Pacific Journal of Mathematics

POINTWISE COMPACTNESS AND MEASURABILITY

Surjit Singh Khurana

Vol. 83, No. 2 April 1979

PACIFIC JOURNAL OF MATHEMATICS
Vol. 83, No. 2, 1979

POINTWISE COMPACTNESS AND MEASURABILITY

SURJIT SINGH KHURANA

Among other results it is proved that if $(X, \mathfrak{A}, \mu)$ is a probability space, E a Hausdorff locally convex space such that $(E', \sigma(E', E))$ contains an increasing sequence of absolutely convex compact sets with dense union, and $f: X \to E$ weakly measurable with $f(X) \subset K$, a weakly compact convex subset of E, then f is weakly equivalent to $g: X \to E$ with $g(X)$ contained in a separable subset of K.

In [8] and [9] some remarkable results are obtained for the pointwise compact subsets of measurable real-valued functions and some interesting applications to strongly measurable Banach space-valued functions are established. In this paper we continue those ideas a little further. We first give a somewhat different proof of ([9], Theorem 1) and then apply it to give a generalization of classical Phillip's theorem ([5]). Also some result about equicontinuous subsets of $C(X)$, the space of all continuous real-valued functions on (X, τ_ρ) (τ_ρ is the lifting topology, [10], p. 59; in [8] this topology is denoted by T_ρ) are obtained.

All locally convex spaces are taken over reals and notations of [6] are used. For a topological space Y, $C(Y)$ (resp. $C_b(Y)$) will denote the set of all (resp. all bounded) real-valued continuous functions of Y. N will denote the set of natural numbers.

In this paper $(X, \mathfrak{A}, \mu)$ is a complete probability measure space. Let $\mathscr{L}$ be the set of all real-valued $\mathfrak{A}$-measurable functions on X, $\mathscr{L}^\infty$, the essentially bounded elements of $\mathscr{L}$, and M^∞, the bounded elements of $\mathscr{L}$. We fix a lifting, [10], $\rho: \mathscr{L}^\infty \to M^\infty$ and on X we always take the lifting topology τ_ρ ([10], p. 59). For $f \in \mathscr{L}$, $g \in \mathscr{L}$, we write $f = g$ if $f(x) = g(x)$, $\forall x \in X$, and $f \equiv g$ if $f(x) = g(x)$, a.e. $[\mu]$. For a Hausdorff locally convex space E, a function $f: X \to E$ is said to be weakly measurable if $h \circ f$ is $\mathfrak{A}$-measurable, $\forall h \in E'$, the topological dual of E. Two weakly measurable functions $f_i: X \to E$, $i = 1, 2$, are said to be weakly equivalent if $h \circ f_1 \equiv h \circ f_2$, $\forall h \in E'$. The space $\mathscr{L}_1$ and norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ have the usual meanings. We shall call a topological space, countably compact if every sequence in it has a cluster point, and sequentially compact if every sequence has a convergent subsequence.

We start with a different proof of the following result of [9].

THEOREM 1 ([9], *Theorem* 1). *Let H be a subset of $\mathscr{L}$ such that for any $h_1 \in H$, $h_2 \in H$, $h_1 \neq h_2$ implies $h_1 \not\equiv h_2$. Then, with the pointwise topology on H, the following are equivalent*:

(i) *H is sequentially compact*;

(ii) *H is compact and metrizable.*

If H is convex, then each of (i) *and* (ii) *is also equivalent to*:

(iii) *H is compact*;

(iv) *H is countably compact.*

Proof. By ([6], Theorem 11.2, p. 187) each of (i), (ii), (iii), (iv) implies that H is relatively compact in R^X, with product topology. Thus each of these conditions implies that H is pointwise bounded. Denote by φ the homeomorphism, $[0, \infty] \to [0, 1]$, $x \to x/(1 + x)$. For any $\alpha \in I$, the directed net of all finite subsets of H, let $h_\alpha = \sup\{|h|: h \in \alpha\}$, and $p_\alpha = \rho(\varphi \circ h_\alpha)$. $\{p_\alpha\}$ is a monotone bounded net in $C_b(X)$, which is boundedly complete. Let $\sup p_\alpha = p \in C_b(X)$. This means there is an increasing sequence $\{\alpha(n)\} \subset I$ such that $p = \sup p_{\alpha(n)}$ (this follows from the fact that $\mu(p) = \sup \mu(p_\alpha)$). Since $p_\alpha \equiv \varphi \circ h_\alpha$, we get $p_\alpha^{-1}\{1\}$ is μ-null, $\forall \alpha$. From this it follows that $K = p^{-1}\{1\}$ is μ-null. Thus $q = (\varphi^{-1} \circ p)\chi_{X/K}$ is a measurable function such that $|h| \leqq q$ a.e. $[\mu]$, $\forall h \in H$.

(i) $\Leftrightarrow$ (ii) is simple ([8], Prop. 1, p. 197), the metric d of (ii) being defined by $d(f, g) = \|(f - g)/1 + q\|_1$. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are trivial. Now we come to the proof of (iv) $\Rightarrow$ (i). Take a sequence $\{f'_n\} \subset H$. Since $1/(1 + q)H$ is relatively weakly compact in $(\mathscr{L}_1, \|\cdot\|_1)$ there exists a subsequence $\{f_n\}$ of $\{f'_n\}$ and an $f_0 \in \mathscr{L}_1$ such that $1/(1 + q)f_n \to f_0$ weakly. Thus there exists a sequence $\{g_n\}$ in the convex hull of $\{f_n: 1 \leqq n < \infty\}$ (note $\{g_n\} \subset H$) such that $1/(1 + q)g_n \to f_0$ a.e. $[\mu]$ (because a convergent sequence in $(\mathscr{L}_1, \|\cdot\|_1)$ has a subsequence converging a.e. $[\mu]$). Taking f to be a cluster point of $\{g_n\}$ in H, we get $1/(1 + q)f \equiv f_0(\mu)$. We claim $f_n \to f$ in H. If $f_n \not\to f$ there exists an $x \in X$, an $\varepsilon > 0$, and a subsequence $\{f''_n\}$ of $\{f_n\}$ such that one of the two following conditions are satisfied:

(i) $f''_n(x) > f(x) + \varepsilon$, $\forall n$;

(ii) $f''_n(x) < f(x) - \varepsilon$, $\forall n$.

Since $1/(1 + q)f''_n \to 1/(1 + q)f$ weakly, proceeding as before we get a sequence $\{g''_n\}$ in the convex hull of $\{f''_n: 1 \leqq n < \infty\}$ such that $1/(1 + q)g''_n \to 1/(1 + q)f$ a.e. $[\mu]$. If f'' is a cluster point of $\{g''_n\}$ in H we get $f'' \equiv f(\mu)$ but because of (i) or (ii), $f''(x) \neq f(x)$, a contradiction. This proves that H is sequentially compact.

This result is also proved in [11] by a different method.

By a classical theorem of Phillips [5], if $f: X \to E$, E being a Banach space, is weakly measurable and $f(X)$ is relatively weakly compact in E, then f is weakly equivalent to a strongly measurable function ([8], Theorem 3, p. 200). What one really needs to do is to find a weakly equivalent function g such that $g(X)$ is separable. The next theorem is a generalization of Phillips' theorem.

THEOREM 2. *Let $(E, \mathscr{T})$ be a Hausdorff locally convex space such that there exists an increasing sequence $\{A_n\}$ of absolutely convex compact subsets of $(E', \sigma(E', E))$ whose union is dense in $(E', \sigma(E', E))$. Suppose $f: X \to E$ is weakly measurable and $f(X) \subset K$, for some weakly compact convex subset of E. Then there exists a weakly measurable function $g: X \to E$, $g \equiv f(w)$ and $g(X) \subset K_0$, a separable closed convex subset of K.*

Proof. Since $(E, \sigma(E, E'))$ can be considered as a subspace of $R^{E'}$, with product topology, f can be considered as $f: X \to R^{E'}$. For each $h \in E'$, define $g(h) = \rho(h \circ f)$ and let $g: X \to R^{E'}$, $(g)_h = g(h)$, $\forall h \in E'$. g is evidently continuous. If $g(x_0) \notin K$ for some $x_0 \in X$, there exists, by separation theorem ([6], p. 65), an $h \in E'$ such that $h \circ g(x_0) >$ $\sup(K)$. This is a contradiction since $h \circ f \leqq \sup h(K)$ implies $\rho(h \circ f) \leqq \sup h(K)$. Evidently $g \equiv f(w)$. Fix $n \in N$. By Theorem 1, $B_n = \{h \circ g: h \in A_n\}$, with the topology of pointwise convergence on X, is a compact metric space. We metrize E by the seminorms p_n, $p_n(x) = \sup\{|h(x)|: h \in A_n\}$. We denote this metric topology by $\mathscr{T}_0$. For each n, $E_n = (C(B_n), ||\cdot||)$ is a separable Banach space (here $||\cdot||$ is sup norm), and so $F = \prod_{n=1}^{\infty} E_n$ is a separable Frechet space. Let X_0 be the quotient space obtained from X by the equivalent relation, $x \equiv y \Leftrightarrow g(x) = g(y)$. Each $x \in X_0$ gives rise to $x \in C(B_n)$, $x(t) = t(x)$ for each $t \in B_n$, for every n. Thus X_0 can be embedded in F, $x_0 \to$ $(x_0, x_0, \cdots) \in F$. Taking, on X_0, the topology induced by F, we easily verify that $g: X_0 \to (E, \mathscr{T}_0)$ is continuous and so $(g(X), \mathscr{T}_0)$ is separable. Let $K_0 =$ the closed convex hull, in $(E, \mathscr{T})$, of a countable dense subset of $(g(X), \mathscr{T}_0)$. If $g(X) \not\subset K_0$, by separation theorem, there exists an $h \in E'$ and $x_0 \in X$ such that $h \circ g(x_0) > \sup h(K_0)$. Since $(E, \mathscr{T}_0)' \supset \bigcup_{n=1}^{\infty} A_n$, $q \circ g(x_0) \leqq \sup q(K_0)$, $\forall q \in \bigcup_{n=1}^{\infty} A_n$. Now there exists a net $\{h_\alpha\} \subset \bigcup_{n=1}^{\infty} A_n$ such that $h_\alpha \to h$ uniformly on each compact convex subset of $(E, \sigma(E, E'))$. From this it follows $h \circ g(x_0) \leqq \sup h(K_0)$, a contradiction. This proves the result.

REMARK 3. If E is metrizable then $(E', \sigma(E', E))$ contains a sequence of compact absolutely convex sets whose union is E'. If Y is a completely regular Hausdorff space containing a σ-compact dense set and $E = C_b(Y)$ with strict topology β_0, β_1, then it is proved in ([3], Theorem 3) that $(E', \sigma(E', E))$ has an increasing sequence of absolutely convex compact sets with dense union — here E is not metrizable.

REMARK 4. The function $g: X \to (E, \sigma(E, E'))$, obtained in this theorem, is measurable in the sense of ([2], Def. 4, p. 89).

The next theorem, in some sense, is a generalization of ([9], Theorem 3).

THEOREM 5. *Let E be a Hausdorff locally convex space such that there exist, in $(E', \sigma(E', E))$, an increasing sequence $\{A_n\}$ of absolutely convex compact sets whose union is E'. Suppose $g: X \to E$ is weakly measurable such that $g \circ f \neq 0$ implies $g \circ f \not\equiv 0$, for every $f \in E'$. Then $g(X)$ is contained in a separable subspace of E.*

Proof. In the notations of Theorem 2, $B_n = \{h \circ g: h \in A_n\}$ are compact and metrizable, with the topology of pointwise convergence, and $\mathscr{T}_0$ is the metric topology, on E, of uniform convergence on A_n. Proceeding exactly as in Theorem 2, we prove that $g(X)$ is a separable subset of $(E, \mathscr{T}_0)$. Let $F = (E, \mathscr{T}_0)'$ and $E_0 =$ the closed separable subspace, in $(E, \mathscr{T})$, generated by a countable dense subset of $(g(X), \mathscr{T}_0)$. If $g(x_0) \notin E_0$ for some $x_0 \in X$ there exists, by separation theorem, an $h \in E'$ such that $h \circ g(x_0) > 0$ and $h \equiv 0$ on E_0. Since $E' = \bigcup_{n=1}^{\infty} A_n \subset F$, $h \circ g(x_0) \leqq \sup (h \circ g(X)) \leqq \sup h(E_0) = 0$, a contradiction. This proves the result.

In the next theorem we do not assume H to be uniformly bounded ([8], Theorem 4, p. 203).

THEOREM 6. *Let H be a pointwise bounded subset of $C(X)$. If H is equicontinuous then, with the topology of pointwise convergence on X, its closure in $C(X)$ is compact and metrizable. Conversely if H is sequentially compact then there is a μ-null set A such that H is equicontinuous at each point of the open set $X \backslash A$ of (X, τ_ρ).*

Proof. If H is equicontinuous then its pointwise closed convex hull H_0, in R^X, lies in $C(X)$ and is compact and convex, and so the result follows from Theorem 1.

Conversely suppose H is sequentially compact. Then, by Theorem 1, H is compact and metrizable. By the generalized Egoroff's theorem ([4], p. 198) there exists a $\mathfrak{A}$-partition of $X = \bigcup_{i=0}^{\infty} X_i$, with $\mu(X_0) = 0$ and $\mu(X_i) > 0$, $\forall i \geqq 1$ such that $H|_{X_i}$ is compact in the topology of uniform convergence on X_i, $\forall i \geqq 1$.

$Y_i = X_i \cap \rho(X_i)$, $i \geqq 1$, are nonvoid, disjoint, open subsets of (X, τ_ρ) and $\mu(A) = 0$, where $A = X \backslash \bigcup_{i=1}^{\infty} Y_i$. By the Ascoli Theorem ([1], Ch. X, §2.5), $H|_{Y_i}$ are equicontinuous for each i. The result follows now.

REFERENCES

1. N. Bourbaki, *General Topology*, Hermann, Paris, 1966. (Translation by Addison-Wesley, Massachusetts.)
2. N. Dinculeanu, *Vector Measures*, Pergamon Press, New York, 1967.
3. S. S. Khurana, *Dunford-Pettis property*, J. Math. Anal. Appl., **65** (1978), 361-364.
4. P. A. Meyer, *Représentation intégral des fonctions excessives*, Résultats de Mokobodzki, Deminaire de Probabilities V, Univ. de Strasbourg, Springer-Verlag, Berlin-New York, Lecture Notes, **198** (1971), 196-208.
5. R. S. Phillips, *On weakly compact sets of a Banach space*, Amer. J. Math., **65** (1943), 108-136.
6. H. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, 1971.
7. A. Ionescu Tulcea, *Topologies compatible with liftings*, Bull, de la Math. Gréce, **8** (1967), 116-126.
8. ———, *On pointwise convergence, compactness and equicontinuity in the lifting topology I*, Zeit Wahrscheinlickheittheorie und Verw. Gebiete, **26** (1973), 197-205.
9. ———, *On pointwise convergence, compactness and equicontinuity II*, Adv. Math., **12** (1974), 171-177.
10. A. Ionescu Tulcea, C. Ionescu Tulcea, *Topics in the Theory of Liftings*, Springer-Verlag, New York, 1969.
11. L. Schwartz, *Certaines propriétés des measures sur les espaces de Banach*, Sem. Maurey-Schwartz, 1975-76, Ecole Polytechnique, No. 23.

Received August 12, 1978.

THE UNIVERSITY OF IOWA
IOWA CITY, IA 52242

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$84.00 a year (6 Vols., 12 issues). Special rate: \$42.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 83, No. 2 April, 1979