

Pacific Journal of Mathematics

**ON COMPACT SUBMANIFOLDS WITH NONDEGENERATE
PARALLEL NORMAL VECTOR FIELDS**

VLADIMIR I. OLIKER

ON COMPACT SUBMANIFOLDS WITH NONDEGENERATE PARALLEL NORMAL VECTOR FIELDS

V. I. OLIKER

In this paper we obtain characterizations of spherical submanifolds in Euclidean space of codimension ≥ 1 . Such characterizations are given here in terms of certain relationships involving the elementary symmetric functions of principal radii of curvature and the support function of a submanifold.

1. Introduction. For hypersurfaces similar characterizations are well known. For example, let M be a closed convex hypersurface in Euclidean space, h the support function of M , and S_l the elementary symmetric function of order l of principal curvatures. It has been proved by several authors (see Simon [8], and further references given there) that if for some integer $l(1 \leq l \leq \dim M)$ everywhere on M $h^l S_l = \text{const}$, then M is a hypersphere. Other results of this type are also known [8], [9].

Our proofs are based on a differential analogue of the Minkowski-Hsiung formulas, relating the support function and elementary symmetric functions of various orders of the principal radii of curvature. Those formulas are obtained for submanifolds which possess a nondegenerate normal vector field parallel in the normal bundle.

Finally, we note that characterizations of spherical submanifolds in terms of the elementary symmetric functions of principal curvatures are obtained by Chen [2] and Chen and Yano [4] (see also Chen [3], Chapter 6).

The author wishes to thank the referee for useful comments.

2. Preliminaries. In this section we shall present local formulas relating the second fundamental form and the support function of a submanifold in Euclidean space. We shall use the following convention on the ranges of indices:

$$1 \leq i, j, k, l, r \leq m, \quad 1 \leq \alpha \leq n,$$

and as usual, it is agreed that repeated lower and upper indices are summed over the respective ranges. We denote by E the Euclidean space of dimension $m + n$, and we fix the origin at some point O . Consider a smooth, orientable submanifold M of dimension $m(\geq 2)$ immersed in E , and represented by the position vector field

$$X = X(u^1, \dots, u^m),$$

where $\{u^i\}$ are the local coordinates on M . Let x be a point of M . We denote by $T_x(M)$ and $N_x(M)$ the restrictions of the tangent bundle $T(M)$ and normal bundle $N(M)$ at x .

Put

$$X_i = \partial_i X, \quad \partial_i = \partial/\partial u^i.$$

The metric I , or the first fundamental form induced on M from E via X , is $G_{ij} = \langle X_i, X_j \rangle$, where \langle, \rangle denotes the inner product in E . Let ξ be an arbitrary unit normal vector field defined in a neighborhood U of $x \in M$. The second fundamental form at x with respect to ξ is $II(\xi) = b_{ij}(\xi) du^i du^j$, where $b_{ij}(\xi) = -\langle X_i, \xi_j \rangle$. Let η be a unit normal vector field in U not necessarily different from ξ . The mixed third fundamental form is $III(\xi, \eta) = g_{ij}(\xi, \eta) du^i du^j$, where $g_{ij}(\xi, \eta) = \langle \xi_i, \eta_j \rangle$. We write $III(\xi) \equiv III(\xi, \xi)$, and $g_{ij}(\xi) \equiv g_{ij}(\xi, \xi)$. Evidently, $g_{ij}(\xi, \eta) = g_{ji}(\eta, \xi)$, but, in general, no other symmetries exist. For a unit normal vector field $\xi \in N(M)$, $h(\xi)$ denotes the support function of M with respect to ξ , that is, $h(\xi) = -\langle X, \xi \rangle$.

Recall that a nondegenerate normal vector field on M is a unit normal vector field ξ such that $\det(b_{ij}(\xi)) \neq 0$ everywhere on M (see [2], and also [3], p. 59).

Vectors $\{X_i\}$ form a basis in $T_x(M)$, $x \in M$, and we denote by $\{N(\alpha)\}$ a field of orthonormal frames in $N(M)$. Put

$$X_{ij} = \partial_{ij} X, \quad \partial_{ij} = \partial^2/\partial u^i \partial u^j.$$

At first we note that $b_{ij}(\xi) = \langle X_{ij}, \xi \rangle$, and $b_{ij}(\xi) = b_{ji}(\xi)$ for a unit normal vector field ξ . Also, $g_{ij}(\xi) = -\langle \xi_{ij}, \xi \rangle = g_{ji}(\xi)$. Suppose that ξ is parallel in $N(M)$, that is, $\xi_i \in T(M)$, $i = 1, \dots, m$, everywhere on M , and let η be an arbitrary unit normal vector field on M . Then $g_{ij}(\xi, \eta) = -\langle \xi_{ij}, \eta \rangle = g_{ji}(\xi, \eta)$.

In the frame $X_1, \dots, X_m, N(1), \dots, N(n)$ we have for an arbitrary unit vector field $\xi \in N(M)$:

$$(1) \quad \xi_i = -b_i^j(\xi) X_j + \sum_{\alpha} \langle \xi_i, N(\alpha) \rangle N(\alpha),$$

where $b_i^j(\xi) = b_{ii}(\xi) G^{ij}$, and G^{ij} being the inverse of G_{ij} . From here, for two unit normal vector fields ξ and η , we find

$$(2) \quad g_{ij}(\xi, \eta) = b_i^r(\xi) b_{rj}(\eta) + \sum_{\alpha} \langle \xi_i, N(\alpha) \rangle \langle \eta_j, N(\alpha) \rangle.$$

If ξ or η is parallel, then

$$(3) \quad g_{ij}(\xi, \eta) = b_i^r(\xi) b_{rj}(\eta).$$

Note that when M , ξ and η are such that $II(\xi)$ and $II(\eta)$ are positive

definite then so is $III(\xi, \eta)$. However, the form $III(\xi)$ is nonnegative definite for an arbitrary $II(\xi)$. If ξ is nondegenerate everywhere on M , then $III(\xi)$ induces a Riemannian metric on M . We denote by $dO(\xi)$ the corresponding volume-element. From formula (1) it follows that if ξ is nondegenerate and parallel, then vectors $\{\xi_i\}$ form a basis in $T_x(M)$, $x \in M$, and according to the Gauss equation we have:

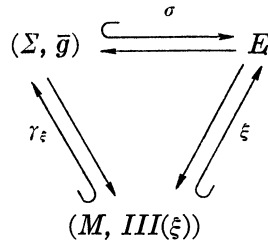
$$(4) \quad \xi_{ij} = \Gamma_{ij}^k(\xi)\xi_k - \sum_{\alpha} g_{ij}(\xi, N(\alpha))N(\alpha),$$

where $\Gamma_{ij}^k(\xi)$ denote the Cristoffel symbols of the second kind with respect to $III(\xi)$.

When ξ is nondegenerate, then translating it parallel to itself in E to the origin O we can define an immersion $\gamma_{\xi}: M \rightarrow \Sigma$, where Σ is a unit hypersphere in E centered at O . In codimension one γ_{ξ} is the standard Gauss map.

PROPOSITION 2.1. *Let M be a submanifold of E and ξ a nondegenerate parallel normal vector field on M . Then γ_{ξ} is an isometric immersion of M with the metric $III(\xi)$ into Σ .*

Proof. Let the symbol \hookrightarrow denote an immersion, and \rightarrow a pull-back of the metric from the ambient space. Then the following diagram is commutative in \hookrightarrow and \rightarrow .



where σ is the standard imbedding of Σ in E , and \bar{g} is the metric induced on Σ from E . The Proposition is proved.

For convenience we write $h(\alpha) \equiv h(N(\alpha))$. The position vector field X of a submanifold M can be decomposed into two parts:

$$(5) \quad X = X_T + X_N,$$

where $X_T \in T(M)$, $X_N \in N(M)$. In the frame $X_1, X_2, \dots, X_m, N(1), \dots, N(n)$ we have

$$(6) \quad \begin{aligned} X_T &= G^{ij} \langle X, X_i \rangle X_j, \\ X_N &= - \sum_{\alpha} h(\alpha) N(\alpha). \end{aligned}$$

If ξ is nondegenerate and parallel, then from (1) we see that

$$(7) \quad X_T = -g^{ij}(\xi)h_i(\xi)\xi_j$$

where $g^{ij}(\xi)$ are the elements of $(g_{ij}(\xi))^{-1}$, and $h_i(\xi) = \partial_i h(\xi)$.

Put

$$h_{ij}(\xi) = \partial_{ij} h(\xi), \nabla_{ij} h(\xi) = h_{ij}(\xi) - \Gamma_{ij}^k(\xi)h_k(\xi).$$

Under the above assumptions on ξ we obtain with the use of (4)

$$(8) \quad b_{ij}(\xi) = \nabla_{ij} h(\xi) + \sum_{\alpha} g_{ij}(\xi, N(\alpha))h(\alpha).$$

3. The elementary symmetric functions of principal radii of curvature and the associated differential equations. Let ξ be a unit normal vector at a point $x \in M$. The principal radii of curvature associated with ξ are denoted by $R_{\xi_1}, \dots, R_{\xi_m}$ and defined as the roots of the determinantal equation

$$\det(b_{ij}(\xi) - Rg_{ij}(\xi)) = 0.$$

If ξ is a restriction to x of a nondegenerate vector field, then $III(\xi)$ is positive definite, and in this case the R_{ξ_i} are well defined. Moreover, in this case they do not vanish. Let $g(\xi) \equiv \det(g_{ij}(\xi))$. The elementary symmetric function of order k in R_{ξ_i} (nonnormed)

$$S_{\xi k}(R) = \sum_{i_1 \neq i_2 \neq \dots \neq i_k} R_{\xi_{i_1}} \dots R_{\xi_{i_k}},$$

and it is the coefficient at $(-R)^{m-k}$ of the polynomial

$$(9) \quad \frac{\det(b_{ij}(\xi) - Rg_{ij}(\xi))}{g(\xi)} = (-R)^m + S_{\xi_1}(R)(-R)^{m-1} + \dots + S_{\xi_m}(R).$$

Set $a_{ij}(\xi) = b_{ij}(\xi) - \lambda g_{ij}(\xi)$, where λ is real. Consider a polynomial in λ defined by the equation

$$(10) \quad \alpha^{ij}(\xi) = \sum_{k=1}^m (-\lambda)^{m-k} S_{\xi k}^{ij},$$

where $\alpha^{ij}(\xi)$ is the cofactor of the element $a_{ij}(\xi)$.

PROPOSITION 3.1. *Let M be a submanifold of E and ξ is a parallel unit normal vector field defined in a neighborhood of $x \in M$ and such that $II(\xi) > 0$ at x . Then the quadratic forms $S_{\xi k}^{ij} \nu_i \nu_j, k=2, \dots, m$, are positive definite at x . Here ν_1, \dots, ν_m are arbitrary real parameters, $\nu^2 = \nu_1^2 + \nu_2^2 + \dots + \nu_m^2 \neq 0$. If M is compact, ξ is defined on M , parallel, and $II(\xi) \neq 0$ everywhere on M , then those quadratic forms are definite everywhere and by selecting a proper orientation of M and E , they can be made positive definite. When $k=1$ this*

assertion is true under the only assumption that ξ is parallel and nondegenerate.

The proof of this Proposition is standard and we omit it here.

Suppose now that ξ is a nondegenerate parallel vector field. Then in view of (8), (9), and (10) we put

$$\begin{aligned}
 P_{\xi k}(h) &\equiv \frac{1}{g(\xi)} S_{\xi k}^{ij} \nabla_{ij} h(\xi), \\
 Q_{\xi k} &= \frac{1}{g(\xi)} \sum_{\alpha} S_{\xi k}^{ij} g_{ij}(\xi, N(\alpha)) h(\alpha), \\
 (11) \quad M_{\xi k}(h) &\equiv P_{\xi k}(h) + Q_{\xi k}.
 \end{aligned}$$

It is not difficult to see that

$$(12) \quad M_{\xi k}(h) = kS_{\xi k}(R).$$

PROPOSITION 3.2. *Let M be a submanifold of E and ξ a nondegenerate parallel normal vector field defined in a neighborhood of $x \in M$. Then*

$$(13) \quad Q_{\xi k} = (m - k + 1)S_{\xi k-1}(R)h(\xi) + \langle H_{\xi k}, X \rangle, \quad (S_{\xi 0} \equiv 1),$$

where $H_{\xi k}$ is a uniquely defined vector in $N'_x(M) = N_x(M) \ominus \xi$ independent on the choice of basis in $N'_x(M)$. If $k = 1$, then $-H_{\xi 1}$ is the m times mean curvature vector of the submanifold $\gamma_{\xi}(M) \subset \Sigma$.

Proof. Since $\xi \in N_x(M)$, we can select an orthonormal basis in $N_x(M)$ so that ξ is one of the vectors in this basis. Let us preserve the old notation for the new basis, and let $\xi = N(1)$. Then

$$\begin{aligned}
 Q_{\xi k} &= \frac{S_{\xi k}^{ij}}{g(\xi)} g_{ij}(\xi) h(\xi) + \sum_{2 \leq \alpha \leq n} \frac{S_{\xi k}^{ij}}{g(\xi)} g_{ij}(\xi, N(\alpha)) h(\alpha) \\
 &= (m - k + 1)S_{\xi k-1}(R)h(\xi) - \left\langle \sum_{2 \leq \alpha \leq n} \frac{S_{\xi k}^{ij}}{g(\xi)} g_{ij}(\xi, N(\alpha)) N(\alpha), X \right\rangle.
 \end{aligned}$$

The form $-(S_{\xi k}^{ij}/g(\xi))g_{ij}(\xi, \eta)$, where $\eta \in N'_x(M)$, is linear in η . Therefore, there exists a unique element $H_{\xi k}$ in $N'_x(M)$ such that

$$-\frac{S_{\xi k}^{ij}}{g(\xi)} g_{ij}(\xi, \eta) = \langle H_{\xi k}, \eta \rangle$$

for any $\eta \in N'_x(M)$. (Strictly speaking, the inner product in the last formula should be taken in $N'_x(M)$. But it is induced in $N'_x(M)$ from E , and, therefore, it is the same in either sense.) Thus, we conclude that

$$-\left\langle \sum_{2 \leq \alpha \leq n} \frac{S_{\xi k}^{ij}}{g(\xi)} g_{ij}(\xi, N(\alpha)) N(\alpha), X \right\rangle = \langle H_{\xi k}, X \rangle .$$

The rest of the Proposition follows from Proposition 2.1 and the fact that

$$\frac{S_{\xi 1}^{ij}}{g(\xi)} = g^{ij}(\xi) .$$

This completes the proof.

COROLLARY 3.1. *Let M be a submanifold of E and ξ a non-degenerate parallel vector field on M . Then (13) holds everywhere on M and*

$$(14) \quad M_{\xi k} = P_{\xi k}(h) + (m - k + 1)S_{\xi k-1}(R)h(\xi) + \langle H_{\xi k}, X \rangle$$

for all $x \in M$.

REMARK 1. The functions $\langle H_{\xi k}, X \rangle$ are similar to the functions $F_k(\xi)$ constructed in [4]. However, the latter are related to principal curvatures and depend on the first and the second fundamental forms, while $\langle H_{\xi k}, X \rangle$ depend on the second and the third fundamental forms in the direction ξ . It is not difficult to point out situations where $H_{\xi k}$ or $\langle H_{\xi k}, X \rangle$ vanish. For example, if $\dim E - \dim M = 1$, then $H_{\xi k} \equiv 0$ for all k . Another example is when the normal component of X has the direction ξ . Then $h(\alpha) = -\langle X, N(\alpha) \rangle \equiv 0$ for $\alpha = 2, \dots, m$. In these examples the functions $F_k(\xi)$ introduced in [4] also vanish. One more example is given by the case where $III(\xi, N(\alpha)) \equiv 0$ for $\alpha = 2, \dots, m$, $(N(1) = \xi)$.

REMARK 2. Let M be a submanifold of E and ξ a nondegenerate parallel normal field on M . Let f and f' be two smooth functions defined on M . Put

$$b_{ij}^f(\xi) = \nabla_{ij} f - \sum_{\alpha} g_{ij}(\xi, N(\alpha)) \langle X, N(\alpha) \rangle .$$

Similarly to (9), (10), construct $S_{\xi k}^{ij}(f)$ and consider

$$M_{\xi k}(f, f') \equiv \frac{1}{g(\xi)} S_{\xi k}^{ij}(f) \nabla_{ij} f f' + \frac{(m - k + 1)}{k - 1} M_{\xi k-1}(f, f') f' + \langle H_{\xi k}(f), X \rangle , \text{ for } k > 1 ,$$

$$M_{\xi 1}(f, f') \equiv g^{ij}(\xi) \nabla_{ij} f f' + m f f' + \langle H_{\xi 1}(f), X \rangle , \text{ for } k = 1 .$$

These differential operators proved to be useful in the study of uniqueness Theorems for convex hypersurfaces in Euclidean space [7]. (In this case they are elliptic, and the last term in the right-

hand side vanishes.) It is plausible that they have applications in establishing uniqueness Theorems for submanifolds of E in codimension >1 . We hope that we come back to it again elsewhere.

4. Applications. We begin with a slight generalization of the formula (14), which leads to an integral formula relating the elementary symmetric functions of arbitrary order. This formula is of Minkowski-Hsiung type, and in the form involving two consecutive elementary symmetric functions of principal curvatures it was derived and studied by many authors (see Chen and Yano [4], and also [3], Chapter 6; in both sources further references can be found). However, the methods of those authors do not seem to generalize so as to obtain the following formulas (16) and (17).

In what follows, unless stated otherwise, it is assumed that M is a compact submanifold without boundary.

The following Lemma is a version of E. Hopf's Lemma on Laplace-Beltrami operator.

LEMMA. *Suppose that M is a submanifold of E , ξ is a nondegenerate parallel normal vector field on M , and h' is a smooth function on M . Put*

$$P_{\xi k}(h') \equiv \frac{1}{g(\xi)} S_{\xi k}^{ij} \nabla_{ij} h' ,$$

where the coefficients $S_{\xi k}^{ij}$ are the same as in (11). Then

$$(15) \quad P_{\xi k}(h') = \frac{1}{\sqrt{g(\xi)}} \partial_i \left(\frac{S_{\xi k}^{ij}}{\sqrt{g(\xi)}} \partial_j h \right) .$$

If $k = 1$ and, in addition, we assume that $P_{\xi k}(h')$ does not change its sign on M , then h' is a constant function on M . The same is true when $k > 1$ provided there exists at least one point on M where $II(\xi) \neq 0$.

Proof. It is easy to see, with the use of formula (4), that $b_{ij}(\xi)$ is a Codazzi tensor with respect to $\Gamma_{ij}^k(\xi)$. Therefore, $P_{\xi k}(h')$ can be written in the divergence form (15) (see [5, 7]). When $k > 1$ and $II(\xi) \neq 0$ at some point of M then $II(\xi) \neq 0$ everywhere on M because ξ is nondegenerate. By Proposition 3.1 the operator $P_{\xi k}(h')$ is uniformly elliptic. Now the rest of the proof runs similarly to the standard proof of E. Hopf's Lemma on the Laplace-Beltrami operator on a compact Riemannian submanifold ([6], p. 338). The Lemma is proved.

THEOREM 4.1. *Let M be a submanifold of E and ξ a nonde-*

generate parallel normal vector field on M . Then for arbitrary k and s , $k = 1, \dots, m$, $s = 1, \dots, k$,

$$(16) \quad kS_{\varepsilon k}(R) = \sum_{l=0}^{s-1} \frac{(m-k+l)! (k-l-1)!}{(m-k)! (k-1)!} [P_{\varepsilon k-l}(h) + \langle H_{\varepsilon k-l}, X \rangle] h^l + \frac{(m-k+s)! (k-s)!}{(m-k)! (k-1)!} S_{\varepsilon k-s}(R) h^s,$$

and

$$(17) \quad k \int_M S_{\varepsilon k}(R) dO(\xi) = - \sum_{l=1}^{s-1} l \cdot \frac{(m-k+l)! (k-l-1)!}{(m-k)! (k-1)!} \int_M h^{l-1} \frac{S_{\varepsilon k-l}^{ij}}{g(\xi)} h_i h_j dO(\xi) + \sum_{l=0}^{s-1} \frac{(m-k+l)! (k-l-1)!}{(m-k)! (k-1)!} \int_M \langle H_{\varepsilon k-l}, X \rangle h^l dO(\xi) + \frac{(m-k+s)! (k-s)!}{(m-k)! (k-1)!} \int_M S_{\varepsilon k-s}(R) h^s dO(\xi),$$

where $h \equiv h(\xi)$ is the support function of M with respect to ξ .

Proof. Formula (16) follows from the formulas (12) and (14); and (17) is obtained from (16) by integrating, applying Green's formula, and the preceding Lemma.

COROLLARY 4.1. *If in Theorem 4.1 $s = k$, then*

$$(18) \quad k \int_M S_{\varepsilon k}(R) dO(\xi) = (m-k+1) \int_M S_{\varepsilon k-1}(R) h(\xi) dO(\xi) + \int_M \langle H_{\varepsilon k}, X \rangle dO(\xi).$$

This formula is an analogue of an integral formula due to Chen and Yano [4].

We recall that if a submanifold M (not necessarily compact) of E is contained in a hypersphere of E centered at the origin, then it is called a spherical submanifold (see [2]).

In the following we often make use of a Theorem due to Chen [2].

THEOREM A. *Let M be a submanifold (not necessarily compact) of E . If there exists a nondegenerate parallel normal vector field ξ such that $h(\xi) = \text{const}$ everywhere on M , then M is a spherical submanifold of E .*

From now on always when $k > 1$ it is assumed that there

exists a point on M where $II(\xi) \neq 0$, and the orientation is such that $II(\xi) > 0$.

Examples of submanifolds with this property can be constructed as follows. Let M_1 and M_2 be two strictly convex hypersurfaces. Then the natural imbedding of $M_1 \times M_2$ in Euclidean space of dimension = $\dim M_1 + \dim M_2 + 2$ gives such example.

The next Theorem is an immediate consequence of formulas (12), (14), the Lemma, and Theorem A.

THEOREM 4.2. *Let M be a submanifold in E and ξ is a non-degenerate parallel normal vector field on M . Assume further that for some $k, k = 1, 2, \dots, m$, at every point of M*

$$(19) \quad cS_{\xi k}(R) = S_{\xi k-1}(R)h(\xi) \quad (S_{\xi 0} \equiv 1),$$

where c is a constant such that the expression

$$[k - c(m - k + 1)]S_{\xi k}(R) - \langle H_{\xi k}, X \rangle$$

is either nonnegative or nonpositive. Then M is a spherical submanifold.

Proof. In the formula (16) set $s = 1$. Then by (19)

$$[k - c(m - k + 1)]S_{\xi k}(R) - \langle H_{\xi k}, X \rangle = P_{\xi k}(h),$$

and the Theorem follows from the Lemma and Theorem A.

In case $k = 1$ a result similar to this Theorem has been given by Wegner [9], Satz 2. His result can be also obtained by our method, and furthermore, it can be generalized for $k > 1$.

Let $M = S^m$, where S^m is a standard m -sphere lying in $m + 1$ -dimensional Euclidean space $E^{m+1} \subset E$. Then, evidently, $H_{\xi k} \equiv 0$ for all k , and ξ is the unit normal vector field on S^m in E^{m+1} . With this fact in mind we state the following

COROLLARY 4.2. *Let M be a submanifold of E and ξ a non-degenerate parallel normal vector field on M . If for some $k, k = 2, \dots, m$, at every point of M*

$$\langle H_{\xi k}, X \rangle = 0,$$

and

$$cS_{\xi k}(R) = S_{\xi k-1}(R)h(\xi),$$

where c is a constant $\neq 0$, then M is a spherical submanifold. Furthermore, in this case it is necessary that $c = k/(m - k + 1)$. The

assertion is also true when $k = 1$, provided $II(\xi) \neq 0$ at some point of M .

Proof. We show at first that the function $S_{\xi k}(R)$ does not change its sign on M . Let A be a point on M where $II(\xi)$ is definite. Then the principal radii of curvature $R_{\xi i}$, $i = 1, \dots, n$, must all be of the same sign at A . Since ξ is nondegenerate $R_{\xi i}$ will all have the same sign everywhere on M . Hence, the function $S_{\xi k}(R)$ can not change its sign on M , and moreover it does not vanish on M .

Now it is clear that the expression

$$[k - c(m - k + 1)]S_{\xi k}(R)$$

is either nonnegative or nonpositive and therefore by Theorem 4.2. M is a spherical submanifold. On the other hand,

$$\int_M S_{\xi k}(R) dO(\xi) \neq 0;$$

hence, the formula (18) implies that $c = k/(m - k + 1)$. The Corollary is proved.

A Theorem similar to Theorem 4.2 can be stated with the use of Theorem 4.1.

We point out only a particular case of it.

THEOREM 4.3. *Let M be a submanifold in E and ξ a nondegenerate parallel normal vector field on M . Suppose that for some k and s , $k = 1, \dots, m$, $s = 1, \dots, k$, the following conditions are satisfied:*

- (a)
$$kS_{\xi k}(R) \geq \frac{(m - k + s)! (k - s)!}{(m - k)! (k - 1)!} S_{\xi k-s}(R) h^s(\xi);$$
- (b)
$$\int_M \langle H_{\xi k-l}, X \rangle h^l(\xi) dO(\xi) \leq 0 \quad \text{for } l = 0, \dots, s - 1;$$
- (c)
$$h(\xi) > 0.$$

Then M is a spherical submanifold.

Proof. The conditions (a), (b), (c) and Proposition 3.1 imply that all integrals in formula (17) must vanish. Hence $h(\xi) = \text{const}$, that is, M is a spherical submanifold.

THEOREM 4.4. *Let M be a submanifold in E and ξ a nondegenerate parallel normal vector field on M . Suppose that for some k and s , $k = 1, \dots, m$, $s = 1, \dots, k$, the following conditions are*

satisfied:

- (a) $cS_{\varepsilon k}(R) = S_{\varepsilon k-s}(R)h^s(\xi)$ everywhere on M , where c is a constant $\neq 0$;
- (b) $\langle H_{\varepsilon k-l}, X \rangle = 0$ for $l = 0, \dots, s - 1$;
- (c) $h(\xi) > 0$.

In case $k = 1$ assume also that $II(\xi) \neq 0$ at some point of M .

Then M is a spherical submanifold and $c = (m - k)!k!/(m - k + s)!(k - s)!$.

Proof. At first we show that c can have only the value indicated in the assertion. In showing that we follow Blaschke [1], p. 233. Let A be a point on M where $h(\xi)$ ($=h$) attains its maximum. Then at A ,

$$\nabla_{ij}h \leq 0.$$

By Proposition 3.1 the forms $S_{\varepsilon k}^{ij}\nu_i\nu_j$, $k = 1, \dots, m$, are definite, and therefore at the point A the expressions

$$P_{\varepsilon k-l}(h) = \frac{1}{g(\xi)} S_{\varepsilon k-l}^{ij} \nabla_{ij}h \quad l = 0, 1, \dots, k - 1,$$

are all of the same sign, and namely nonpositive. On the other hand, by Theorem 4.1 (formula (16)) in view of the conditions (a) and (b), we obtain

$$\begin{aligned} & \left[k - c \frac{(m - k + s)! (k - s)!}{(m - k)! (k - 1)!} \right] S_{\varepsilon k}(R) \\ &= \sum_{l=0}^{s-1} \frac{(m - k + l)! (k - l - 1)!}{(m - k)! (k - 1)!} h^l P_{\varepsilon k-l}(h). \end{aligned}$$

The right-hand side is nonpositive at A , and similar to the proof of Corollary 4.2 one shows that $S_{\varepsilon k}(R) > 0$ everywhere on M . Therefore,

$$k - c \frac{(m - k + s)! (k - s)!}{(m - k)! (k - 1)!} \leq 0.$$

Considering the point where h attains its minimum we arrive at the opposite inequality. Thus, $c = (m - k)!k!/(m - k + s)!(k - s)!$.

Now, making use of the second part of Theorem 4.1 (formula 17)) and the conditions (a), (b), (c) with constant c taken as above, we obtain

$$-\sum_{l=1}^{s-1} \frac{(m - k + l)! (k - l - 1)!}{(m - k)! (k - 1)!} \int_M h^{l-1} \frac{S_{\varepsilon k-l}^{ij}}{g(\xi)} h_i h_j dO(\xi) = 0.$$

From here, it follows that $h = \text{const}$. Hence, M is a spherical submanifold. The Theorem is proved.

COROLLARY 4.3. *Let M be a closed strictly convex hypersurface in Euclidean space E and ξ is the unit normal vector field on M . Suppose that for some k and s , $k = 1, \dots, m$, $s = 1, \dots, k$,*

$$cS_k(R) = S_{k-s}(R)h^s(\xi)$$

everywhere on M , where c is a constant $\neq 0$. Then M is a hypersphere, and c is as in Theorem 4.4. (In the last equality the subscript ξ is omitted for the obvious reason.)

Proof. For a hypersurface in E , ξ is always parallel, and since M is strictly convex, ξ is nondegenerate. Also $H_{\xi^l} \equiv 0$ for $l = 1, \dots, k$. The support function $h(\xi)$ can always be made strictly positive by placing the origin of the coordinate system in E inside M . Now the Corollary follows from Theorem 4.4.

REMARK 1. As was mentioned in the introduction, this Corollary is known. In particular, the condition quoted earlier can be expressed in terms of the elementary symmetric functions of principal radii of curvature as follows:

$$cS_m(R) = S_{m-s}(R)h^s(\xi).$$

If in Corollary 4.3 we take $k = m$, then we obtain the above result. It is due to Süss; see [8], Korollar 6.3, and other references there.

REMARK 2. Theorem 4.4 does not contain Corollary 4.2, since in the latter it is not required that $h(\xi) > 0$.

REFERENCES

1. W. Blaschke, *Vorlesungen Über Differentialgeometrie*, Berlin, Verlag von Julius Springer, 1930.
2. B.-Y. Chen, *On the support functions and spherical submanifolds*, Math. J. Okayama Univ., **15** (1971), 15-23.
3. ———, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.
4. B.-Y. Chen and K. Yano, *Integral formulas for submanifolds and their applications*, J. Differential Geometry, **5** (1971), 467-477.
5. S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann., **225** (1977), 195-204.
6. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 2, Interscience, New York, 1969.
7. V. I. Oliker, *On certain elliptic differential equations on a hypersphere and their geometric applications*, Indiana Univ. Math. J., **28** no. 1 (1979), 35-51.
8. U. Simon, *Minkowskische Integralformeln und ihre Anwendungen in der Differential Geometrie im Großen*, Math. Annalen, **173** (1967), 307-321.

9. B. Wegner, *Codazzi-Tensoren und kennzeichnungen spharischer immersionen*, J. Differential Geometry, **9** (1974), 61-70.

Received June 20, 1978. This work was supported by a research grant from the University of Iowa.

THE UNIVERSITY OF IOWA
IOWA CITY, IA 52242

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)

University of California
Los Angeles, CA 90024

HUGO ROSSI

University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ANDREW OGG

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$84.00 a year (6 Vols., 12 issues). Special rate: \$42.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1979 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Pacific Journal of Mathematics

Vol. 83, No. 2

April, 1979

Patrick Robert Ahern, <i>On a theorem of Hayman concerning the derivative of a function of bounded characteristic</i>	297
Walter Allegretto, <i>Finiteness of lower spectra of a class of higher order elliptic operators</i>	303
Leonard Asimow, <i>Superharmonic interpolation in subspaces of $C_c(X)$</i>	311
Steven F. Bellenot, <i>An anti-open mapping theorem for Fréchet spaces</i>	325
B. J. Day, <i>Locale geometry</i>	333
John Erik Fornæss and Steven Krantz, <i>Continuously varying peaking functions</i>	341
Joseph Leonide Gerver, <i>Long walks in the plane with few collinear points</i>	349
Joseph Leonide Gerver and Lawrence Thom Ramsey, <i>On certain sequences of lattice points</i>	357
John R. Graef, Yuichi Kitamura, Takaši Kusano, Hiroshi Onose and Paul Winton Spikes, <i>On the nonoscillation of perturbed functional-differential equations</i>	365
James A. Huckaba and James M. Keller, <i>Annihilation of ideals in commutative rings</i>	375
Anzelm Iwanik, <i>Norm attaining operators on Lebesgue spaces</i>	381
Surjit Singh Khurana, <i>Pointwise compactness and measurability</i>	387
Charles Philip Lanski, <i>Commutation with skew elements in rings with involution</i>	393
Hugh Bardeen Maynard, <i>A Radon-Nikodým theorem for finitely additive bounded measures</i>	401
Kevin Mor McCrimmon, <i>Peirce ideals in Jordan triple systems</i>	415
Sam Bernard Nadler, Jr., Joseph E. Quinn and N. Stavrakas, <i>Hyperspaces of compact convex sets</i>	441
Ken Nakamura, <i>An explicit formula for the fundamental units of a real pure sextic number field and its Galois closure</i>	463
Vassili Nestoridis, <i>Inner functions invariant connected components</i>	473
Vladimir I. Olikier, <i>On compact submanifolds with nondegenerate parallel normal vector fields</i>	481
Lex Gerard Oversteegen, <i>Fans and embeddings in the plane</i>	495
Shlomo Reisner, <i>On Banach spaces having the property $G.L$</i>	505
Gideon Schechtman, <i>A tree-like Tsirelson space</i>	523
Helga Schirmer, <i>Fix-finite homotopies</i>	531
Jeffrey D. Vaaler, <i>A geometric inequality with applications to linear forms</i>	543
William Jennings Wickless, <i>T as an \mathcal{G} submodule of G</i>	555
Kenneth S. Williams, <i>The class number of $Q(\sqrt{-p})$ modulo 4, for $p \equiv 3 \pmod{4}$ a prime</i>	565
James Chin-Sze Wong, <i>On topological analogues of left thick subsets in semigroups</i>	571