \ i \geq 1$, are nonvoid, disjoint, open subsets of $(X, \tau_p)$ and $\mu(A) = 0$, where $A = X \setminus \bigcup_{i=1}^{\infty} Y_i$. By the Ascoli Theorem ([1], Ch. X, §2.5), $H|_{\gamma_i}$ are equicontinuous for each $i$. The result follows now.

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