FIX-FINITE HOMOTOPIES

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A well-known result by H. Hopf states that every selfmap $f$ of a polyhedron $|K|$ can be deformed into a selfmap $f'$ which has only a finite number of fixed points and is arbitrarily close to the given one. In addition one can locate all fixed points of $f'$ in maximal simplexes. A map which has a finite fixed point set is here called a fix-finite map, and a homotopy $F: |K| \times I \rightarrow |K|$ is called a fix-finite homotopy if the map $f_t = F(\cdot, t)$ is fix-finite for every $t \in I$. We extend Hopf's result to homotopies, and show that two homotopic selfmaps $f_0$ and $f_1$ of a polyhedron $|K|$ which are fix-finite and have all their fixed points located in maximal simplexes can be related by a homotopy which is fix-finite and arbitrarily close to the given one. All fixed points of $F$ can again be located in as high-dimensional simplexes as possible. Some simple properties are derived from the fact that the fix-finite homotopy is constructed in such a way that its fixed point set is a one-dimensional polyhedron in $|K| \times I$.

A. Introduction. In 1929 H. Hopf [2], Satz V, proved a well-known theorem which states that every selfmap $f$ of a polyhedron can be deformed into a selfmap $f'$ which is arbitrarily close to $f$ and has only a finite number of fixed points. The construction of $f'$ can be carried out so that all fixed points of $f'$ are, in Hopf's terminology, "regular", i.e., they are located in maximal simplexes. We call a map which has only a finite number of fixed points a fix-finite map, and formulate Hopf's result accordingly.

**Theorem 1 (Hopf).** Let $f$ be a selfmap of a polyhedron $|K|$. Given $\varepsilon > 0$, there exists a selfmap $f'$ of $|K|$ such that

1. $f'$ is fix-finite,
2. all fixed points of $f'$ are contained in maximal simplexes of $|K|$,
3. the distance $d(f, f') < \varepsilon$.

We ask in this paper whether a similar result can be obtained for homotopies. We call a map $F: |K| \times I \rightarrow |K|$ (where $I$ is the unit interval) a fix-finite homotopy if the map $f_t: |K| \rightarrow |K|$ defined by $f_t(x) = F(x, t)$ is a fix-finite map for every $t \in I$, and ask therefore whether two selfmaps $f_0$ and $f_1$ of a polyhedron $|K|$ which are fix-finite and homotopic can be related by a homotopy which is fix-finite
and arbitrarily close to the given one. We shall show that this is possible if all fixed points of \( f_0 \) and \( f_1 \) are contained in maximal simplexes, and we shall construct the fix-finite homotopy so that its fixed points are again located as nicely as possible. They clearly cannot all be located in maximal simplexes of \( |K| \), but they can be located in simplexes which are either maximal, or faces of maximal dimension. Let us make these notions precise.

We denote by \( |K| \) a polyhedron which is the realization of a finite simplicial complex \( K \), by \( \sigma \) an open simplex of \( K \), by \( \overline{\sigma} \) its closure, and by \( \dim \sigma \) its dimension. \( \sigma < \tau \) means that \( \sigma \) is a face of the simplex \( \tau \). The (open) star \( \text{st} \sigma \) of \( \sigma \) consists of all simplexes \( \tau \) of \( |K| \) with \( \sigma < \tau \). A simplex \( \sigma \) is called maximal if \( \sigma = \text{st} \sigma \), and we call it a hyperface if \( \dim \text{st} \sigma = \dim \sigma + 1 \). A fixed point of a homotopy \( F: |K| \times I \to |K| \) is defined as a point \( x \in |K| \) with \( F(x, t) = x \) for some \( t \in I \). If \( f, f' \) are maps and \( d \) is the metric of \( |K| \), then the sup metric is given by

\[
d(f, f') = \sup \{d(f(x), f'(x)) \mid x \in X\}.
\]

We use this terminology to state our main result.

**Theorem 2.** Let \( F \) be a homotopy between two selfmaps \( f_0 \) and \( f_1 \) of a polyhedron \( |K| \), let \( f_0 \) and \( f_1 \) be fix-finite, and let all their fixed points be contained in maximal simplexes. Given \( \varepsilon > 0 \), there exists a homotopy \( F' \) from \( f_0 \) to \( f_1 \) such that

1. \( F' \) is fix-finite,
2. all fixed points of \( F' \) are contained in maximal simplexes or hyperfaces of \( |K| \),
3. \( d(F, F') < \varepsilon \).

Special cases of Theorem 2 are known. Weier [6] constructed a fix-finite homotopy satisfying (1) and a condition related to (2) if \( |K| \) is a 2-dimensional pseudomanifold satisfying a certain connectedness condition, and in [4], Satz III we constructed a fix-finite homotopy satisfying (1) and (3) if \( |K| \) is an orientable and triangulable finite dimensional manifold without boundary.

The proof of Theorem 2 given below is related to Hopf's proof of Theorem 1. Hopf started with a simplicial approximation of the given map, and then carried out a succession of changes on simplexes of increasing dimension which freed the simplicial approximation of fixed points on all but maximal simplexes. The final result is a map which is again simplicial and satisfies Theorem 1. Hopf's proof is readily available in [1], pp. 117–118, where the successive changes are called "Hopf constructions".
In our proof of Theorem 2 a homotopy is altered successively on simplexes of increasing dimension by a "Hopf construction for homotopies" which is described in §B. As this construction can only be applied to simplicial homotopies, it is first necessary to approximate the given homotopy by a simplicial one. This leads to a proof of Theorem 2 in three steps. In the first, the given maps $f_0$ and $f_1$ are, with the help of the Hopf construction, approximated by fix-finite simplicial maps $g_0$ and $g_1,$ and fix-finite homotopies $H_i$ from $f_i$ to $g_i$ (where $i = 0, 1$) are obtained in a manner reminiscent of [4]. A homotopy between the simplicial maps $g_0$ and $g_1$ has a simplicial approximation relative to $|K| \times \{0\} \cup |K| \times \{1\},$ on which a succession of Hopf constructions for homotopies is carried out in Step 2, leading to a fix-finite homotopy $G'$ from $g_0$ to $g_1.$ Finally, in Step 3, the desired homotopy $F'$ is obtained by constructing a homotopy from $g_0$ to $g_1$ as the composite of $H_0^{-1}, F,$ and $H_1,$ changing it to a homotopy $G'$ as in Step 2, and forming the composite of $H_0, G',$ and $H_1^{-1},$ where all compositions are made with suitable scale changes to ensure closeness between $F$ and $F'.$

The homotopy $F'$ is constructed in such a way that the set

$$\text{Fix } F' = \{(x, t) \in |K| \times I | F'(x, t) = x\}$$

is a finite one-dimensional polyhedron. Some simple consequences of this fact are given in §D. One of them is the existence of an upper bound $M$ so that the number of fixed points of $f_1'$ is $\leq M$ for every $t \in I.$

B. A Hopf construction for homotopies. Let $G$ be the realization of a simplicial function $P \rightarrow K,$ where $P$ is a suitable complex with $|P| = |K| \times I,$ and let $\tau$ be a given simplex of $|P|.$ The Hopf construction for homotopies, which frees $G$ of all fixed points on $\tau$ as long as $G(\tau)$ is not maximal in $|K|,$ will be the basic tool in the second step of the proof of Theorem 2 and we shall embody its results in the rather technical Lemma 1 below. We write $G: |P| \rightarrow |K|$ to indicate that $G$ is the realization of a simplicial function from $P$ to $K.$ The construction of $K_\tau,$ the barycentric subdivision of $K$ modulo the subcomplex $L,$ can e.g. be found in [3], p. 49. If $L = \phi,$ then it is the ordinary barycentric subdivision of $K.$ A refinement of $K$ is a complex obtained from $K$ by means of a finite number of subdivisions modulo subcomplexes. $\mu(K)$ denotes the mesh of $|K|,$ i.e., the maximum of the diameters of its simplexes.

**Lemma 1.** Let $P$ be a complex with $|P| = |K| \times I,$ let $G: |P| \rightarrow |K| \text{ be simplicial and } \pi: |P| \rightarrow |K| \text{ be the first projection. If } \tau \text{ is a simplex of } |P| \text{ for which } \pi(\tau) \text{ is contained in a simplex } \rho \text{ of } |K'|,$
where $K'$ is a refinement of $K$, if $\tau \cap \text{Fix } G \neq \emptyset$ where $\text{Fix } G = \{(x, t) \in P \mid G(x, t) = \pi(x, t)\}$, and if $G(\tau)$ is not maximal in $|K|$, then there exists a simplicial map $G' : |P_q| \rightarrow |K|$, with $Q = P \setminus \text{st } \tau$, so that

(1) $\tau \cap \text{Fix } G' = \emptyset$,
(2) $G = G'$ on $|Q|$,
(3) $d(G, G') \leq 2\mu(K)$.

Proof. Let $\rho^*$ be a maximal simplex of $K'$ with $\rho < \rho^*$, and $\sigma^*$ be a maximal simplex of $K$ with $\rho^* \subset \sigma^*$. Then

$$\pi(\tau) \subset \rho \subset \bar{\rho}^* \subset \bar{\sigma}^*.$$

If $\sigma = G(\tau)$, then $\pi(\tau) \cap \sigma \neq \emptyset$ implies $\sigma \subset \sigma^*$.

Define $G : |P_q| \rightarrow |K|$ on the vertices of $P_q$ as follows: If $v \in Q$, let $G'(v) = G(v)$. If $\tau' \in \text{st } \tau \setminus \tau$ and $v$ is the vertex of $P_q$ contained in $\tau'$, let $G'(v)$ be any vertex of $\sigma$, and if $v$ is the vertex of $P_q$ contained in $\tau$, let $G'(v)$ be any vertex of $\sigma^*$ which is not a vertex of $\sigma$. (As $\sigma$ is not maximal, such a vertex exists.) It can be checked that $G'$ extends to a simplicial map $G' : |P_q| \rightarrow |K|$. The proof that $G'$ satisfies the conditions (1), (2), and (3) closely parallels arguments in [1], p. 117-118, and is omitted.

C. The proof.

Step 1. Construction of fix-finite simplicial maps $g_i$ which are fix-finitely homotopic to the given maps $f_i$.

We begin with a simple lemma.

**Lemma 2.** Let $|K|$ be a connected polyhedron, $x \in |K|$, and the carrier $\sigma$ of $x$ in $|K|$ maximal. Given $\delta > 0$, there exists a $y \in \sigma$ with $d(x, y) < \delta$ whose carrier in any refinement of $K$ is maximal.

**Proof.** $|K|$ is connected, therefore $\sigma$ is of dimension $p > 0$. As the number of refinements of $\bar{\sigma}$ is countable, the dimension of the union $A$ of the $(p - 1)$-skeletons of all refinements is $p - 1$, and $y \in \sigma \setminus A$ with $d(x, y) < \delta$ exists and satisfies the lemma.

The result of Step 1 is given as the next lemma, where

$$\text{diam } H = \sup \{d(H(x, t), H(x, t')) \mid x \in |K|, t, t' \in I\}$$

denotes the diameter of a homotopy $H : |K| \times I \rightarrow |K|$.

**Lemma 3.** Let $f_i : |K| \rightarrow |K|$, $i = 0, 1$, be two selfmaps of a polyhedron $|K|$ which are fix-finite and have all their fixed points located in maximal simplexes of $|K|$. Given $\varepsilon > 0$, there exist a
refinement $K'$ of $K$, refinements $K''$ of the first barycentric subdivision of $K'$, simplicial maps $g_i: |K''| \to |K'|$, and homotopies $H_i$ from $f_i$ to $g_i$ so that

1. $H_i$ is fix-finite and has all its fixed points located in the maximal simplexes of $|K|$, 
2. the fixed points of $g_i$ are located in distinct maximal simplexes of $|K''|$, 
3. $\text{diam } H_i < \varepsilon/4$, 
4. $\mu(K') < \varepsilon/8(n+1)$, where $n = \dim |K|$. 

Proof. We can assume that $|K|$ is connected, otherwise the construction is made on each component.

(i) We first construct two maps $f_0: |K| \to |K|$ and homotopies $H'_i$ from $f_i$ to $f_0$ such that all carriers of fixed points of $f'_i$ are maximal in every refinement of $K$, all carriers of fixed points of $H'_i$ are maximal in $|K|$, and $\text{diam } H'_i < \varepsilon/2$.

Consider $f_0$, and let $\text{Fix } f_0 = \{c_j\}$ be its fixed point set. As $f_0$ is uniformly continuous, we can select $\beta$ with $0 < \beta < \varepsilon/16$ so that, for all $c_j \in \text{Fix } f_0$, the open $\beta$-balls $U(c_j, \beta)$ are pairwise disjoint and each $U(c_j, \beta)$ is contained in the carrier of $c_j$ in $|K|$. Now select $\gamma$ with $0 < \gamma < \beta/2$ such that $d(x, f_0(x)) < \beta/2$ for all $x \in \bigcup \{U(c_j, \gamma) | c_j \in \text{Fix } f_0\}$. According to Lemma 2 each $U(c_j, \gamma)$ contains a point $c'_j$ whose carrier in all refinements of $|K|$ is maximal. If $x \in \overline{U(c_j, \gamma)} \setminus \{c'_j\}$, let $y$ be the point in which the ray from $c'_j$ to $x$ intersects the boundary $\text{Bd } U(c_j, \gamma)$, and $z$ the point on the segment from $c'_j$ to $y$ for which

$$d(c'_j, z) = \frac{d(c'_j, y)}{d(c'_j, y)} \cdot d(c'_j, x).$$

To define a map $f''_{ij}$ from $\overline{U(c_j, \gamma)}$ to $U(c_j, \beta)$, denote by $\overrightarrow{ab}$ the (free) vector from $a$ to $b$ in $U(c_j, \beta)$, and determine $f''_{ij}(x)$ for $x \neq c'_j$ by

$$c'_j f''_{ij}(x) = c'_j x + z f_0(z);$$

also let $f''_{ij} = c'_j$.

As we have for all $x \in \overline{U(c_j, \gamma)}$

$$d(f''_{ij}(x), c_j) \leq d(f''_{ij}(x), x) + d(x, c_j) = d(f_0(x), z) + d(x, c_j) < \beta/2 + \gamma < \beta,$$

this construction is well defined.

Now define $f'_0: |K| \to |K|$ by

$$f'_0(x) = \begin{cases} f''_{ij}(x) & \text{if } x \in \bigcup \{U(c_j, \gamma) | c_j \in \text{Fix } f_0\}, \\ f_0 & \text{otherwise.} \end{cases}$$

$f'_0$ is continuous, and its fixed point set is $\text{Fix } f'_0 = \{c'_j\}$. Hence all
carriers of its fixed points are maximal in every refinement of $|K|$.

If $f'_0(x) \neq f_0(x)$, then $x \in U(c_j, \gamma)$ for some $c_j \in \text{Fix } f_0$. Denote, for $0 < t \leq 1$, by $c_j(t)$ the point which divides the segment from $c_j$ to $c'_j$ in the ratio $t:(1-t)$, and define $H'_0(x, t)$ as the point in $U(c_j, \beta)$ which is obtained in a manner analogous to $f'_0(x)$ but with the use of $c_j(t)$ instead of $c'_j$. Also put $H'_0(x, 0) = f_0(x)$. Then a homotopy $H'_0$ from $f_0$ to $f'_0$ can be constructed from the $H'_0$ in the same way in which $H'_0$ was constructed from the $f'_0$. If $f'_0(x) = f_0(x)$, then $H'_0$ is the constant homotopy, if $f'_0(x) \neq f_0(x)$, then the set $(H'_0(x, t) | 0 \leq t \leq 1)$ lies in some $U(c_j, \beta)$. Hence $\text{diam } H'_0 < 2\beta < \varepsilon/8$. The construction of $H'_0$ shows that all carriers of its fixed points are maximal in $K$.

The map $f'_i$ and the homotopy $H'_i$ from $f_i$ to $f'_i$ are obtained analogously.

(ii) We now describe the construction of the maps $g_i$ and the homotopies $H''_i$ from $f'_i$ to $g_i$.

Choose $\rho_0$ with $0 < \rho_0 < \varepsilon/32$ so that for each $c'_j \in \text{Fix } f'_0$ with carrier $\kappa_j$ in $|K|$ the set $\bar{U}(c'_j, 4\rho_0) \subset \kappa_j$, and so that the $\bar{U}(c'_j, 4\rho_0)$ are pairwise distinct. As $f'_0$ is uniformly continuous, there exists a $\delta_0$ with $0 < \delta_0 \leq \rho_0$ so that

$$f'_0(\bar{U}(c'_j, \delta_0)) \subset \bar{U}(c'_j, \rho_0) \quad \text{for all } c'_j \in \text{Fix } f'_0.$$ 

Furthermore choose $\eta_0$ with $0 < \eta_0 \leq \rho_0$ so that

$$d(x, f'_0(x)) \geq \eta_0 \quad \text{if} \quad d(x, \text{Fix } f'_0) \geq \delta_0.$$ 

Determine $\rho_0$, $\delta_0$, $\eta_0$ analogously for $f'_i$ and select a refinement $K'$ of $K$ so that $\mu(K') < \min \{\delta_0, \delta_i, \eta_0/(2n+1), \eta_i/(2n+1)\}$, where $n$ is the dimension of $K$.

Let $\psi_0$ be a simplicial approximation of $f'_0$ which maps a refinement of the first barycentric subdivision of $K'$ into $K'$, and choose $g_0$ as a map which is obtained from $|\psi_0|$ by a succession of Hopf constructions in the same way in which $f'_0$ is obtained from $|\psi|$ in the proof of Theorem 2 on p. 118 of [1]. Then $g_0$ is a simplicial map $|K''| \rightarrow |K'|$, where $K''$ again refines the first barycentric subdivision of $K'$. It is fix-finite, has all its fixed points located in distinct maximal simplexes of $|K''|$, and $d(\psi_0, g_0) \leq 2n\mu(K')$. As $d(f'_0, \psi_0) \leq \mu(K')$, we have $d(f'_0, g_0) \leq (2n+1)\mu(K') < \eta_0$.

Next, let us construct a homotopy $H''_0$ from $f'_0$ to $g_0$. If $x \in U(c'_j, \delta_0) \cap c'_j \in \text{Fix } f'_0$, then it follows from [1], p. 118 that $g_0(x) = |\psi_0|(x)$. As $\psi_0$ is a simplicial approximation of $f'_0$, it is possible to define $H''_0(x, t)$ by

$$H''_0(x, t) = tf'_0(x) + (1-t)g_0(x).$$
From \( d(x, f'_0(x)) \geq \eta \) and \( d(f'_\theta, g_0) < \eta \) follows \( H''_0(x, t) \neq x \) for all \( 0 \leq t \leq 1 \).

Now consider one of the sets \( \bar{U}(c'_j, \delta_0) \) contained in a maximal simplex \( \kappa_j \) of \(|K|\). \( H''_0 \) has already been defined on \( \text{Bd} \, \bar{U}(c'_j, \delta_0) \times I \) such that

\[
d(c'_j, H''_0(x, t)) \leq d(c'_j, f'_0(x)) + d(f'_\theta(x), g_0(x)) \leq 2\rho_0.
\]

Let further \( H''_0'(x, 0) = f'_0(x) \) and \( H''_0'(x, 1) = g_0(x) \) for all \( x \in \bar{U}(c'_j, \delta_0) \).

Then \( H''_0 \) is defined on \( \text{Bd} \, (\bar{U}(c'_j, \delta_0) \times I) \), has values in \( \bar{U}(c'_j, 2\rho_0) \), and its fixed point set consists of \( c'_j \times \{0\} \) and finitely many points in \( U(c'_j, \delta_0) \times \{1\} \). To extend \( H''_0 \) over all of \( \bar{U}(c'_j, \delta_0) \times I \), let \( \bar{c}_j = (c'_j, 1/2) \), and determine for every point \( \bar{x} = (x, t) \in (\bar{U}(c'_j, \delta_0) \times I) \backslash \{c_j\} \) the point \( \bar{y} = (y, s) \) as the one in which the ray from \( \bar{c}_j \) to \( \bar{x} \) intersects \( \text{Bd} \, (\bar{U}(c'_j, \delta_0) \times I) \). Let \( d \) denote the product metric in \( |K| \times I \), and define \( H''_0'(x, t) \) by

\[
c'_jH''_0'(x, t) = c'_jx + \lambda yH''_0(y, s),
\]

where

\[
\lambda = \tilde{d}(\bar{c}_j, \bar{x})/\tilde{d}(\bar{c}_j, \bar{y}).
\]

As \( d(c'_j, x) \leq \delta_0 \), \( 0 < \lambda \leq 1 \), and \( d(y, H''_0(y, s)) \leq \delta_0 + 2\rho_0 \leq 4\rho_0 \), we obtain in this way a point \( H''_0'(x, t) \in \bar{U}(c'_j, 4\rho_0) \). Finally, let \( H''_0'(c'_j, 1/2) = c'_j \).

In this way \( H''_0 \) is extended over \( \cup \{\bar{U}(c'_j, \delta_0) \mid c'_j \in \text{Fix} \, f'_0\} \), yielding a homotopy \( H''_0' : |K| \times I \to |K| \) from \( f'_0 \) to \( g_0 \) which is fix-finite and has all its fixed points located in the maximal simplex \( \kappa_j \) of \(|K|\). If \( x \in \cup \{\bar{U}(c'_j, \delta_0) \mid c'_j \in \text{Fix} \, f'_0\} \), then \( \sup \{H''_0'(x, t), H''_0'(x, t') \mid t, t' \in I\} \leq d(f'_0, g_0) < \eta \), and if \( x \in \bar{U}(c'_j, \delta_0) \) for some \( c'_j \in \text{Fix} \, f'_0 \), then \( \{H''_0'(x, t) \mid t \in I\} \subset \bar{U}(c'_j, 4\rho_0) \), so \( \sup \{H''_0'(x, t), H''_0'(x, t') \mid t, t' \in I\} \leq 8\rho_0 \). Hence diam \( H''_0 < \varepsilon/4 \). The construction of \( H''_0' : |K| \times I \to |K| \) is analogous.

(iii) Define finally a homotopy \( H_i \) from \( f_i \) to \( g_i \) by

\[
H_i(x, t) = \begin{cases} H'_i(x, 2t) & \text{for } 0 \leq t \leq 1/2, \\ H''_i(x, 2t - 1) & \text{for } 1/2 \leq t \leq 1. \end{cases}
\]

Then diam \( H_i \leq \text{diam} \, H_i' + \text{diam} \, H''_i < \varepsilon/4 \), and \( H_0 \) and \( H_1 \) satisfy Lemma 3.

Step 2. Construction of a fix-finite homotopy between two fix-finite simplicial maps.

The aim of Step 2 is the construction of a fix-finite homotopy between the fix-finite and simplicial maps \( g_i \) of Lemma 3. It will be achieved with the help of a succession of Hopf constructions for
homotopies. For this purpose, we need to realise $|K| \times I$ as a suitable simplicial complex $P$. If $K'$, $K''$ and $K'''$ are the complexes obtained in Lemma 3, then we require that $P$ is a simplicial complex with $|P| = |K| \times I$ and satisfies the following two conditions:

(P1) $K'' \times \{0\}$ and $K''' \times \{1\}$ are subcomplexes of $P$,

(P2) if $\tau \in |P|$ is a simplex and $\pi: |P| \to |K|$ the first projection, then $\pi(\tau) \subset \rho$, where $\rho$ is a simplex of $K'$.

$P$ can easily be obtained by starting with the complex usually associated with the polyhedron $|K'| \times I$ and then refining it modulo the complements of the simplicial neighborhoods of those simplexes in $K' \times \{0\}$ and $K' \times \{1\}$ which are subdivided in $K''$ resp. $K'''$.

We state one more technical detail as a lemma.

**Lemma 4.** Let $P'$ be a refinement of $P$, let $G_s: |P'| \to |K'|$ be a simplicial map, and $\tau \in |P'|$ so that $\tau \cap \text{Fix} G_s \neq \emptyset$. If $\tau$ is neither maximal nor a hyperface in $|P'|$, then $G_s(\tau)$ is not maximal in $|K'|$.

**Proof.** Let $G_s(\tau) = \sigma$, where $\sigma$ is a simplex of $|K'|$, and $\pi(\tau) \subset \rho$, where $\rho \in |K'|$. As $\tau \cap \text{Fix} G_s \neq \emptyset$ implies $\pi(\tau) \cap \sigma \neq \emptyset$, we have $\rho = \sigma$, and $\dim \rho \leq \dim \tau$. By assumption there exists a simplex $\tau^* \in |P'|$ with $\tau < \tau^*$ and $\dim \tau \leq \dim \tau^* - 2$, therefore

$$\dim \rho + 1 \leq \dim \tau^* - 1 \leq \dim \pi(\tau^*),$$

so $\pi(\tau^*) \not\supset \rho$. But $\pi(\tau) \subset \rho$ implies $\pi(\tau^*) \cap \rho \neq \emptyset$, hence $\rho$ cannot be maximal in $|K'|$. As $\rho = \sigma$, $G_s(\tau)$ cannot be maximal either.

The next lemma contains the result of Step 2.

**Lemma 5.** Let $K'$, $K''$ and $g: |K''| \to |K'|$ be as in Lemma 3. If $g_0$ and $g_1$ are related by a homotopy $G$, then there exists a homotopy $G'$ relating them such that

(i) $G'$ is fix-finite and has all its fixed points located in maximal simplexes or hyperfaces of $|K|$, 

(ii) $d(G, G') < \varepsilon/4$.

**Proof.** Again we can assume that $|K|$ is connected. Let $P$ satisfy (P1) and (P2). We first select as a simplicial approximation of $G$ a simplicial map $G_s: |P'| \to |K'|$, where $P'$ is a refinement of $P$ obtained by a finite number of subdivisions modulo $(K'' \times \{0\}) \cup (K'' \times \{1\})$, so that $G_s$ satisfies $G_s = G$ on $(|K''| \times \{0\}) \cup (|K''| \times \{1\})$ and $d(G, G_s) < \mu(K')$. The existence of $G_s$ follows from [3], p. 55.

If $\bar{x}_0 = (x_0, t_0)$ is a vertex of $|P'|$ with $G_s(\bar{x}_0, t_0) = x_0$, then $x_0$ is a vertex of $|K'|$ and hence not maximal. Lemma 1 allows us to
make a Hopf construction which results in a simplicial map $G'_s:\ |P''| \to \l[K'\r]$, where $P''$ refines $P'$, for which $G'_s(x_0, t_0) \neq x_0$ and $G'_s = G_s$ on $|P''| \setminus \{\bar{x}_0\}$. Hence any vertex $\bar{x} \in |P''| \cap \text{Fix} G'_s$ must also be a vertex of $|P\setminus\{\bar{x}_0\}|$. We can therefore make further Hopf constructions for all such vertices until we arrive at a simplicial map, denoted again by $G'_s:\ |P''| \to \l[K'\r]$, where $P''$ refines $P'$, which is fixed point free on all vertices of $|P''|$. As $G_s$ is fixed point free on the vertices of $\{(K'_s \times \{0\}) \cup (K' \times \{1\})\}$, we have $G'_s = G_s$ on this subcomplex.

Next we carry out a succession of Hopf constructions for all one-dimensional simplexes $\tau \in |P''|$ for which $\tau \cap \text{Fix} G'_s \neq \emptyset$ and $G'_s(\tau)$ is not maximal in $|K'|$, then for all two-dimensional simplexes with the same property, and so on. According to (P2) and Lemmas 1 and 4 we can continue until we arrive at a simplicial map $G'_s:\ |P''| \to |K'|$, which equals $G_s$ on the subpolyhedron $\{(K'_s \times \{0\}) \cup (K' \times \{1\})\}$ of $|P''|$ and is fixed point free on all simplexes of $|P''|$ which are neither maximal nor hyperfaces.

If $\tau$ is a hyperface of $|P''|$ for which $\tau \cap \text{Fix} G'_s \neq \emptyset$, then it follows (as in [1], pp. 118–119) from the fact that $G'_s$ is linear on $\bar{\tau}$ and that $\text{Bd} \ \tau \cap \text{Fix} G'_s = \emptyset$ that $G'_s$ has at most one fixed point on $\tau$. Now consider a maximal simplex $\tau \in |P''|$ with $\tau \cap \text{Fix} G'_s \neq \emptyset$. Then $\text{Bd} \ \tau \cap \text{Fix} G'_s$ is empty or a finite set $\{\bar{x}_j\}$. Let $\bar{x}_j = (x_j, t_j)$, and select $\bar{x}_0 = (x_0, t_0) \in \tau$ so that $t_0 \neq t_j$ for all $t_j$. For any $\bar{x} = (x, t) \in \bar{\tau}\setminus\{\bar{x}_0\}$, let $\bar{y} = (y, u)$ be the point in which the ray from $\bar{x}_0$ to $\bar{x}$ intersects $\text{Bd} \ \tau$, and modify $G'_s$ on $\bar{\tau}$ to $G'$ by defining $G'(x, t)$ as the point in $\bar{\sigma} = G'_s(\bar{\tau})$ with

$$x, G'(x, t) = x, x + \lambda y G'_s(y, u),$$

where $\lambda = (x_0, \bar{x}) / (x_0, \bar{y})$. As $\pi(\bar{\tau}) \subset \bar{\sigma}$ and $\bar{\sigma}$ is convex, this yields a point $G'(x, t) \in \bar{\sigma}$. Also let $G'(x_0, t_0) = x_0$. Then $\bar{\tau} \cap \text{Fix} G'$ consists of the union of the segments from $\bar{x}_0$ to all the $\bar{x}_j$ if $\text{Bd} \ \tau \cap \text{Fix} G' \neq \emptyset$, and otherwise of the point $\bar{x}_0$ alone. If we carry out this construction on all maximal simplexes of $|P''|$ on which $G'_s$ has fixed points, we obtain a fix-finite homotopy $G' : |P''| \to |K'|$, where $P''$ refines $P'$ and hence $P$. By construction $G'(x, 0) = g_s(x)$ and $G'(x, 1) = g_s(x)$ for all $x \in |K|$. If $\bar{x} = (x, t) \in \text{Fix} G'$, then $\bar{x}$ is contained in a maximal simplex or hyperface of $|P''|$ and hence of $|P|$. It follows from (P2) that $x$ is contained in a maximal simplex or hyperface of $|K'|$ and hence of $|K|$. Each point $\bar{x} \in |P|$ is moved during the succession of Hopf
constructions at most \( n \) times, where again \( n \) is the dimension of \(|K|\), and by a distance of at most \( 2\mu(K') \) on each move. During the last change of \( G' \) to \( G' \) it is moved by a distance of at most \( \mu(K) \). So we have

\[
d(G_s, G') \leq (2n + 1)\mu(K'),
\]

and hence, according to (4) of Lemma 3,

\[
d(G, G') \leq 2(n + 1)\mu(K') < \varepsilon/4.
\]

We see that \( G' \) satisfies Lemma 5.

**Step 3.** Construction of a fix-finite homotopy between the given maps.

It remains to paste the constructed homotopies together in a suitable way to find a homotopy \( F'' \) satisfying Theorem 2. Given \( F: |K| \times I \rightarrow |K| \) as in Theorem 2 and \( \varepsilon > 0 \), we can choose \( \delta \) with \( 0 < \delta < 1 \) so that \( d(F(x, t), F(x, t')) < \varepsilon/4 \) for all \( x \in |K| \) and \( t, t' \in I \) with \( |t - t'| < \delta \). Use the homotopies \( H_0, H_1 \) obtained in Lemma 3 and define \( F'': |K| \times I \rightarrow |K| \) as a homotopy which equals \( H_0H_0^{-1}FH_1H_1^{-1} \) apart from a scale change by

\[
F''(x, t) = \begin{cases} 
H_0(x, 2t/\delta) & \text{if } 0 \leq t \leq \delta/2, \\
H_0(x, 2(1 - t/\delta)) & \text{if } \delta/2 \leq t \leq \delta, \\
F(x, (t - \delta)/(1 - 2\delta)) & \text{if } \delta \leq t \leq 1 - \delta, \\
H_1(x, \delta(t + \delta - 1)/2) & \text{if } 1 - \delta \leq t \leq 1 - \delta/2, \\
H_1(x, \delta(1 - t)/2) & \text{if } 1 - \delta/2 \leq t \leq 1.
\end{cases}
\]

Then \( d(F, F'') < \varepsilon/2 \).

The homotopy \( G: |K| \times I \rightarrow |K| \) defined by \( G(x, t) = F''(x, t(1 - \delta) + \delta/2) \) for all \( (x, t) \in |K| \times I \) equals \( H_0^{-1}FH_1 \) apart from a scale change and is hence a homotopy from \( g_0 \) to \( g_1 \). Replace it by a homotopy \( G' \) according to Lemma 5, and define \( F': |K| \times I \rightarrow |K| \) by

\[
F'(x, t) = \begin{cases} 
H_0(x, 2t/\delta) & \text{if } 0 \leq t \leq \delta/2, \\
G'(x, (t - \delta/2)/(1 - \delta)) & \text{if } \delta/2 \leq t \leq 1 - \delta/2, \\
H_1(x, \delta(1 - t)/2) & \text{if } 1 - \delta/2 \leq t \leq 1.
\end{cases}
\]

It is easy to check that \( F' \) is a homotopy satisfying Theorem 2.

**D.** Some properties of the fix-finite homotopy. The proof of Theorem 2 allows an easy description of \( \text{Fix } F' \).

**Proposition 1.** The homotopy \( F' \) in Theorem 2 can be chosen
so that $\text{Fix } F'$ is a one-dimensional finite polyhedron in $|K| \times I$ without horizontal edges.

Here a horizontal edge means an edge contained in a section $|K| \times \{t\}$, for some $t \in I$. Note that $\text{Fix } F'$, though constructed as a polyhedron, was not constructed as a subpolyhedron of $|P|$, and its projection $\pi(\text{Fix } F')$ is not a subpolyhedron of $|K|$.

As $\text{Fix } F'$ has a simple structure, it has simple properties. We collect a few. The first two are immediate consequences of the homotopy and additivity axioms of the fixed point index $i(f, x)$ of the selfmap $f$ of a polyhedron at the isolated fixed point $x$.

**Proposition 2.** Let $e$ be an edge of $\text{Fix } F'$. Then the index of $f'$ along $e$ is constant, i.e.,

$$i(f', x) = i(f', y) \quad \text{if} \quad (x, t) \in e \quad \text{and} \quad (y, s) \in e.$$

**Proposition 3.** Let $v = (x, t)$ be a vertex of $\text{Fix } F'$. Then the index of $f'$ at $x$ is the sum of the indices of fixed points chosen on all edges of $\text{Fix } F'$ either leading towards $v$ or away from $v$, i.e.,

$$i(f', x) = \sum i(f'_{t_k}, x_k),$$

where all $(x_k, t_k)$ lie on edges $e_k \in \text{st } v$, with $e_k$ distinct, and the sum taken over all edges in $\text{st } v \cap (|K| \times [0, t])$ (resp. in $\text{st } v \cap (|K| \times (t, 1])$).

Finally we note that $F'$ is "uniformly" fix-finite.

**Proposition 4.** There exists a positive integer $M$ so that the number of fixed points of $f'$ is $\leq M$ for all $t \in I$.

**Proof.** It suffices to choose $M$ as the number of edges in $\text{Fix } F'$, as no section $|K| \times \{t\}$ can intersect the closure of an edge of $\text{Fix } F'$ more than once.

E. Conclusion. For a single selfmap $f$ of a polyhedron $|K|$ the construction of a fix-finite map which is arbitrarily close to $f$ and has all its fixed points contained in maximal simplices is only a first step on the road to the construction of a map homotopic to $f$ which has a minimal number of fixed points. It is, in fact, possible to obtain a map $g$ homotopic to $f$ which has exactly $N(f)$ fixed points, where $N(f)$ is the Nielsen number of $f$, as long as $|K|$ satisfies the Shi condition, which is a somewhat stronger connectedness condition. (See [5] or [1], p. 140.) Hence a similar
question arises for homotopies.

Problem. If \( f_0 \) and \( f_1 \) are two self maps of a polyhedron \(|K|\) which satisfies the Shi condition, if \( f_0 \) and \( f_1 \) are homotopic and have each exactly \( N(f_0) \) fixed points, does there exist a homotopy \( F \) from \( f_0 \) to \( f_1 \) so that, for every \( t \in I \), the map \( f_t = F(\cdot, t) \) has exactly \( N(f_0) \) fixed points?

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Received August 23, 1978 and in revised form January 14, 1979. This research was partially supported by the National Research Council of Canada (Grant A-7579).