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HYPERSPACES OF COMPACT CONVEX SETS

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The purpose of this paper is to develop in detail certain aspects of the space of nonempty compact convex subsets of a subset X (denoted cc(X)) of a metric locally convex T.V.S. It is shown that if X is compact and dim $(X) \geqq 2$ then cc(X) is homeomorphic with the Hilbert cube (denoted $\mathrm{cc}(X) \cong I_\infty$). It is shown that if $n \geqq 2$, then $\mathrm{cc}(R^n)$ is homeomorphic to I_∞ with a point removed. More specialized results are that if $X \subset R^2$ is such that $\mathrm{cc}(X) \cong I_\infty$ then X is a two cell; and that if $X \subset R^3$ is such that $\mathrm{cc}(X) \cong I_\infty$ and X is not contained in a hyperplane then X must contain a three cell.

For the most part we will be restricting ourselves to compact spaces X although in the last section of the paper, § 7, we consider some fundamental noncompact spaces.

We will be using the following definitions and notation. For each $n = 1, 2, \cdots, R^n$ will denote Euclidean n-space, $S^{n-1} = \{x \in R^n: ||x|| = 1\}$, $B^n = \{x \in R^n: ||x|| \leqq 1\}$, and ${}^0B^n = \{x \in R^n: ||x|| < 1\}$. A *continuum* is a nonempty, compact, connected metric space. An *n-cell* is a continuum homeomorphic to B^n. The symbol I_∞ denotes the *Hilbert cube*, i.e., $I_\infty = \prod_{i=1}^\infty [-1/2^i, 1/2^i]$. By I_∞^0 we will denote the *pseudo interior* of the Hilbert cube, $I_\infty^0 = \prod_{i=1}^\infty (-1/2^i, 1/2^i)$. We let I^+ denote the set of natural numbers. We use cl and $\overline{\mathrm{co}}$, respectively, to denote closure and closed convex hull. If Y is a subset of a space Z, then int$[Y]$ means the union of all open subsets of Z which are contained in Y. The notation $X \cong Y$ will mean that the space X is homeomorphic to the space Y.

All spaces are considered in this paper to be subsets of a real topological vector space. Since we are restricting our attention in this paper to separable metric spaces this is no restriction topologically or geometrically (cf. Vol. I of [14, p. 242]). If X is a space, by *cc*(X) we will mean the hyperspace of all nonempty compact convex subsets of X (with the Hausdorff metric). We will call *cc*(X) the *cc-hyperspace of* X.

If x and y are points in a real topological vector space V, then $\widehat{xy}$ or $[x, y]$ denotes the *convex segment or point* (if $x = y$) *determined by* x and y, i.e., $\widehat{xy} = \{tx + (1-t)y\colon 0 \leqq t \leqq 1\} = [x, y]$. Let $X \subset V$. If $x \in X$, we let $S(x)$ denote $\{y \in X\colon \widehat{xy} \subset X\}$, and we let Ker($X$) denote $\bigcap_{x \in X} S(x)$; the set Ker(X) is called the *kernel of* X. We say X is *starshaped* if and only if $\mathrm{Ker}(X) \neq \varnothing$. For $A \subset Y$, a point p in A is called an *extreme point of* A if and only if no convex segment lying in A has p in its (relative) interior. The symbol $ext[A]$ denotes the set of all extreme points of A. If X is a subset of R^n, for some n, a point $p \in X$ is said to be a *point of local nonconvexity of* X if every neighborhood of p in X fails to be convex. We will denote the set of all points of local nonconvexity of a set X by $LN(X)$. For spaces X and Y with $X \subset Y$ the boundary of X, denoted $\mathrm{Fr}(X)$, is defined by $\mathrm{Fr}(X) = \mathrm{cl}(X) \cap \mathrm{cl}(Y - X)$. A closed subset A of a metric space X is a *Z-set in* X if for any nonnull and homotopically trivial open set $U \subset X$ it is true that $U - A$ is nonnull and homotopically trivial (see [1]).

The paper is organized as follows: In § 2 we give some general results which are closely related to early work of Klee. One of the results of this section establishes that if K is a compact convex subset of a metrizable locally convex topological vector space and $\dim[K] \geqq 2$, then $\mathrm{cc}(K) \cong I_\infty$. This sets the stage for the remainder of the paper, as one of our major concerns becomes obtaining answers to the following question:

(1.1) For what continua K is $\mathrm{cc}(K) \cong I_\infty$? In § 3, we show that if $K \subset R^2$ is as in (1.1), then K is a 2-cell. Thus, for R^2, a complete answer to (1.1) becomes a matter of determining which 2-cells K in R^2 have their cc-hyperspace homeomorphic to I_∞. Results about this are in § 5, where we show that there is a 2-cell in R^2 whose cc-hyperspace is not homeomorphic to I_∞ and we obtain some geometric results which give sufficient conditions on a continuum X in order that $\mathrm{cc}(X) \cong I_\infty$. Many of the results in § 5 are for continua more general than 2-cells in the plane.

Though $K \subset R^2$ as in (1.1) must be a 2-cell, $K \subset R^3$ as in (1.1) need not be a 2-cell or 3-cell (see (4.7)). However, in § 4, we show that if $K \subset R^3$ is as in (1.1) and K is not contained in a 2-dim hyperplane in R^3, then K must *contain* a 3-cell (see (4.1)). Some lemmas about arcs of convex arcs in R^2 and arcs of convex 2-cells in R^3, which we use to prove (4.1), seem to be of interest in themselves.

In § 6 we give some examples and state some problems. Many of these help to delineate the status of the problem of which 2-cells in R^2 have their cc-hyperspace homeomorphic to I_∞. The technique used in (6.4) is particularly noteworthy since using it in combination with suitable results for 2-cells with polygonal boundary can, perhaps, lead to a satisfactory solution of (1.1).

The final section, § 7, begins to touch on the problems connected with determining the topological type of the cc-hyperspace of some noncompact subsets of topological vector spaces. The main result of this section is that, for $n \geqq 2$, $\mathrm{cc}(R^n) \cong I_\infty - \{p\}$ for $p \in I_\infty$. Several open questions are also posed in this section.

2. Some basic results.

(2.1) LEMMA. *Let K be a compact convex subset of a metrizable locally convex real topological vector space L, $\dim[K] \geqq 2$. Then there exists a countable family $\{\zeta_i: i = 1, 2, \cdots\}$ of continuous linear functionals ζ_i such that given $A \in cc(K)$ and $x \in [K - A]$, there exists a $j \in I^+$ such that $\zeta_j(x) \notin \zeta_j(A)$.*

Proof. The compact metric space K in the relative topology has a countable base of convex sets $Q = \{V_i\}_{i=1}^{\infty}$. Define a family F by $F = \{(V_1, V_2, \cdots, V_n) | n \in I^+, V_i \in Q \text{ and } \overline{\text{co}}(\bigcup_{i=1}^{n-1} V_i) \cap \text{cl}[V_n] = \emptyset\}$. Given any $(V_1, V_2, \cdots, V_n) \in F$, by a (well known) separation theorem there exists a continuous linear functional strictly separating $\overline{\text{co}}(\bigcup_{i=1}^{n-1} V_i)$ and $\text{cl}[V_n]$. For each member of F, select one such functional thus obtaining a countable family $\{\zeta_i\}_{i=1}^{\infty}$ of functionals. The proof is completed by noting that for $x \in K$ and $A \in cc(K)$ with $x \notin A$ there exists a $(V_1, V_2, \cdots, V_n) \in F$ with $A \subset \overline{\text{co}}(\bigcup_{i=1}^{n-1} V_i)$ and $x \in \text{cl}[Y_n]$.

(2.2) THEOREM. *Let K be a compact convex subset of a metrizable locally convex real topological vector space L, $\dim[K] \geqq 2$. Then $cc(K) \cong I_\infty$.*

Proof. For each $A \in cc(K)$, let $\zeta_i(A) = [a_i, b_i]$ where the ζ_i are as in (2.1) such that, without loss of generality, $\sup\{|\zeta_i(x)|: x \in K\} \leqq 1$ for each i. Let $F: cc(K) \to I_\infty$ be defined by

$$F(A) = (a_1/2, b_1/2^2, a_2/2^3, b_2/2^4, \cdots, a_n/2^{2n-1}, b_n/2^{2n}, \cdots) .$$

Since $\{\zeta^2\}_{i=1}^{\infty}$ is a separating family, F is one-to-one. Furthermore, for each j, the co-ordinate functions $F_{2j-1} = a_j/2^{2j-1}$ and $F_{2j} = b_j/2^{2j}$ are continuous since ζ_j is continuous. Thus, F is continuous (we are mapping into I_∞). Let $A^1, A^2 \in cc(K)$, $\lambda \in [0, 1]$, and $j \in I^+$; then, using the linearity of ζ_j,

$$\begin{aligned} \zeta_j(\lambda A^1 + (1 - \lambda)A^2) &= \lambda\zeta_j(A^1) + (1 - \lambda)\zeta_j(A^2) \\ &= \lambda[a_j^1, b_j^1] + (1 - \lambda)[a_j^2, b_j^2] \\ &= [\lambda a_j^1 + (1 - \lambda)a_j^2, \lambda b_j^1 + (1 - \lambda)b_j^2] , \end{aligned}$$

where $[a_j^k, b_j^k] = \zeta_j(A^k)$ for $k = 1$ and 2. Thus, $F_t(\lambda A^1 + (1 - \lambda)A^2) = \lambda F_t(A^1) + (1 - \lambda)F_t(A^2)$ where $t = 1, 2, \cdots,$. This says that the set $F(cc(K))$ is convex. Now, since $\dim[K] \geqq 2$ K contains a convex 2-cell, say D. Thus, for each n, K contains a regular n-sided polygon P_n with sides $s_1, s_2, \cdots, s_n$ which lies in the "interior" of the 2-cell D. For each i, let A_i be a convex arc which lies in the exterior of P_n along the perpendicular bisector of s_i in D. For each n-tuple $(t_1, t_2, \cdots, t_n)$ in $\prod_{i=1}^n A_i$ let $G((t_1, t_2, \cdots, t_n)) = \overline{\text{co}}(\{t_1, t_2, \cdots, t_n\})$. It is clear that the mapping G is a homeomorphism of the n-cell $\prod_{i=1}^n A_i$ into $\text{cc}(K)$. Thus, $\text{cc}(K)$ contains an n-cell for every n and, tnerefore, is infinite dimensional. Thus, $F(\text{cc}(K))$ is a compact and infinite dimensional convex subset of l_2. Hence, by Keller's theorem [10], $F(\text{cc}(K)) \cong I_\infty$. Therefore, $\text{cc}(K) \cong I_\infty$.

We point out that the proof of Theorem 2.2 is a slight modification of a proof used by Klee [12] to generalize Keller's theorem. Also Klee, in a conversation with the authors, has pointed out a different proof of Theorem 2.2 in the case when L is a normed linear space. This consists of using a theorem in [17] to embed the compact convex subsets of a normed linear space into a normed linear space, noting that for a fixed $K \subset L$, $\text{cc}(K)$ is embedded convexly, and then using Klee's generalization [12] of Keller's theorem.

Let L be as in (2.2) and let $F \subset \text{cc}(L)$. We say that the *family F is convex* if and only if for all $A, B \in F$ and λ, $0 \leqq \lambda \leqq 1$, $(\lambda A + (1-\lambda)B) \in F$ (where λA means $\{\lambda \cdot a \colon a \in A\}$).

(2.3) THEOREM. *Let L be as in* (2.2) *and let $F \subset \text{cc}(L)$ be such that F is compact, convex, and infinite dimensional. Then, $F \cong I_\infty$.*

Proof. By (2.2) $\text{cc}(L)$ and hence F can be affinely embedded into l_2. But then F is a compact, convex, infinite dimensional subset of l_2 and Keller's theorem applies to give $F \cong I_\infty$ (see [10]).

As a consequence of (2.3) and the part of the proof of (2.2) showing that $\text{cc}(K)$ is infinite dimensional, we have the following two corollaries.

(2.4) COROLLARY. *Let K and L be as in* (2.2). *Let Q be a given compact subset of K such that $\overline{\text{co}}[Q] \neq K$. Then, $\{A \in \text{cc}(K) \colon Q \subset A\} \cong I_\infty$.*

(2.5) COROLLARY. *Let K and L be as in* (2.2). *Let K_0 be a given nonempty compact convex subset of K. Then $\{A \in \text{cc}(K) \colon A \cap K_0 \neq \varnothing\} \cong I_\infty$.*

It follows, in particular, from (2.3) or (2.4) that the space of compact convex subsets of the unit disc in R^2 which contain the origin is homeomorphic to I_∞.

3. A topological converse to (2.2) for the plane. In the plane, (2.2) says that the cc-hyperspace of a convex 2-cell is homeomorphic to the Hilbert cube. The question arises as to which subsets of the plane have their cc-hyperspaces homeomorphic to I_∞. A complete answer to this problem will involve both topological and geometric considerations. The topological considerations are the subject of this section. Our result is

(3.1) THEOREM. *If X is a continuum in R^2 such that $\mathrm{cc}(X) \cong I_\infty$, then X is a two cell.*

To prove (3.1) we will make use of the following lemmas. The first three lemmas are stated in more generality than explicitly needed for proving (3.1).

(3.2) LEMMA. *Let E be a Banach space which admits a topologically equivalent norm that is strictly convex. Then there is a continuous selection from $\mathrm{cc}(E)$ to E. Thus, for any separable Banach space, there is such a selection.*

Proof. Let $\|\cdot\|$ denote a strictly convex norm on E and let $p \in E$. Define η: $\mathrm{cc}(E) \to E$ by letting $\eta(A)$ denote the unique point $a_0 \in A$ such that $\inf\{\|p - a\| \, a \in A\} = \|p - a_0\|$ (see [3, p. 19]). It is easy to see that η is continuous and is a selection. The second part of (3.2) follows from the fact that any separable Banach space admits an equivalent strictly convex norm [3, p. 18].

(3.3) LEMMA. *Let X be a dendrite. Then* $\dim[\mathrm{cc}(X)] \leqq 2$.

Proof. Let X be a dendrite (in some real topological vector space) and note that any member of $\mathrm{cc}(X)$ is either a (convex) arc or a singleton. Hence, the barycenter map $g: \mathrm{cc}(X) \to X$ is continuous where g is defined by: if a and b are the endpoints of a convex arc A in X or if $a = b$, in which case let $A = \{a\}$, then $g(A) = (a + b)/2$. Let $p \in X$. Since p belongs to arbitrarily small open subsets of X with finite boundaries [21, p. 99], there are at most countably many convex arcs $A_i = [a_i, b_i]$, $i = 1, 2, \cdots$, maximal with respect to the property that $g(A_i) = p$. For each p let $D_i = \{[s_i, t_i] \subset A_i\colon g([s_i, t_i]) = p\}$. Since the map $s_i \to [s_i, t_i]$ is a homeomorphism of $[a_i, p]$ onto D_i, $D_i \cong [a_i, p]$ (note: D_i could be just $\{p\}$). Also, it is clear that $g^{-1}(p) = \bigcup_{i=1}^{\infty} D_i$. Hence, by III 2 of [9], $\dim[g^{-1}(p)] \leqq 1$. Therefore, from the statement on p. 92 of [9] which is verified in order to prove VI 7 of [9], $\dim[\mathrm{cc}(X)] \leqq 1 + \dim[X] = 2$.

(3.4) LEMMA. *Let X be a continuum lying in a Banach space*

E. If $\mathrm{cc}(X) \cong I_\infty$, *then* X *is an absolute retract and* $\dim[X] \geqq 2$.

Proof. Let F denote the closed linear span of X. Since X is separable, F is a separable subspace of E. Hence, by (3.2), we have a continuous selection $\eta\colon \mathrm{cc}(F) \to F$. Since the restriction of η to $\mathrm{cc}(X)$ is a retraction of $\mathrm{cc}(X)$ onto X, the fact that X is an AR now follows from the well known fact that [**14**, Vol. II, Th. 7, p. 341] a retract of I_∞ is an AR. For the remainder of the proof, suppose $\dim[X] \leqq 1$. If $\dim[X] = 0$, in which case X consists of only one point, then $\mathrm{cc}(X) \cong X$. So, for the purpose of proof, assume $\dim[X] = 1$. Then X is a one-dimensional AR and, hence, a dendrite (cf. Brosuk's "Theory of Retracts" p. 138). By (3.3) this implies $\dim[\mathrm{cc}(X)] \leqq 2$ which contradicts the assumption that $\mathrm{cc}(X) \cong I_\infty$.

(3.5) *Conjecture.* If A is a dendrite, then $\mathrm{cc}(A)$ is embeddable in the plane.

(3.6) LEMMA. *The space of singletons and convex arcs in* $R^n (n \geqq 2)$ *denoted* $AS(R^n)$, *is homeomorphic to* $R^n \times ([0, \infty) \times P^{n-1} / 0 \times P^{n-1})$. *In the special case that* $n = 2$, $AS(R^2) \cong R^4$.

Proof. We note that the space of lines through the origin in R^n is homeomorphic to projective $n-1$ space P^{n-1}. For each convex arc or point $\widehat{ab}$ in R^n define $F(\widehat{ab})$ in $R^n \times ([0, \infty) \times p^{n-1}/0 \times p^{n-1})$ by $F(\widehat{ab}) = (a+b)/2, [(\|b-a\|, s)]$ where s is the point of p^{n-1} determined by the line parallel to $\widehat{ab}$ if $\widehat{ab}$ is nondegenerate and s is the point of p^{n-1} determined by the first axis if $\widehat{ab}$ is a singleton. In this proof we have used $[\circ]$ to denote "equivalence class." It is a straightforward matter to check that F is a homeomorphism. If $n = 2$, then $R^2 \times ([0, \infty) \times p^1/0 \times p^1) \cong R^2 \times ([0, \infty) \times S^1/0 \times S^1) \cong R^2 \times R^2 \cong R^4$. The lemma is proved.

(3.7) LEMMA. *If* X *is a continuum in* R^2 *such that* $\mathrm{cc}(X) \cong I_\infty$, *then* $\mathrm{int}[X] \neq \emptyset$ *and* $X = \mathrm{cl}(\mathrm{int}[X])$.

Proof. Suppose there is a point p in $X - \mathrm{cl}(\mathrm{int}(X))$. Clearly, we may then choose a neighborhood N in $\mathrm{cc}(X)$ about $\{p\}$ such that N consists only of singletons and convex arcs. Hence, N is embeddable in R^4 (by (3.6)) and, therefore, finite dimensional. This contradicts the assumption that $\mathrm{cc}(X) \cong I_\infty$.

(3.8) LEMMA. *If* X *is a continuum in* R^2 *such that* $\mathrm{cc}(X) \cong I_\infty$, *then* $\mathrm{int}[X]$ *is connected.*

Proof. Let p and q be distinct points of $\mathrm{int}[X]$. We show that there is an arc in $\mathrm{int}[X]$ from p to q. Let $\Lambda = \{A \in \mathrm{cc}(X) | A$ is a singleton or a convex arc$\}$. By virtue of (3.6), Λ is finite dimensional. Therefore, since $\mathrm{cc}(X) \cong I_\infty$ and Λ is compact, $\mathrm{cc}(X) - \Lambda$ is arcwise connected (that no finite dimensional continuum can separate I_∞) (arc separate is equivalent to separate for locally connected continua) follows from the fact that, for each n, I_n is a Cantor manifold (see Corollary 2 on p. 48 of [9]) and the set of all points of the form $\bigcup_{n=1}^{\infty} I_n^1$ is dense in I_∞ (here $I_n^1 = \prod_{i=1}^{n} I_i \times (1/2, 1/2, \cdots))$. Let $K, L \in \mathrm{cc}(X)$ be 2-cells with $[K \cup L] \subset \mathrm{int}[X]$ and $\beta(K) = p$ and $\beta(L) = q$ (where $\beta\colon \mathrm{cc}(X) \to X$ is the barycenter map). Now, let α be an arc in $\mathrm{cc}(X) - \Lambda$ with endpoints K and L. Since $\alpha \subset [\mathrm{cc}(X) - \Lambda]$ each point of α is a 2-cell and thus, the restriction of β to α is continuous. Thus, $\beta(\alpha)$ is a locally connected continuum and hence $\beta(\alpha)$ is arcwise connected. Since $X \subset R^2$ and each member M of α is a 2-cell, it follows that $\beta(M) \in \mathrm{int}(M) \subset \mathrm{int}[X]$. Therefore, we now have that $\beta(\alpha)$ is arcwise connected and $p, q \in \beta(\alpha) \subset \mathrm{int}[X]$. The lemma follows.

Proof of Theorem 3.1. By (3.4), X is an absolute retract and therefore $R^2 - X$ is connected [7, p. 364]. Therefore, (since X is a locally connected continuum in R^2), $\mathrm{Bd}[R^2 - X]$ is a locally connected continuum (see 2.2 of [21, p. 106]). Let N denote $\mathrm{Bd}[R^2 - X]$. Direct computation using only definitions yields

$$(*) \qquad R^2 - N = [R^2 - X] \cup \mathrm{int}\, X\,.$$

Thus we have that N is a locally connected continuum and, by (3.9), and $(*)$ $E^2 - X$ and $\mathrm{int}[X]$ are the components of $E^2 - N$. It now follows from 2.51 of [21, p. 107] that there is a simple closed curve $J \subset N$. Let G denote the bounded component of $E^2 - J$. By (3.8), $\mathrm{int}[X] \subset G$, and hence, $\mathrm{cl}(\mathrm{int}[X]) \subset [G \cup J]$. Therefore, by (3.7), $X \subset [G \cup J]$. However, since $E^2 - X$ is connected and $J \subset X$, we have $G \subset X$, i.e., $[G \cup J] \subset X$. This proves $X = G \cup J$ and, thus, X is a 2-cell. This proves (3.1).

REMARK. The part of the proof of Theorem 3.1 which follows the lemmas is devoted entirely to showing that if Z is a planar compact absolute retract such that $Z = \mathrm{cl}(\mathrm{int}[Z])$ and $\mathrm{int}[Z]$ is connected, then Z is a 2-cell. This characterization of 2-cells among continua in the plane does not seem to be explicitly stated in the literature.

4. Analogue to the 2-cell theorem for 3-space. In this section we will establish

(4.1) THEOREM. *If X is a continuum in R^3 such that* $\mathrm{cc}(X) \cong I_\infty$ *and X is not contained in any 2-dimensional hyperplane, then* $\mathrm{int}[X] \neq \varnothing$.

We use the following lemmas to prove (4.1).

(4.2) LEMMA. *Let $\sigma: [0, 1] \to cc(R^2)$ be an arc of convex arcs in R^2. Suppose that L is a straight line in R^2 such that, for $0 \leqq t \leqq s$ where $s > 0$, $L \cap \sigma(t)$ consists of only one point. Then the convex segment with noncut points $\sigma(0) \cap L$ and $\sigma(s) \cap L$ is contained in $\bigcup_{0 \leqq t \leqq s} \sigma(t)$.*

(4.3) REMARK. It is easy using (4.2) to prove that if $\sigma[0, 1] \to \mathrm{cc}(R^2)$ is a one-to-one continuous mapping such that, for each $x \in [0, 1]$, $\sigma(s)$ is a convex arc and such that there exist s_1 and s_2 such that $\sigma(s_1)$ and $\sigma(s_2)$ are not co-linear, then $\bigcup_{s \in [0,1]} \sigma(s)$ contains a 2-cell.

Proof. Consider the mapping $\tilde{\sigma}$: $[0, s] \to L$ defined by $\tilde{\sigma}(t) = \sigma(t) \cap L$. Using the single valuedness of $\tilde{\sigma}$, it is easy to show that $\tilde{\sigma}$ is continuous. Thus, $\tilde{\sigma}([0, s])$ is connected in L and the result follows.

(4.4) LEMMA. *Let $\sigma: [0, 1] \to \mathrm{cc}(R^3)$ be an arc of convex 2-cells in R^3 such that there is a sequence $s_r \to 0$ such that $\sigma(s_r)$ and $\sigma(0)$ are not co-planar. Then, $\bigcup_{s \in [0,1]} \sigma(s)$ contains a 3-cell.*

Proof. Let $\prod_i$ $(i = 1, 2, 3)$ be the standard projection onto the ith factor of R^3. Since $\sigma(0)$ is nondegenerate, there exist i_1 and i_2 such that neither $\prod_{i_1}[\sigma(0)]$ nor $\prod_{i_2}[\sigma(0)]$ is a single point. Without loss of generality, we will assume that $i_1 = 1$ and $i_2 = 2$. Let $[a_1, a_2] \subset \mathrm{int}[\prod_1(\sigma(0))]$. Note that, for $x \in [a_1, a_2]$, $\prod_1^{-1}(x) \cap \sigma(0)$ is a nondegenerate arc. Let c be chosen so that $\prod_2^{-1}(c) \cap \prod_1^{-1}((a_1 + a_2)/2) \cap \sigma(0)$ is an interior point of the arc $\sigma(0) \cap \prod_1^{-1}((a_1 + a_2)/2)$. Let $a_1 \leqq a'_1 < (a_1 - a_2)/2 < a'_2 \leqq a_2$ be chosen so that, for each $x \in [a'_1, a'_2]$, $\prod_2^{-1}(c) \cap \prod_1^{-1}(x) \cap \sigma(0)$ is an interior point of the arc $\prod_1^{-1}(x) \cap \sigma(0)$. Let $c_1 < c < c_2$ be chosen so that, for $y \in [c_1, c_2]$ and $x \in [a'_1, a'_2]$ it is true that $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(0)$ is an interior point of the arc $\prod_1^{-1}(x) \cap \sigma(0)$. Let $t > 0$ be chosen so that:

(1) for $s \in [0, t]$ and $x \in [a'_1, a'_2]$, $\prod_1^{-1}(x) \cap \sigma(s)$ cuts $\sigma(s)$, and

(2) for $s \in [0, t]$, $x \in [a'_1, a'_2]$ and $y \in [c_1, c_2]$, $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(s)$ is an interior point of the arc $\prod_1^{-1}(x) \cap \sigma(s)$.

Let $0 < t' < t$ be chosen so that $\sigma(0)$ and $\sigma(t')$ arc not co-planar. Note, since there can be at most one x in $[a'_1, a'_2]$ for which $\sigma(0) \cap$

$\prod_1^{-1}(x)$ and $\sigma(t') \cap \prod_1^{-1}(x)$ are co-linear, we may assume without loss of generality that, for $x \in [a_1', a_2']$, $\prod_1^{-1}(x) \cap \sigma(0)$ and $\prod_1^{-1}(x) \cap \sigma(t')$ are not co-linear. Since, for each $x \in [a_1', a_2']$, there can be at most one $y \in [c_1, c_2]$ such that $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(0) \cap \sigma(t') \neq \varnothing$, we may now choose $a_1' \leqq a_1'' < a_2'' \leqq a_2'$ and $c_1 \leqq c_1' < c_2' \leqq c_2$ so that, for $x \in [a_1'', a_2'']$ and $y \in [c_1', c_2']$, (*) $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(0) \cap \sigma(t') = \varnothing$. Consider now the set of points $D = \{\prod_2^{-1}(c_i') \cap \prod_1^{-1}(a_j'') \cap \sigma(z)\colon i, j = 1, 2, z = 0 \text{ or } z = t'\}$. We claim that $\overline{\text{co}}(D) \subset \bigcup_{s \in [0,1]} \sigma(s)$. To see this, note first that if $D_0 = \{\prod_2^{-1}(c_i) \cap \prod_1^{-1}(a_j'') \cap \sigma(0)\}$ where $i, j = 1, 2\}$ and $D_{t'} = \{\prod_2^{-1}(c_i) \cap \prod_1^{-1}(a_j'') \cap \sigma(t')\colon i, j = 1, 2\}$ then $\overline{\text{co}}(D_z) \subset \sigma(z) \subset [\bigcup_{s \in [0,1]} \sigma(s)]$ where $z \in \{0, t'\}$. Now, if $p \in \overline{\text{co}}(D)$ then, for some $x \in [a_1'', a_2'']$, we have that $p \in \prod_1^{-1}(x)$. Also, for some $y \in [c_1', c_2']$ we have that $p \in \prod_2^{-1}(y)$. Since $p \in \overline{\text{co}}(D)$ we have that p is on the convex segment in $\prod_2^{-1}(y) \cap \prod_1^{-1}(x)$ which joins $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(0)$ and $\prod_2^{-1}(y) \cap \prod_1^{-1}(x) \cap \sigma(t')$. This is true because $\overline{\text{co}}(D_0) \cap \overline{\text{co}}(D_t) = \varnothing$ (otherwise we would contradict (*)). Now, the mapping $\sigma_x\colon [0, t'] \to \text{cc}(\prod_1^{-1}(x))$ defined by $\sigma_x(s) = \sigma(s) \cap \prod_1^{-1}(x)$ is easily seen to be continuous. Also, $\sigma_x(0)$ and $\sigma_x(t')$ are not co-linear and the line $\prod_2^{-1}(y) \cap \prod_1^{-1}(x)$ in $\prod_1^{-1}(x)$ cuts each of the arcs $\sigma_x(s)$ for $s \in [0, t']$. It now follows from (4.2) that $p \in \bigcup_{s \in [0,t']} \sigma_x(s)$. The lemma is proved.

The following lemma is a simple consequence of (4.4).

(4.5) LEMMA. *Let $\sigma\colon [0, 1] \to \text{cc}(R^3)$ be a one-to-one continuous mapping of $[0, 1]$ into $cc(R^3)$ such that $\sigma(s)$ is a (convex) 2-cell for each s and such that there exist s_1 and s_2 such that $\sigma(s_1)$ and $\sigma(s_2)$ are not co-planar. Then, $\bigcup_{s \in [0,1]} \sigma(s)$ contains a 3-cell. We are now ready to establish* (4.1).

Proof of (4.1). It can be seen that the space of convex arcs and points in a compact subset of R^3 is of dimension less than or equal to 6 (see (3.6)). If X satisfies the conditions of (4.1) and $AS(X)$ denotes the space of arcs and singletons in $\text{cc}(X)$ then $\text{cc}(X) - AS(X)$ must be arcwise connected (see the remark in the proof of (3.8)). Let p_1 and p_2 be points in X which lie in the interior of two cells P_1 and P_2, respectively, such that P_1 and P_2 are not co-planar. Now, $[\text{cc}(X) - AS(X)] \supset \{P_1, P_2\}$ and, hence, there is a one-to-one continuous mapping $\sigma\colon [0, 1] \to [\text{cc}(X) - AS(X)]$ such that $\sigma(0) = P_1$ and $\sigma(1) = P_2$. If $\sigma(s)$ is not a 2-cell for some s, then $\sigma(s)$ is a 3-cell and we are done. Hence, without loss of generality, we may assume $\sigma(s)$ is a 2-cell for each $s \in [0, 1]$. Thus, by virtue of (4.5), $X \supset \bigcup_{s \in [0,1]} \sigma(s)$ contains a 3-cell. The theorem is proved.

(4.6) EXAMPLE. We show that the natural analogue to (4.1) does not hold in R^n, $n > 3$. Let Y be the continuum in R^4 defined by $Y = Y_1 \cup Y_2$ where $Y_1 = \{(x, y, z, w): |x| \leqq 1, |y| \leqq 1, |z| \leqq 1, w=0\}$ and $Y_2 = \{(x, y, z, w): |x| \leqq 1, |y| \leqq 1, z = 0, |w| \leqq 1\}$. Now, $\mathrm{cc}(Y) = \mathrm{cc}(Y_1) \cup \mathrm{cc}(Y_2)$ and $\mathrm{cc}(Y_1 \cap Y_2) = \mathrm{cc}(Y_1) \cap \mathrm{cc}(Y_2) \cong I_\infty$. A theorem of Anderson [20] asserts that the union of two Hilbert cubes which intersect in a Hilbert cube is a Hilbert cube provided the intersection has property Z in each. We thus want to see that $\mathrm{cc}(Y_1 \cap Y_2)$ has property Z in $\mathrm{cc}(Y_1)$ and $\mathrm{cc}(Y_2)$.

To this end, let U be a homotopically trivial subset of $\mathrm{cc}(Y_1)$. Let $g: S^{k-1} \to U - \mathrm{cc}(Y_1 \cap Y_2)$ and let $\bar{g}: B^k \to U$ be an extension of g. For each $p \in U$ let $d(p) = \inf\{d(p, q): q \in \mathrm{cc}(Y_1) - U\}$. For each $t \in [0, 1]$ and each b in the sphere of radius t in B^k, let $G(b) = \mathrm{co}(N((1-t)(d(\bar{g}(b)))/2, g(b)))(N(\varepsilon, \bar{g}(b))) = \{x: \text{for some } a \in \bar{g}(b), \|x - a\| < \varepsilon\})$. Clearly $G(b) \in U$ for each $b \in B^k$ and, even more, since $G(b)$ is a 3-cell for each b, we have $G(b) \in U - \mathrm{cc}(Y_1 \cap Y_2)$. Also $G|S^{k-1} = g$. We have established that $\mathrm{cc}(Y_1) \cap \mathrm{cc}(Y_2)$ has property Z in $\mathrm{cc}(Y_1)$. The proof for $\mathrm{cc}(Y_2)$ is the same. It now follows that $\mathrm{cc}(Y) = I_\infty$. This shows that the analogue to (4.1) does *not* hold in R^4. Actually, it is clear that similar examples exist in dimensions $n > 4$ as well.

This next example is of a 3-dimensional continuum in R^3 which is not a 3-cell but whose cc-hyperspace is homeomorphic to I_∞.

(4.7) Example. Let X be the continuum in R^3 defined by $X = X_1 \cup X_2$ where

$$X_1 = \{(x, y, z): \|(x, y, z)\| \leqq 1\}$$

and

$$X_2 = \{(x, y, 0): \max\{|x|, |y|\} \leqq 1\} .$$

Now, $\mathrm{cc}(X) = \mathrm{cc}(X_1) \cup \mathrm{cc}(X_2)$ is a union of two convex Hilbert cubes. Also, $\mathrm{cc}(X_1) \cap \mathrm{cc}(X_2) = \mathrm{cc}(X_1 \cap X_2)$ is a convex Hilbert cube. Using the same techniques as were used in Example (4.6) one can easily show that $\mathrm{cc}(X_1) \cap \mathrm{cc}(X_2)$ is a Z-set in $\mathrm{cc}(X_1)$. By Handel's result [8], it follows that $\mathrm{cc}(X_1) \cap \mathrm{cc}(X_2) = \mathrm{cc}(X)$ is a Hilbert cube.

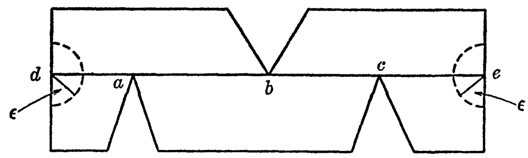
5. Some geometric considerations.

In view of Theorem (3.1), it is natural to ask the question:

Which 2-cells X in R^2 have the property that $\mathrm{cc}(X) \cong I_\infty$?

The following example shows that not every 2-cell in R^2 has this property.

(5.1) Example. Let X be the 2-cell in R^2 pictured below.



The three points a, b and c of local nonconvexity of X all lie on the convex arc $\widehat{de}$. It is clear that any compact convex subset of X which is within ε of the arc $\widehat{de}$ (in the Hausdorff metric) must be a subarc of $\widehat{de}$. Hence, it follows that $\widehat{de}$ has small 2-cell neighborhoods in $\mathrm{cc}(X)$. Therefore, $\mathrm{cc}(X)$ is 2-dimensional at $\widehat{de}$ and, thus, $\mathrm{cc}(X) \not\cong I_\infty$.

The remainder of this section is devoted to proving two results which can be used to establish that some rather wide classes of 2-cells do have the property that their hyperspaces of nonempty compact convex subsets are topologically I_∞. We begin with some definitions.

(5.2) DEFINITION. Let K be a starshaphed subset of l^2 and let $p \in \mathrm{Ker}(K)$. The point $x \in K$ will be called a *p-relative interior point of K* if there exists an $x^* \in K$ such that, for some $\lambda \in (0, 1)$, $\lambda x^* + (1 - \lambda)p = x$. A point in K which is not a p-relative interior point will be called a *p-relative extreme point of K.*

(5.3) DEFINITION. Let $K_1 \subseteqq K_2$ be two starshaped subsets of l_2 such that $\mathrm{Ker}(K_1) \cap \mathrm{Ker}(K_2) \neq \varnothing$. Let $p \in [\mathrm{Ker}(K_1) \cap \mathrm{Ker}(K_2)]$. Then p is called a *K_2 inside point of K_1* if, for every $x \in K_2$, $\{\lambda p + (1-\lambda)x: \lambda \in (0, 1)\} \cap K_1 \neq \varnothing$.

(5.4) THEOREM. *Let $K_1 \subseteqq K_2$ be two compact, starshaped subsets of l_2 and suppose that there exists a point $p \in K_1$ such that:*

(i) $p \in \mathrm{Ker}(K_1) \cap \mathrm{Ker}(K_2)$,

(ii) *p is a K_2-inside point of K_1,*

(iii) *the set of all p-relative interior points of K_1 (resp., K_2) is an open subset of K_1 (resp., K_2). Then, K_1 and K_2 are homeomorphic.*

Proof. Let the hypothesis of the theorem be satisfied. We will assume without loss of generality that $p = (0, 0, 0, \cdots)$. For each point $x \in K_2 - \{p\}$ (clearly, the theorem is valid if $K_2 - \{p\} = \varnothing$) let $\bar{x}$ be that p-relative extreme point of K_2 defined by $\bar{x} = \alpha_x x$ where $\alpha_x = \sup\{\alpha \in (0, \infty): \alpha x \in K_2\}$. To each p-relative extreme point y of K_2, let $\lambda_y = \sup\{\lambda \in [0, 1]: \lambda y \in K_1\}$. Let $f: K_2 \to K_1$ be the function defined by

$$f(x) = \begin{cases} \lambda_2 x, & \text{if } x \in K_2 - \{p\} ; \\ p, & \text{if } x = p . \end{cases}$$

It is easy to see that f is one-to-one. We wish now to show that f is onto and continuous. To see that f is onto, let $x \in K_1$. If $x = p$, we are done since $f(p) = p$. If $x \neq p$, then $1/\lambda_2 \leqq \alpha_x$. Hence, $y = x/\lambda_2 \in K_2$ and, clearly, $f(y) = x$. We have seen that f is onto. To see that f is continuous, let $\{x_i\}_{i=1}^{\infty}$ be a sequence in K_2 such that $\lim_{i\to\infty} x_i = x \in K_2$. If $x = p$, it is clear that $\lim_{i\to\infty} f(x_i) = f(p) = p$. So, assume that $x \neq p$. We may then assume that $x_i \neq p$ for all i. We will first show that $\lim_{i\to\infty} \bar{x}_i = \bar{x}$. Since K_2 is compact, we must have that some subsequence $\{\bar{x}_{i_j}\}_{j=1}^{\infty}$ of the sequence $\{\bar{x}_i\}_{i=1}^{\infty}$ converges to an $x_0 \in K_2$. Without loss of generality, we may assume that the sequence $\{\bar{x}_i\}_{i=1}^{\infty}$ converges to x_0. Now, it follows from condition (iii) that x_0 must be a p-relative extreme point of K_2. To see that $x_0 = \bar{x}$, we need now only show that, for some $\lambda > 0$, $\lambda x_0 = x$. Let λ_i be such that $\lambda_i x_i = \bar{x}_i$ and consider $\lambda_i x$. Now, the λ_i's are bounded and since $||\lambda_i x - \lambda_i x_i|| = |\lambda_i| \, ||x - x_i||$ we have that $\lim_{i\to\infty} \lambda_i x = x_0$. It is now not difficult to see that, for some $\lambda_0 > 0$, $\lim_{i\to\infty} \lambda_i = \lambda_0$ and $\lambda_0 x = x_0 = \bar{x}$. To establish the continuity of f, we need now only show that $\lim_{i\to\infty} \lambda_{\bar{x}_i} = \lambda_{\bar{x}}$. First consider $\{\lambda_{\bar{x}_i}\bar{x}_i\}_{i=1}^{\infty}$. Since, for each i, $\lambda_{\bar{x}_i}\bar{x}_i$ is a p-relative extreme point of K_1, we have that some subsequence converges to a p-relative extreme point of K_1. Without loss of generality, we will assume that $\lim_{i\to\infty} \lambda_{\bar{x}_i}\bar{x}_i = x'$ where x' is a p-relative extreme point of K_1. But, $||\lambda_{\bar{x}_i}\bar{x} - \lambda_{\bar{x}_i}\bar{x}|| = |\lambda_{\bar{x}_i}| \, ||\bar{x} - \bar{x}_i|| \leqq ||\bar{x} - \bar{x}_i||$. Hence, $\lim_{i\to\infty} \lambda_{\bar{x}_i}\bar{x} = x'$. But, the fact that the sequence $\{\lambda_{\bar{x}_i}\bar{x}\}_{i=1}^{\infty}$ is Cauchy implies that $\{\lambda_{\bar{x}_i}\}_{i=1}^{\infty}$ is Cauchy and, hence, that there exists a λ' such that $\lim_{i\to\infty} \lambda_{\bar{x}_i} = \lambda'$. Thus, $\lambda'\bar{x} = x'$ which says that $\lambda' = \lambda_{\bar{x}}$. We have now established the continuity of f. Since K_1 and K_2 are compact, it follows that f is a homeomorphism.

(5.5) COROLLARY. *Let X be a compact starshaped subset of R^n such that* $\operatorname{int}[\operatorname{Ker}(X)] \neq \emptyset$. *Then,* $\operatorname{cc}(X) \cong I_\infty$.

Proof. For simplicity, we will assume that the origin $0 \in \operatorname{int}[\operatorname{Ker}(X)]$. Let $\varepsilon > 0$ be such that $\bar{B}_\varepsilon = \{x \in R^n\colon ||x|| \leqq \varepsilon\}$ is contained in $\operatorname{Ker}(X)$. Since X is compact, there exists an $r > 0$ such that $X \subset \bar{B}_r$. Let F be an affine embedding of $\operatorname{cc}(\bar{B}_r)$ into l_2 such that $F(0) = 0$ (as in the proof of (2.2)). Let $K_1 = F(\operatorname{cc}(\bar{B}_\varepsilon))$ and let $K_2 = F(\operatorname{cc}(X))$. Then, $K_1 \subseteqq K_2$. Since we have already seen that $\operatorname{cc}(\bar{B}_\varepsilon) \cong I_\infty$ (Theorem (2.2)), the result will now follow provided conditions (i), (ii) and (iii) of (4.4) are shown to be satisfied for $p = 0$. It is easy to see that conditions (i) and (ii) are satisfied. That condition (iii) is satisfied will follow if we can show that the

p-relative extreme points of K_1 (resp., K_2) are precisely those elements of the form $F(G)$ where $G \cap \mathrm{Fr}(\bar{B}_\varepsilon) \neq \varnothing$ (resp., $G \cap \mathrm{Fr}(X) \neq \varnothing$). We will show this only for K_2 since it is obvious for K_1. It is clear that if $G \in \mathrm{cc}(X)$ is such that $G \cap \mathrm{Fr}(X) = \varnothing$ then $F(G)$ is not a p-relative extreme point of K_2. It remains only to show that if $G \in \mathrm{cc}(X)$ is such that $G \cap \mathrm{Fr}(X) \neq \varnothing$ then $F(G)$ is a p-relative extreme point of K_2. Suppose not, then there exists a $\lambda > 1$ such that $\lambda F(G) \in K_2$. Let $G' \in \mathrm{cc}(X)$ be such that $F(G') = \lambda F(G)$. By the one-to-oneness and the convexity of F, it follows that $\lambda G = G'$. If $c \in G \cap \mathrm{Fr}(X)$, then $\lambda c \in X$. But $\overline{\mathrm{co}}(\lambda c, \bar{B}_\varepsilon) \subset X$ and contains c as an interior point. This contradicts the fact that $c \in \mathrm{Fr}(X)$. The corollary now follows. T. A. Chapman showed (see Theorem 10 of [5]) that a compact Hilbert cube manifold is homeomorphic to the Hilbert cube if and only if it is homotopically trivial. This enables one to "localize" the problem of showing the cc-hyperspace of a given 2-cell is homeomorphic to I_∞.

(5.6) THEOREM. (1) *If X is a contractible continuum lying in a Banach space, then* $\mathrm{cc}(X)$ *is contractible.*

(2) *Thus, in particular, if X is a 2-cell (or n-cell),* $\mathrm{cc}(X) \cong I_\infty$ *if and only if* $\mathrm{cc}(X)$ *is a Hilbert cube manifold.*

Proof. The closed linear span L of X is a separable Banach space. By (3.2), there is a continuous selection η from $\mathrm{cc}(X)$ to X. Define $g: \mathrm{cc}(X) \times [0, 1] \to \mathrm{cc}(X)$ by $g(A, t) = t\{\eta(A)\} + (1 - t)A$. It follows using g and the contractibility of X that $\mathrm{cc}(X)$ is contractible. This proves (1). The proof of (2) uses (1) and Theorem 10 of [5].

These next results will show that a fairly large class of 2-cells have the property that their hyperspaces of compact convex subsets are homeomorphic to I_∞. We begin with a notational agreement and a definition.

If A is a nondegenerate, convex arc in the plane then by $A^\sim$ we will denote the unique line in R^2 which contains A. If $p \in R^n$ and $\varepsilon > 0$ then $B(\varepsilon, p) = \{x \in R^n : \|x - p\| < \varepsilon\}$.

(5.7) DEFINITION. Let X be a 2-cell in R^2 and let $A \in \mathrm{cc}(X)$ be an arc. Suppose that one complementary domain of $A^\sim$ has been designated the right side of $A^\sim$ and the other the left side of $A^\sim$. A point $p \in LN(X) \cap A$ will be said to lie on the left side (right side) of A if, for every $\varepsilon > 0$, $B(\varepsilon, p) - X$ contains points on the left side (right side) of $A^\sim$. If for some $\varepsilon > 0$, $B(\varepsilon, p) - X$ contains no points on the right side (left side) of $A^\sim$ then p will be said to lie strictly on the left side (right side) of A.

(5.8) LEMMA. *Let X be an n-cell. If $A \in \mathrm{cc}(X)$ is an n-cell then A is contained in a closed starshaped subset N of X with* $\mathrm{int}[\mathrm{Ker}(N)] \neq \emptyset$ *such that* $\mathrm{cc}(N)$ *is a neighborhood of A in* $\mathrm{cc}(X)$.

Proof. Let $A \in \mathrm{cc}(X)$ be an n-cell and let $q \in \mathrm{int}[A]$. Let $\varepsilon > 0$ be chosen so that $\mathrm{cl}(B(\varepsilon, q)) \subset \mathrm{int}[A]$. Let $\Gamma = \{K \in \mathrm{cc}(X)\colon \mathrm{cl}(B(\varepsilon, q)) \subset K\}$ and let $D = \cup \Gamma$. It is not difficult to see that D is a closed starshaped subset of X and that $\mathrm{Ker}(D) \supseteq \mathrm{cl}(B(\varepsilon, q))$. It is also not difficult to see that $\mathrm{cc}(D)$ is a neighborhood of A in $\mathrm{cc}(X)$. The lemma is proved.

(5.9) LEMMA. *If X is an n-cell in R^n then the following are equivalent:*

(i) *Every $A \in \mathrm{cc}(X)$ lies in a starshaped subset of X whose kernel has nonvoid interior.*

(ii) *Every maximal convex subset of X is an n-cell.*

Proof. Suppose (i) is satisfied. Let $A \in \mathrm{cc}(X)$ be maximal. By (i) there exists an n-ball $B \subset X$ such that $\overline{\mathrm{co}}\{B, A\} \subset X$. But, by maximality of A, $\overline{\mathrm{co}}\{B, A\} = A$. Hence A is an n-dimensional compact convex subset of R^n and thus must be an n-cell. We have that (i) implies (ii). Now, if (ii) holds and $A \in \mathrm{cc}(X)$, then let $M(A)$ be a maximal convex subset of X which contains A. As $M(A)$ is a starshaped set whose kernel has nonvoid interior, we are done.

(5.10) LEMMA. *Let X be a 2-cell in R^2. Let $A \in \mathrm{cc}(X)$ be an arc with noncut points p and q. Suppose there exists a closed ball $D \subset X$ and neighborhoods P of p and Q of q in X such that for each $d \in D$ we have $P \cup Q \subset S(d)$. Then A is contained in a closed starshaped subset Y of X with* $\mathrm{int}[\mathrm{Ker}(Y)] \neq \emptyset$ *such that* $\mathrm{cc}(Y)$ *is a neighborhood of A in* $\mathrm{cc}(X)$.

Proof. We can assume that D lies in the interior of a convex 2-cell $B \subset X$ such that A is on the boundary of B. We may also assume that $A - (P \cup Q) \neq \emptyset$ (we would be done in this case anyway as will become evident at the end of the proof). Let P' and Q' be balls in R^2 centered at p and q, respectively, which satisfy

(a) the radii of P' and Q' are less than 1/2 min {radius of P, radius of Q}, and

(b) for each $a \in [A - (P \cup Q)]$, $r \in \mathrm{cl}(P')$, $s \in \mathrm{cl}(Q')$ and $d \in D$, the ray through a from d must intersect the segment $\widehat{rs}$ in a cut point. Now, for each $a \in A - (P \cup Q)$, choose a ball B_a about a such that

(**) if $r \in \mathrm{cl}(P')$, $s \in \mathrm{cl}(Q')$, $t \in B_a$ and $d \in D$, then the ray from

d through t must intersect the segment $\widehat{rs}$ in a cut point.

Let $\sum$ be the collection of all convex sets C in X such that C inter sects both P' and Q' and is contained in the union of P, Q and the balls B_a for $a \in A - (P \cup Q)$. It is clear that $\sum$ is a neighborhood of A in cc(X). We wish to show now that if $d \in D$, then d sees each point of any C in $\sum$. So, let $C \in \sum$ and let $r \in [P' \cap C]$ and $s \in [Q' \cap C]$. Let $\alpha \in C - (P \cup Q)$ (note, if $\alpha \in [P \cup Q]$ we are done) and let $a \in A - (P \cup Q)$ be such that $\alpha \in B_a$. Since $\alpha \in B_a$, by (**) we have that the ray from d through $\alpha (d \in D)$ must intersect $\widehat{rs}$. By simple connectivity of X, it follows that the 2-cell (rds) and $(rs\alpha)((rs\alpha)$ may be an arc) lie in X. If the segment $\widehat{d\alpha}$ intersects $\widehat{rs}$ then $\widehat{d\alpha} = [\widehat{d\alpha} \cap (rsd)] \cup [\widehat{d\alpha} \cap (rs\alpha)] \subset X$. If the segment $\widehat{d\alpha}$ does not intersect $\widehat{rs}$, then $\widehat{d\alpha} \subset (rsd) \subset X$. Thus, $\widehat{d\alpha} \subset X$ and we have the desired conclusion. Now, let $\Gamma = \{K \in \mathrm{cc}(X)\colon K \supset D\}$. Let $Y = \cup \Gamma$. We have just seen that the starshaped set Y has the property that $\mathrm{cc}(Y) \supset \sum$. Also, we have that $\mathrm{Ker}[Y] \supset \mathrm{int}[D]$ and hence $\mathrm{int}[\mathrm{Ker}(Y)] \neq \varnothing$. The lemma is proved.

(5.11) LEMMA. *Let X be a polygonal 2-cell in R^2 and let $A \in \mathrm{cc}(X)$ be an arc such that no two points in $LN(X) \cap A$ lie strictly on opposite sides of A. Then there exists a closed starshaped subset N of X with* $\mathrm{int}[\mathrm{Ker}(N)] \neq \varnothing$ *such that* cc(N) *is a neighborhood of A in* cc(X).

Proof. Let A be an arc in cc(X) such that no two points of $LN(X) \cap A$ lie strictly on opposite sides of A. Consider the noncut points, say p and q, of A. If at least one of p and q is not a point in $LN(X)$ which lies strictly on one side of A then it can be seen that there is a closed ball D in X and neighborhoods $B(\alpha, p) \cap X$ and $B(\gamma, q) \cap X$ such that, for any $d \in D$, $(B(\alpha, p) \cup B(r, q)) \cap X \subset S(d)$. The result now follows from (5.10). Suppose now that both p and q are points in $LN(X)$ which lie strictly on one side of A. It is geometrically clear that, in this event, ono can obtain balls P, Q and M such that

(a) $p \in P$, $q \in Q$ and $\mathrm{cl}(M) \subset \mathrm{int}[X]$,

(b) $\mathrm{cl}(M) \cap A = \varnothing$, and

(c) if C is a convex set in X such that $C \cap P \neq \varnothing$ and $C \cap Q \neq \varnothing$ then $C \cap (P \cup Q) \subset S(m)$ for every $m \in \mathrm{cl}(M)$.

The proof from here proceeds as it did in the proof of (5.10).

(5.12) THEOREM. *Let X be a polygonal 2-cell in R^2. Then the following are equivalent*:

(i) *Every maximal convex subset of* X *is a* 2-*cell.*

(ii) *Each* $A \in \mathrm{cc}(X)$ *is contained in a closed starshaped subset* N *of* X *for which* $\mathrm{int}[\mathrm{Ker}(N)] \neq \varnothing$ *and* $\mathrm{cc}(N)$ *is a neighborhood of* A *in* $\mathrm{cc}(X)$.

Furthermore, if (i) *or* (ii) *holds then* $\mathrm{cc}(X) \cong I_\infty$.

Proof. That condition (ii) implies condition (i) follows from (5.9). Now, assume that (i) holds. If $A \in \mathrm{cc}(X)$ is a singleton then it is easy to see that A is contained in a closed starshaped neighborhood N in X. But then $\mathrm{cc}(N)$ is a neighborhood of A in $\mathrm{cc}(X)$ and we are done in this case. If $A \in \mathrm{cc}(X)$ is a 2-cell, then we are done by virtue of (5.8). If A is an arc, then by (5.11) we will be done if we can show that no two points in $LN(X) \cap A$ lie strictly on opposite sides of A. Let $p_1, p_2 \in LN(X) \cap A$ lie strictly on opposite sides of A. If both p_1 and p_2 are cut points of A then it is clear that no convex 2-cell in X can contain A and this contradicts (i). If one or more of p_1 and p_2 are noncut points of A then one can obtain an arc $A' \supset A$ with $A' \in \mathrm{cc}(X)$ for which both p_1 and p_2 are cut points. This again leads to a contradiction of condition (i). Thus, no two points of $LN(X) \cap A$ can lie strictly on opposite sides of A and we have the desired result. We have now established the equivalence of (i) and (ii).

To complete the proof we need only see that if (ii) holds then $\mathrm{cc}(X) \cong I_\infty$. So, suppose that (ii) holds. Let $A \in \mathrm{cc}(X)$ by virtue of (ii) there exists a closed starshaped subset N of X with $\mathrm{int}[\mathrm{Ker}(N)] \neq \varnothing$ for which $\mathrm{cc}(N)$ is a neighborhood of A in $\mathrm{cc}(X)$. But, $\mathrm{cc}(N) \cong I_\infty$ by (5.5). Thus, $\mathrm{cc}(X)$ is homeomorphic to I_∞ by virtue of (5.6). The theorem is proved.

(5.13) THEOREM. *Let* X *be a* 2-*cell in* R^2 *such that* $(*)$ *whenever* $p, q \in X$ *are such that* $p \in S(q)$ *and* N *is a neighborhood of* p *in* X, *then there exists an open set* $M \subset N$ *and a neighborhood* Q *of* q *such that for each point* m *in* M *we have* $S(m) \supset Q$.

The following are equivalent:

(i) *Every maximal convex subset of* X *is a* 2-*cell.*

(ii) *Each* $A \in \mathrm{cc}(X)$ *is contained in a starshaped subset* N *of* X *for which* $\mathrm{int}[\mathrm{Ker}(N)] \neq \varnothing$ *and* $\mathrm{cc}(N)$ *is a neighborhood of* A *in* $\mathrm{cc}(X)$.

Furthermore, if (i) *or* (ii) *holds then* $\mathrm{cc}(X) \cong I_\infty$.

Proof. All aspects of the proof for this result are the same as the proof of (5.12) with the exception of showing that condition (i) implies condition (ii). So, suppose that condition (i) holds and let $A \in \mathrm{cc}(X)$. If A is a singleton, it is easy to use $(*)$ to obtain the desired set N. If A is a 2-cell we are again done by virtue of (5.8). Suppose, that $A = [p, q]$ is an arc. Let B be a 2-cell in $\mathrm{cc}(X)$ which contains A (condition (i) implies B exists). Let $b \in \mathrm{int}(B)$. Since $p \in S(b)$ there is by $(*)$ a ball $C \subset B$ and a neighborhood P of p such that for each $m \in C$ we have $S(m) \supset P$. Let $m_1 \in C$. Since $m_1 \in B$ we have $S(m_1) \supset q$. Thus, by $(*)$, there exists a closed ball $D \subset C$ and a neighborhood Q of q such that, for any $d \in D$. $S(d) \supset Q$. Now application of (5.10) gives the existence of the starshaped subset N of X with the desired properties. The result is established.

6. Some problems and examples. While at present we have some large classes of nonconvex 2-cells whose cc-hyperspaces are homeomorphic to I_∞, we still do not know exactly which 2-cells have their cc-hyperspaces homeomorphic to I_∞. The following problems are connected with this.

(6.1) *Problem.* Let X be a 2-cell in R^2. If every point of $\mathrm{cc}(X)$ has arbitrarily small infinite dimensional neighborhoods, is it true that $\mathrm{cc}(X) \cong I_\infty$?

(6.2) *Problem.* Let X be a 2-cell in R^2. If every maximal convex subset of X is either a point or a 2-cell, is it true that $\mathrm{cc}(X) \cong I_\infty$?

(6.3) *Problem.* Let X be a 2-cell in R^2. If every maximal convex subset of X is a 2-cell, is it true that $\mathrm{cc}(X) \cong I_\infty$?

An affirmative answer to (6.1) would provide a satisfactory characterization. This is true since it would then follow that Example 5.1 is, in a sense, canonical. An affirmative answer to (6.1) would imply an affirmative answer to (6.2) and an affirmative answer to (6.2) would imply an affirmative answer to (6.3).

The following two examples give a bit more insight into the above problems. The technique used in this next example is one which has become standard in infinite dimensional topology. It was first used by Schori and West in [18]. For the difinition of shape see [4]. An onto map $f: X \to Y$ where X and Y are homeomorphic metric spaces, is a *near homeomorphism* if f can be uniformly approximated by homeomorphisms. For terminology related to inverse limits it is suggested that the reader see [13] or [18]. In the discussion of the example we use a characterization by T. A. Chapman of near homeomorphisms between Hilbert cubes as being those continuous surjections for which point inverses have trivial shape.

(6.4) EXAMPLE. Consider the planar 2-cell X formed by intersecting the planar regions A, B and C where $A = \{(x, y): x \leqq 1/2, y \geqq 0\}$, $B = \{(x, y): (x + 1/2)^2 + y^2 \geqq 1/4\}$ and $C = \{(x, y): x^2 + y^2 \leqq 1\}$ (see Fig. 6.6 below).

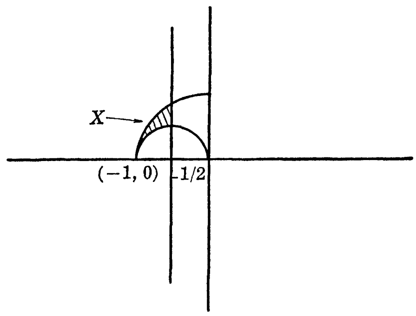


FIGURE 6.6

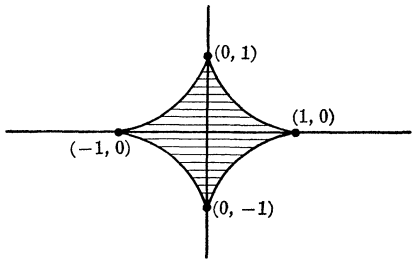


FIGURE 6.7

Note that the point $(-1, 0)$ is a maximal convex subset of X. Now, for each $3\pi/4 \leqq \theta \leqq \pi$ let $X_\theta = X \cap \{(r, \varphi): \pi/2 \leqq \varphi \leqq \theta\}$. For each pair (θ_1, θ_2) with $\pi/2 \leqq \theta_1 \leqq \theta_2 \leqq \pi$, let the mapping $g_{\theta_2\theta_1}: X_{\theta_2} \to X_{\theta_1}$ be defined by $g_{\theta_2\theta_1}(r, \varphi) = (r, \theta_1)$ for $\theta_1 \leqq \varphi \leqq \theta_2$, and $g_{\theta_2\theta_1}(r, \varphi) = (r, \varphi)$ if $\pi/2 \leqq \varphi \leqq \theta_1$. Define, for (θ_1, θ_2) as above, the retraction $r_{\theta_2\theta_1}$: $\mathrm{cc}(X_{\theta_2}) \to \mathrm{cc}(X_{\theta_1})$ by $r_{\theta_2\theta_1}(A) = \overline{\mathrm{co}}(g_{\theta_2\theta_1}(A))$. Also, for a compact convex subset A of X_θ which intersects $\{(r, \theta): r \geqq 0\}$ define $p_1(A, \theta) = \inf\{r: (r, \theta) \in A\}$. For each $n = 1, 2, \cdots$, let $\theta_n = \pi - \pi/2^{n+1}$ and let $r_n = r_{\theta_{n+1}\theta_n}$ and $X_n = X_{\theta_n}$. For $A \in \mathrm{cc}(X_n)$, let $Y \in r_n^{-1}(A)$ and define $\theta_Y = \sup\{\theta: r_n(r_{\theta_{n+1}\theta}(Y)) = A\}$. For each $\theta \in [\theta_n, \theta_{n+1}]$, let

$$H(Y, \theta) = r_{\theta_{n+1},\theta}(Y) \quad \text{if} \quad \theta_Y \leqq \theta \leqq \theta_{n+1}\,,$$
$$\mathrm{co}(r_{\theta_{n+1}\theta_Y}(Y) \cap X_\theta) \cup \{(p_1(r_{\theta_{n+1}\theta}(r_{\theta_{n+1}\theta_Y}(Y)), \theta)\} \quad \text{if} \quad \theta_n \leqq \theta \leqq \theta_Y\,.$$

It is geometrically clear that $H: r_n^{-1}(A) \times [\theta_n, \theta_{n+1}] \to r_n^{-1}(A)$ is a homotopy of the identity on $r_n^{-1}(A)$ to a constant map. Thus, for each $A \in \mathrm{cc}(X_n)$, $r_n^{-1}(A)$ is contractible and, hence [4, (5.5) p. 28], of trivial shape. It now follows that r_n is a near homeomorphism and, hence, (since each X_n satisfies the conditions of Theorem 5.13) that $\varprojlim_n(\mathrm{cc}(X_n), r_n) = \mathrm{cc}(X_1) = I_\infty$. Furthermore, the inverse sequence

$\{(\mathrm{cc}(X_n), r_n)\}$ also satisfies the conditions that

(a) $\mathrm{cc}(X_n) \subset \mathrm{cc}(X_{n+1})$ and $\overline{\bigcup_n \mathrm{cc}(X_n)} = \mathrm{cc}(X)$,

(b) $\sum_{n=1}^{\infty} d(r_n, id_{\mathrm{cc}(X_{n+1})}) < \infty$,

(c) for each j, $\{r_j \circ \cdots \circ r_i: \mathrm{cc}(X_{i+1}) \to \mathrm{cc}(X_j) \mid i \geqq j\}$

is an equi-uniformly continuous family of functions.

That condition (a) holds is immediate. The fact that condition (b) holds rests on the fact that if $d(A, B) < \varepsilon$ and B is convex then $d(\overline{\mathrm{co}}(A), A) \leqq \varepsilon$.

To see that (c) holds, let, for each n, $r^n: X \to X_n$ be the retraction $g_{\pi\theta_n}$.

Let $j \in I^+$ be given and let $\varepsilon > 0$. Choose j_0 so that if $A \notin \mathrm{cc}(\mathrm{int}[X_{j_0+1}])$ then $A \cap X_j = \varnothing$. Choose $\delta_1 > 0$ so that if $d(A, B) < \delta_1$ then $d(r^n(A), r^n(B)) < \varepsilon$. Let $\delta_2 > 0$ be chosen so that, if $d(A, B) < \delta_2$ and $A, B \in \mathrm{cc}(X_{j_0+1})$, then $d(r_j \circ \cdots \circ r_{j_0}(A), r_j \circ \cdots \circ r_{j_0}(B)) < \varepsilon$. Let δ_3 be chosen so that, if $A \notin \mathrm{cc}(\mathrm{int}[X_{j_0+1}])$ and $d(A, B) < \delta_3$, then $B \cap X_j = \varnothing$. Now, if $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and $d(A, B) < \delta$ then, either $A, B \in \mathrm{cc}(X_{j_0+1})$ in which case $d(r_j \circ \cdots \circ r_k(A), r_j \circ \cdots \circ r_k(B)) \leqq d(r_j \circ \cdots \circ r_{j_0}(A), r_j \circ \cdots \circ r_{j_0}(B)) < \varepsilon$ or $A \cap X_j = \varnothing$ and $B \cap X_j = \varnothing$ in which case $r_j \circ \cdots \circ r_k(A) = r^j(A)$ and $r_j \circ \cdots \circ r_k(B) = r^j(B)$ and, hence, $d(r_j \circ \cdots \circ r_k(A), r_j \circ \cdots \circ r_k(B)) < \varepsilon$. We have established that condition (c) holds. Thus, by [13, Lemma B], $\mathrm{cc}(X) \cong \lim_i(\mathrm{cc}(X_i), r_i)$ and thus $\mathrm{cc}(X) \cong I_\infty$.

(6.5) EXAMPLE. Consider the 2-cell X in R^2 which is the closure of the bounded complementary domain of $\bigcup_{i=1}^4 C_i$, where

$$C_1 = \{(x, y): (x-1)^2 + (y-1)^2 \leqq 1\}\ ,\ C_2 = \{(x, y): (x-1)^2 + (y+1)^2 \leqq 1\}$$
$$C_3 = \{(x, y): (x+1)^2 + (y+1)^2 \leqq 1\} \text{ and } C_4 = \{(x, y): (x+1)^2 + (y-1)^2 \leqq 1\}\ .$$

(Fig. 6.7.) Note, the convex segment with noncut points $(0, -1)$ and $(0, 1)$ is a maximal convex subset of X and the kernel of X consists only of the origin $(0, 0)$. In spite of this, if one takes $Y = \{(x, y): x^2 + y^2 \leqq 1/4\}$ and sets $K_1 = \mathrm{cc}(Y), K_2 = \mathrm{cc}(X)$ and $p = (0, 0)$ then all the conditions of Theorem 4.4 are satisfied. It follows that $\mathrm{cc}(X) \cong \mathrm{cc}(Y) \cong I_\infty$.

The 2-cell of Example (6.4) illustrates the validity of (6.1) and (6.2) for a specific 2-cell. The 2-cell of Example (6.5) illustrates that though the hypotheses in (6.2) and (6.3) may be sufficient, they are definitely not necessary.

7. The cc-hyperspaces of ${}^0B^n$ and R^n, $n \geqq 2$. In this section we show that $\mathrm{cc}({}^0B^n)$ and $\mathrm{cc}(R^n)$, $n \geqq 2$, are homeomorphic to the Hilbert cube with a point removed. We also state some problems.

Let U be a nonempty proper open subset of $\mathrm{cc}(B^n)$. For each

$A \in U$ let $A\acute{u} = \inf\{d(A, D) \mid D \in [\mathrm{cc}(X) - U]\}$, where d denotes the Hausdorff metric. Note that $0 < Au \leqq 2$.

(7.1) LEMMA. *Let U be a proper open subset of* $\mathrm{cc}(B^n)$. *Let $A \in U$ and let α be real,* $0 < \alpha \leqq 1$. *Then* $(1 - \alpha Au/2)A \in [U \cap \mathrm{cc}({}^0B^n)]$.

Proof. For any $a \in A$ and $\beta > 0, \beta \neq 1$, note that $\|a - \beta a\| = |1 - \beta| \, \|a\| \leqq |1 - \beta| < 2|1 - \beta|$. Thus, setting $\beta = 1 - \alpha Au/2$, it follows that

$$d\left(A, \left(1 - \frac{\alpha Au}{2}\right)A\right) < 2\left|1 - \left(1 - \frac{\alpha Au}{2}\right)\right| = \alpha Au \leqq Au ,$$

which implies $(1 - \alpha Au/2)A \in U$. Note that $(1 - \alpha Au/2)A \in \mathrm{cc}({}^0B^n)$ since $(1 - \alpha Au/2) < 1$.

(7.2) THEOREM. *If $n \geqq 2$, then* $\mathrm{cc}({}^0B^n) \cong I_\infty - \{p\}$ *for* $p \in I_\infty$.

Proof. Let $K = \{A \in \mathrm{cc}(B^n) \mid A \cap S^{n-1} \neq \varnothing\}$. We show K has property Z in $\mathrm{cc}(B^n)$. Let U be a nonempty homotopically trivial open subset of $\mathrm{cc}(B^n)$. Let $f\colon S^{k-1} \to U - K$ be continuous, and let $F\colon B^k \to U$ be a continuous extension of f. Let $h\colon [0, 1] \to [0, 1]$ be a homeomorphism such that $h(0) = 1$ and $h(1) = 0$. Define a function F^* on B^k by $F^*(x) = (1 - [h(\|x\| \, F(x)u/2)])F(x)$. Note F^* is continuous and F^* extends f since if $\|x\| = 1$, $F^*(x) = F(x) = f(x)$. If $\|x\| < 1$ note that $F^*(x) \in [U \cap \mathrm{cc}({}^0B^n)]$ by (7.1), and hence $F^*(x) \in [U - K]$. Thus, K has property Z in $\mathrm{cc}(B^n)$. Hence, by (2.2) above and a theorem of Anderson [1], we assume without loss of generality that $K \subset I^0_\infty$. For each $t \in [0, 2]$ and $A \in K$ let $g(A, t) = \mathrm{cl}(N(t, A) \cap B^n)(N(t, A) = \bigcup_{a \in A} \{x \mid \|x - a\| < t\})$. Note g is continuous and that $g(A, 0) = A$ and $g(A, 2) = B^n$. (See Borsuk [4].) By a result of Chapman [6] it follows that $\mathrm{cc}(B^n) - K \cong \mathrm{cc}(B^n) - \{M\}$ for $M \in \mathrm{cc}(B^n)$. Hence, by (2.2) above, $\mathrm{cc}({}^0B^n) \cong I_\infty - \{p\}$, and this completes the proof.

(7.3) THEOREM. *If* $n \geqq 2$, $\mathrm{cc}(R^n) \cong I_\infty - \{p\}$ *for* $p \in I_\infty$.

Proof. Using the proof of (5.4), it is easy to see that $\mathrm{cc}(R^n) \cong \mathrm{cc}({}^0B^n)$. Therefore, by (7.2) $\mathrm{cc}(R^n) \cong I_\infty - \{p\}$. Theorem 7.3 suggests the following.

(7.4) *Problem.* If H is a separable Hilbert space, is $\mathrm{cc}(H) \cong H$?

We will now discuss and state two problems which arise out of our previous work. Problem 7.5 is motivated in part by the result of Schori and West [16] that $2^I \cong I_\infty$.

Let D be the semidisc in R^2 given by $\{(x, y)|x^2 + y^2 \leqq 1, y \geqq 0\}$ and let K be the semicircle $D \cap S^1$. Let $R = \{A \in \mathrm{cc}(D)|\mathrm{ext}[A] \subset K\}$. The mapping $f: 2^K \to R$ given by $f(E) = \overline{\mathrm{co}}(E)$ is a homeomorphism. Let $R^* = \mathrm{cc}(D) - R$. Note that R^* is an open convex subset of $\mathrm{cc}(D)$ and that $I_\infty \cong R = \mathrm{cc}(D) - R^*$. This suggests the following problem:

(7.5) *Problem.* Let M be an open convex subset of a convex Hilbert cube Q. What are necessary and sufficient conditions on M in order that $I_\infty \cong Q - M$?

Several times in our work we encountered infinite dimensional compact convex subsets P of I_∞ such that $P \cong \mathrm{ext}[P] \cong I_\infty$. The countable product of semidiscs is such an example. This suggests the following problem.

(7.6) Let Q be a convex Hilbert cube. What are necessary and sufficient conditions for Q to be homeomorphic with ext[Q]?

We remark that a theorem answering the above question may by considered as a compact analogue of the theorem of Klee [11] that in separable Hilbert space the unit sphere is homeomorphic with the closed unit ball.

REMARK. After this paper was written, certain developments occurred which may be of interest to the reader. D. W. Curtis in a forthcoming paper entitled "Growth hyperspaces" investigates, among other things, subspaces G of the cc-hyperspace having the property that if $A \in G$ and $A \subset B$ then $B \in G$. D. W. Curtis, J. Quinn and R. M. Schori in a forthcoming paper entitled "On the cc-hyperspace of a polyhedral two-cell" show that the cc-hyperspace of a polyhedral two cell in R^2 is I_∞ with perhaps a finite number of two cell flanges. J. Quinn and R. Y. T. Wong in a forthcoming paper entitled "Unions of convex Hilbert cubes" show that the union of finitely many convex Hilbert cube manifolds each subcollection of which intersects vacuously or in a Hilbert cube is a Hilbert cube manifold, and, as a corollary, obtain the result that if A and B are infinite dimensional compact convex sets in l_2 such that $A \cap B$ is infinite dimensional then $A \cup B \cong I_\infty$. Reiter and Stavrakas in a forthcoming paper entitled "On the compactness of the hyperspace of faces" and Quinn and Stavrakas in a forthcoming paper "Selections in the hyperspace of faces" investigate certain topological aspects of the hyperspace of faces of a compact convex set.

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