A GEOMETRIC INEQUALITY WITH APPLICATIONS TO LINEAR FORMS

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Let $C_N$ be a cube of volume one centered at the origin in $R^N$ and let $P_K$ be a $K$-dimensional subspace of $R^N$. We prove that $C_N \cap P_K$ has $K$-dimensional volume greater than or equal to one. As an application of this inequality we obtain a precise version of Minkowski's linear forms theorem. We also state a conjecture which would allow our method to be generalized.

1. Introduction. Let $C_N = [-1/2, 1/2]^N$ be the $N$-dimensional cube of volume one centered at the origin in $R^N$ and suppose that $P_K$ is a $K$-dimensional linear subspace of $R^N$. Dr. Anton Good has conjectured that the $K$-dimensional volume of $P_K \cap C_N$ is always greater than or equal to one. In case $K = N - 1$ this has recently been proved by Hensley [6], who also obtained upper bounds for this volume. Our purpose in this paper is to prove the conjecture for arbitrary $K$ and to give some applications to Minkowski's theorem on linear forms. In fact we prove a more general inequality for the product of spheres of various dimensions which contains the conjecture as a special case.

We write $\bar{x}$ for the column vector $\left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right)$ in $R^n$ and

$$ |\bar{x}| = \left( \sum_{j=1}^{n} (x_j)^2 \right)^{1/2} $$

for its length. We define the sphere $S_n$ by

$$ S_n = \{ \bar{x} \in R^n : |\bar{x}| \leq \rho_n \} $$

where $\rho_n = \pi^{-1/2}(\Gamma(n/2 + 1))^{1/n}$. It follows that $\mu_n(S_n) = 1$ where $\mu_n$ is Lebesgue measure on $R^n$. Also we let $\chi_U(\bar{x})$ denote the characteristic function of a subset $U$ in $R^n$.

Our first main result is contained in the following theorem.

**Theorem 1.** Suppose that $n_1, n_2, \ldots, n_J$ are positive integers, $Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_J}$ is in $R^N$, $N = n_1 + n_2 + \cdots + n_J$, and $A$ is a real $N \times K$ matrix, $\text{rank}(A) = K$. Then

$$ |\det A^T A|^{-1/2} \leq \int_{R^K} \chi_{Q_N}(A\bar{x}) d\mu_K(\bar{x}) , $$

where $A^T$ is the transpose of $A$. 543
We note that if \( \text{rank}(A) < K \) then each side of (1.1) is infinite. From Theorem 1 we easily deduce a lower bound for \( \mu_K(Q_N \cap P_K) \).

**Corollary.** Let \( Q_N \) be as in Theorem 1 and let \( P_K \) be a \( K \)-dimensional subspace of \( \mathbb{R}^N \). Then \( \mu_K(Q_N \cap P_K) \geq 1 \).

**Proof.** Choose \( A \) in Theorem 1 so that the columns of \( A \) form an orthonormal basis for \( P_K \) in \( \mathbb{R}^N \). Then the left hand side of (1.1) is 1 while the right hand side is \( \mu_K(Q_N \cap P_K) \).

The corollary clearly contains Good’s conjecture since \( Q_N = C_N \) if \( n_0 = 1 \) and \( J = N \).

Next we suppose that \( L_j(\bar{x}) \), \( j = 1, 2, \cdots, N \) are \( N \) linear forms in \( K \) variables,

\[
L_j(\bar{x}) = \sum_{k=1}^{K} a_{jk} x_k ,
\]

so that \( A = (a_{jk}) \) is an \( N \times K \) matrix. We assume that the forms \( L_j \) are real for \( j = 1, 2, \cdots, r \) and that the remaining forms consist of \( s \) pairs of complex conjugate forms arranged so that \( L_{r+2j-1} = \overline{L_{r+2j}} \) for \( j = 1, 2, \cdots, s \). Thus \( N = r + 2s \). Let \( \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N \) be positive with \( \varepsilon_r+2j-1 = \varepsilon_{r+2j} \) for \( j = 1, 2, \cdots, s \). We define the \( N \times N \) diagonal matrix \( E \) by \( E = (c_j \delta_{jk}) \) where \( c_j = \varepsilon_j^{-1} \) if \( j = 1, 2, \cdots, r \), \( c_j = (2/\pi)^{1/2} \varepsilon_j^{-1} \) if \( j = r + 1, r + 2, \cdots, N \) and \( \delta_{jk} \) is the Kronecker delta. Theorem 1 allows us to prove the following precise version of Minkowski’s classical result on linear forms.

**Theorem 2.** Let \( M \) be a positive integer and suppose that

\[
M |\det A^* E^2 A|^{1/2} \leq 1 ,
\]

where \( A^* \) is the complex conjugate transpose of the matrix \( A \). Then there exist at least \( M \) distinct pairs of nonzero lattice points \( \pm \bar{v}_m \), \( m = 1, 2, \cdots, M \), such that

\[
|L_j(\pm \bar{v}_m)| \leq \varepsilon_j
\]

for each \( j \) and each \( m \). In particular if \( |\det A^* A| > 0 \) then there exists a pair of nonzero lattice points \( \pm \bar{v} \) such that

\[
|L_j(\pm \bar{v})| \leq |\det A^* A|^{1/2K}
\]

for \( j = 1, 2, \cdots, r \), and

\[
|L_j(\pm \bar{v})| \leq \left( \frac{2}{\pi} \right)^{1/2} |\det A^* A|^{1/2K}
\]

for \( j = r + 1, r + 2, \cdots, N \).
Theorem 2 was first proved in the case $N \leq K$ and $M = 1$ by Minkowski [8, p. 104]. Subsequently the extension of Minkowski's convex body theorem by van der Corput [5] allowed Theorem 2 to be proved for $N \leq K$ and arbitrary $M$. Of course if $N = K$ then (1.2) becomes the more familiar condition

$$M \left( \frac{2}{\pi} \right)^k | \det A | \leq \varepsilon_1 \varepsilon_2 \cdots \varepsilon_N ,$$

and if $N < K$ then (1.2) is trivially satisfied since the left hand side is zero. The novelty in our result is that Theorem 2 now holds for $1 \leq K < N$. Previously in the case $1 \leq K < N$ we knew only that (1.3) held if

$$2^k M \leq \mu_K(\{ \bar{x} \in \mathbb{R}^K : | L_j(\bar{x}) | \leq \varepsilon_j, j = 1, 2, \cdots, N \}) .$$

We prove Theorem 2 by showing that the right hand side of (1.6) is bounded from below by $2^k | \det A^* E^2 A |^{-1/2}$. As will be clear from the proof, Theorem 2 could be generalized to include linear forms with values in $\mathbb{R}^n$ for various $n$.

In §5 we state a conjecture which would allow us to obtain a significant improvement in Theorem 1. Specifically, we deduce from this conjecture an analogue of Theorem 1 in which $Q_N$ is replaced by an arbitrary closed, convex, symmetric subset of $\mathbb{R}^n$ having $N$-dimensional volume equal to one.

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2. Preliminary results. In this section we briefly summarize some facts about logarithmically concave measures and functions. A more detailed discussion can be found in the papers of Kanter [7] and Prékopa [9].

A function $f: \mathbb{R}^n \to [0, \infty)$ is said to be log-concave if for every pair of vectors $\bar{x}_1, \bar{x}_2$ in $\mathbb{R}^n$ and every $\lambda$, $0 < \lambda < 1$, we have

$$f(\lambda \bar{x}_1 + (1 - \lambda) \bar{x}_2) \geq (f(\bar{x}_1))^\lambda (f(\bar{x}_2))^{1-\lambda} .$$

A probability measure $\nu$ defined on the measurable subsets of $\mathbb{R}^n$ is log-concave if for every pair of open convex sets $U_1$ and $U_2$ in $\mathbb{R}^n$ and every $\lambda$, $0 < \lambda < 1$, we have

$$\nu(\lambda U_1 + (1 - \lambda) U_2) \geq (\nu(U_1))^\lambda (\nu(U_2))^{1-\lambda} ,$$

where $+$ on the left hand side of (2.1) indicates Minkowski addition of sets. Clearly (2.1) holds for all open convex sets $U_1$ and $U_2$ if and only if it holds for all closed convex sets $U_1$ and $U_2$. The relationship
between log-concave measures and log-concave functions is contained in the following lemma.

**Lemma 3.** Let \( \nu \) be a log-concave probability measure on \( \mathbb{R}^n \) and suppose that the support of \( \nu \) spans the \( k \)-dimensional subspace \( P_k \) in \( \mathbb{R}^n \). Then there is a log-concave probability density function \( f \) defined on \( P_k \) such that \( d\nu = f \, d\mu_k \), where \( \mu_k \) is \( k \)-dimensional Lebesgue measure on \( P_k \). Conversely for any log-concave probability density function \( f \) defined on a \( k \)-dimensional subspace \( P_k \) in \( \mathbb{R}^n \), the probability measure defined by \( d\nu = f \, d\mu_k \) is log-concave, where \( \mu_k \) is Lebesgue measure on \( P_k \).

The first part of Lemma 3 is a result of Borell [2, p. 123] while the converse was proved by Prékopa [9], (see also Kanter [7, Lemma 2.1]).

Let \( \nu_1 \) and \( \nu_2 \) be probability measures on \( \mathbb{R}^n \). We say that \( \nu_2 \) is more peaked than \( \nu_1 \) if

\[
\nu_1(U) \leq \nu_2(U)
\]

for all closed, convex, symmetric subsets \( U \) in \( \mathbb{R}^n \). (We recall that \( U \subseteq \mathbb{R}^n \) is symmetric if \( U = -U \).) If \( f_1 \) and \( f_2 \) are probability density functions on \( \mathbb{R}^n \) we say that \( f_2 \) is more peaked than \( f_1 \) if the measure \( f_2 \, d\mu_n \) is more peaked than the measure \( f_1 \, d\mu_n \). The notion of peakedness was introduced by Birnbaum [1] and Sherman [10]. A complementary relation is that of symmetric dominance in the sense of Kanter [7]. If \( \nu_3 \) and \( \nu_4 \) are measures on \( \mathbb{R}^n \) then \( \nu_3 \) symmetrically dominates \( \nu_4 \) if

\[
\nu_3(\mathbb{R}^n \setminus U) \geq \nu_4(\mathbb{R}^n \setminus U)
\]

for all closed, convex, symmetric subsets \( U \) in \( \mathbb{R}^n \). It is clear that if \( \nu_3 \) and \( \nu_4 \) are both probability measures then \( \nu_3 \) symmetrically dominates \( \nu_4 \) if and only if \( \nu_4 \) is more peaked than \( \nu_3 \). For our purposes it is more convenient to work with the relation of peakedness.

If \( \nu_1 \) and \( \nu_2 \) are log-concave probability measures on \( \mathbb{R}^n \) then the convolution \( \nu_1 \ast \nu_2 \) is also log-concave on \( \mathbb{R}^n \) (Kanter [7, Lemma 2.3]). It follows that if \( \nu_1 \) and \( \nu_2 \) are log-concave probability measures on \( \mathbb{R}^{n_1} \) and \( \mathbb{R}^{n_2} \) respectively then the product measure \( \nu_1 \times \nu_2 \) is log-concave on \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \). Forming product measures also preserves the peakedness relation.

**Lemma 4.** Suppose that \( \nu_1, \nu_2, \nu'_1 \) and \( \nu'_2 \) are all log-concave probability measures such that \( \nu_1 \) is more peaked than \( \nu'_1 \) on \( \mathbb{R}^{n_1} \) and
\(v_2\) is more peaked than \(v'_2\) on \(R^{n_2}\). Then \(v_1 \times v_2\) is more peaked than \(v'_1 \times v'_2\) on \(R^{n_1} \times R^{n_2}\).

For the proof of Lemma 4 we refer to Kanter [7, Corollary 3.2] where the result is obtained for the more general class of unimodal measures.

3. Proof of Theorem 1. We begin by proving the following lemma.

**Lemma 5.** Suppose that \(n_1, n_2, \ldots, n_J\) are positive integers and 
\(Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_J}\) is in \(R^N\), \(N = n_1 + n_2 + \cdots + n_J\). Then \(\mathcal{L}_{Q_N}(\bar{x})\) is more peaked than the normal density function \(\exp\{-\pi |\bar{x}|^2\}\) on \(R^N\).

**Proof.** Since the measures \(\mathcal{L}_{Q_N}(\bar{x})d\mu_N(\bar{x})\) and \(\exp\{-\pi |\bar{x}|^2\}d\mu_N(\bar{x})\) are both product measures which factor in \(R^{n_1} \times R^{n_2} \times \cdots \times R^{n_J}\) it suffices to prove the peakedness relation in each factor space and then apply Lemma 4. Thus we need only show that for each positive integer \(n\), \(\mathcal{L}_{S_n}(\bar{x})\) is more peaked than \(\exp\{-\pi |\bar{x}|^2\}\) on \(R^n\). Of course it is trivial to verify that both of the density functions \(\mathcal{L}_{S_n}(\bar{x})\) and \(\exp\{-\pi |\bar{x}|^2\}\) are log-concave on \(R^n\).

Let \(\Sigma_{n-1} = \{\bar{x} \in R^n : |\bar{x}| = 1\}\) so that for each \(\bar{x} \neq 0\) in \(R^n\) we have the unique polar decomposition \(\bar{x} = r \bar{x}'\) where \(r = |\bar{x}|\) and \(\bar{x}' \in \Sigma_{n-1}\). If \(U\) is a closed, convex, symmetric subset of \(R^n\) then it follows that

\[
(3.1) \quad \int_U \exp\{-\pi |\bar{x}'|^2\}d\mu_n(\bar{x}) = \int_{\Sigma_{n-1}} \int_0^\infty \mathcal{L}_U(r\bar{x}') \exp\{-\pi r^2\}r^{n-1}drd\bar{x}',
\]

where \(d\bar{x}'\) is the induced Lebesgue measure on \(\Sigma_{n-1}\). Now for each fixed \(\bar{x}' \in \Sigma_{n-1}\) we have either

\[
(3.2) \quad \mathcal{L}_U(r\bar{x}') \leq \mathcal{L}_{S_n}(r\bar{x}') , \quad 0 \leq r < \infty
\]

or

\[
(3.3) \quad \mathcal{L}_{S_n}(r\bar{x}') \leq \mathcal{L}_U(r\bar{x}') , \quad 0 \leq r < \infty,
\]

since \(S_n\) and \(U\) are convex. If (3.2) holds at \(\bar{x}'\) then

\[
(3.4) \quad \int_0^\infty \mathcal{L}_U(r\bar{x}') \exp\{-\pi r^2\}r^{n-1}dr \leq \int_0^\infty \mathcal{L}_U(r\bar{x}') \mathcal{L}_{S_n}(r\bar{x}')r^{n-1}dr.
\]

If (3.3) holds at \(\bar{x}'\) then
\[
\int_{0}^{\infty} \chi_{U}(r \bar{x}') \exp \{-\pi r^2\} r^{n-1} dr = \int_{0}^{\infty} \chi_{S_n}(r \bar{x}') r^{n-1} dr = \int_{0}^{\infty} \chi_{U}(r \bar{x}') \chi_{S_n}(r \bar{x}') r^{n-1} dr.
\]

Combining (3.1), (3.4) and (3.5) we obtain

\[
\int_{U} \exp\{-\pi |\bar{x}|^2\} d\mu_n(\bar{x}) \leq \sum_{n=1}^{\infty} \int_{0}^{\infty} \chi_{U}(r \bar{x}') \chi_{S_n}(r \bar{x}') r^{n-1} dr d\bar{x}' = \int_{U} \chi_{S_n}(\bar{x}) d\mu_n(\bar{x}).
\]

Thus \(\chi_{S_n}(\bar{x})\) is more peaked than \(\exp\{-\pi |\bar{x}|^2\}\) on \(\mathbb{R}^n\) and the lemma is proved.

We now prove Theorem 1. If \(N = K\) then (1.1) is trivial so we may suppose that \(K' = N - K\) is positive. Let \(P_K\) be the \(K\)-dimensional subspace of \(\mathbb{R}^N\) spanned by the columns of \(A\). Next let \(W\) be an \(N \times N\) matrix whose first \(K\) columns are the columns of \(A\) and whose next \(K'\) columns are the columns of an \(N \times K'\) matrix \(B\). We choose the columns of \(B\) so that they form an orthonormal basis in \(\mathbb{R}^N\) of the \(K'\)-dimensional subspace which is orthogonal to \(P_K\). Identifying \(\mathbb{R}^N\) with \(\mathbb{R}^K \times \mathbb{R}^{K'}\) we may write each \(\bar{z} \in \mathbb{R}^N\) as \(\bar{z} = (\bar{x}/\bar{y})\) where \(\bar{x} \in \mathbb{R}^K\) and \(\bar{y} \in \mathbb{R}^{K'}\). For each \(\varepsilon, 0 < \varepsilon \leq 1\) we define

\[
H_{\varepsilon} = \{\bar{z} \in \mathbb{R}^N: z = (\bar{x}/\bar{y}), \max_{1 \leq j \leq K} |y_j| \leq \frac{\varepsilon}{2}\}
\]

and

\[
H_{\varepsilon}' = \{\bar{y} \in \mathbb{R}^{K'}: \max_{1 \leq j \leq K'} |y_j| \leq \frac{\varepsilon}{2}\}.
\]

Clearly \(H_{\varepsilon}\) is a closed, convex, symmetric subset of \(\mathbb{R}^N\) and so is the image of \(H_{\varepsilon}\) under the nonsingular linear transformation determined by \(W\). Thus by Lemma 5,

\[
\int_{H_{\varepsilon}} \exp\{-\pi |W \bar{z}|^2\} d\mu_N(\bar{z}) \leq \int_{H_{\varepsilon}} \chi_{Q_N}(W \bar{z}) d\mu_N(\bar{z}).
\]

Multiplying each side of (3.6) by \(\{\mu_{K'}(H_{\varepsilon}')\}^{-1} = \varepsilon^{-K'}\) and factoring \(H_{\varepsilon}\) into \(\mathbb{R}^K \times H_{\varepsilon}'\) we find that

\[
\varepsilon^{-K'} \int_{\mathbb{R}^K} \int_{H_{\varepsilon}'} \exp\{-\pi |A \bar{x} + B \bar{y}|^2\} d\mu_{K'}(\bar{y}) d\mu_K(\bar{x}) \leq \varepsilon^{-K'} \int_{\mathbb{R}^K} \int_{H_{\varepsilon}'} \chi_{Q_N}(A \bar{x} + B \bar{y}) d\mu_{K'}(\bar{y}) d\mu_K(\bar{x}).
\]
By the orthogonality condition $|A\bar{x} + B\bar{y}|^2 = |A\bar{x}|^2 + |B\bar{y}|^2$ and so as $\varepsilon \to 0^+$ the left hand side of (3.7) clearly converges to
$$
\int_{R^K} \exp(-\pi |A\bar{x}|^2) d\mu_K(\bar{x}) = |\det A^T A|^{-1/2}.
$$

To evaluate the corresponding limit on the right hand side of (3.7) we observe that for $0 < \varepsilon \leq 1$ and each $\bar{x} \in R^K$,
$$
\varepsilon^{-K'} \int_{H'_i} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) \leq 1.
$$

Since $Q_N$ and $H'_i$ are both bounded we have
$$
\varepsilon^{-K'} \int_{H'_i} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) = 0
$$
for sufficiently large $|\bar{x}|$ independent of $\varepsilon$. Thus by dominated convergence the limit on the right of (3.7) as $\varepsilon \to 0^+$ is

$$
(3.8) \quad \int_{R^K} \left\{ \lim_{\varepsilon \to 0^+} \varepsilon^{-K'} \int_{H'_i} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) \right\} d\mu_K(\bar{x}).
$$

Clearly
$$
\lim_{\varepsilon \to 0^+} \varepsilon^{-K'} \int_{H'_i} \chi_{Q_N}(A\bar{x} + B\bar{y}) d\mu_{K'}(\bar{y}) = \chi_{Q_N}(A\bar{x})
$$
except possibly when $A\bar{x}$ is a boundary point of $Q_N \cap P_K$. Since this boundary has $K$-dimensional measure zero we see that (3.8) is equal to
$$
\int_{R^K} \chi_{Q_N}(A\bar{x}) d\mu_K(\bar{x}).
$$

We have now shown that as $\varepsilon \to 0^+$ on each side of (3.7) we obtain (1.1) and this proves the theorem.

**4. Proof of Theorem 2.** By van der Corput's extension of Minkowski's convex body theorem [5] (see also Cassels [4, Chapter III, Theorem II]) the condition (1.6) implies that there exist at least $M$ distinct pairs $\pm \bar{v}_m$, $m = 1, 2, \ldots, M$, of nonzero lattice points such that (1.3) holds. If $\text{rank}(A) < K$ then (1.2) and (1.6) are both trivially satisfied. Thus to establish the first part of Theorem 2 it suffices to show that if $\text{rank}(A) = K$ then

$$
(4.1) \quad 2^K |\det A^* E^2 A|^{-1/2} \leq \mu_K(\{\bar{x} \in R^K: |L_j(\bar{x})| \leq \varepsilon_j, j = 1, 2, \ldots, N\}).
$$

Let $G_j(\bar{x})$, $j = 1, 2, \ldots, N$ be linear forms defined by $G_j(\bar{x}) = L_j(\bar{x})$ for $j = 1, 2, \ldots, r$ and
\[ G_{r+2j-1}(\bar{x}) = \sqrt{2} \text{Re}\{L_{r+2j-1}(\bar{x})\}, \]
\[ G_{r+2j}(\bar{x}) = \sqrt{2} \text{Im}\{L_{r+2j-1}(\bar{x})\}. \]

for \( j = 1, 2, \ldots, s \). We write \( B = (b_{jk}) \) for the corresponding real \( N \times K \) matrix so that
\[ G_j(\bar{x}) = \sum_{k=1}^K b_{jk}x_k. \]

Next we let \( Q_N = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r+s} \) where \( n_j = 1 \) for \( j = 1, 2, \ldots, r \) and \( n_j = 2 \) for \( j = r+1, r+2, \ldots, r+s \). It follows that \( |L_j(\bar{x})| \leq \varepsilon_j \) if and only if \( 1/2\varepsilon_j^{-1}G_j(\bar{x}) \in S_{n_j}, j = 1, 2, \ldots, r \), and
\[ |L_{r+2j}(\bar{x})| = |L_{r+2j}(\bar{x})| \leq \varepsilon_{r+2j} \]
if and only if
\[ \left(2\pi\right)^{-1/2} \varepsilon_{r+2j}^{-1/2} \begin{pmatrix} G_{r+2j-1}(\bar{x}) \\ G_{r+2j}(\bar{x}) \end{pmatrix} \in S_{n_{r+j}}. \]

\( j = 1, 2, \ldots, s \). Therefore
\[ \mu_K(\{\bar{x} \in R^K : |L_j(\bar{x})| \leq \varepsilon_j, j = 1, 2, \ldots, N\}) = \mu_K(\{\bar{x} \in R^K : \frac{1}{2}EB\bar{x} \in Q_N\}) = \int_{R^K} \chi_{Q_N}(\frac{1}{2}EB\bar{x})d\mu_K(\bar{x}) \]
\[ \leq \left| \det \left( \frac{1}{2}EB \right)(\frac{1}{2}EB)^{-1/2} \right| = 2^K |\det B^TE^2B|^{-1/2}. \]

An easy computation shows that \( B^TE^2B = A^*E^2A \) and so completes the proof of (4.1).

To prove the second part of Theorem 2 we choose \( \varepsilon_j = |\det A^*A|^{1/2K} \) for \( j = 1, 2, \ldots, r \) and \( \varepsilon_j = (2/\pi)^{1/2} |\det A^*A|^{1/2K} \) for \( j = r+1, r+2, \ldots, N \). Then
\[ |\det A^*E^2A| = 1 \]
and so (1.4) and (1.5) follow from the first part of the theorem.

5. Lower bounds for arbitrary convex bodies. In this section we suppose that \( Q_N \) is a closed, convex, symmetric subset of \( R^N \) with \( \mu_N(Q_N) = 1 \). If \( A \) is an \( N \times K \) matrix, rank(\( A \)) = \( K \), we will be interested in the problem of finding a lower bound for
\[ \int_{R^K} \chi_{Q_N}(A\bar{x})d\mu_K(\bar{x}). \]

The method used to deduce Theorem 1 from Lemma 5 will also lead to a lower bound in this more general situation, provided that we
can find a suitable normal density function on $\mathbb{R}^N$ which is less peaked than $\mathcal{L}_{Q_N}(\bar{x})$. We succeeded in proving Lemma 5 because the special structure imposed on $Q_N$ allowed us to appeal to Lemma 4. We now describe an alternative method which leads to a conjectured lower bound for (5.1).

We write $Q$ for $Q_N$ and we assume that $Q$ is a fixed, closed, convex, symmetric subset of $\mathbb{R}^N$, $\mu_N(Q) = 1$. For each positive integer $m$ let

$$\mathcal{L}_Q^{(m)}(\bar{x}) = \mathcal{L}_Q \ast \mathcal{L}_Q \ast \cdots \mathcal{L}_Q(\bar{x})$$

be the $m$-fold convolution of $\mathcal{L}_Q$. We define the dilation operator $D_\lambda$ for $\lambda > 0$ and for integrable real valued functions $f$ on $\mathbb{R}^N$ by

$$D_\lambda(f)(\bar{x}) = \lambda^N f(\lambda \bar{x}) .$$

Next we define a sequence of positive numbers $\lambda_m$, $m = 1, 2, \cdots$ by

$$(\lambda_m)^n \mathcal{L}_Q^{(m)}(\bar{0}) = 1 .$$

With this notation we have the following

**Conjecture 6.** For each positive integer $m$, $\mathcal{L}_Q(\bar{x})$ is more peaked than $D_{\lambda_m}(\mathcal{L}_Q^{(m)})(\bar{x})$.

Now let $\Omega$ be the $N \times N$ covariance matrix determined by a random vector which is uniformly distributed on the convex body $Q$. That is $\Omega = (\omega_{rs})$ is the $N \times N$ matrix defined by

$$\omega_{rs} = \int_{\mathbb{R}^N} y_r y_s \mathcal{L}_Q(\tilde{y}) d \mu_N(\tilde{y}) ,$$

where $y_r$ and $y_s$ are the $r$th and $s$th co-ordinate functions of $\tilde{y}$, $r = 1, 2, \cdots, N$, and $s = 1, 2, \cdots, N$. It is clear that $\Omega$ is symmetric and nonsingular since $Q$ has a nonempty interior. By the Central Limit Theorem (Breiman [3, Theorem 11.10]) we have

$$\lim_{m \to \infty} D_{\lambda_m}(\mathcal{L}_Q^{(m)})(\bar{x}) = (2\pi)^{-N/2}(\det \Omega)^{-1/2} \exp \left\{ -\frac{1}{2} \bar{x}^T \Omega^{-1} \bar{x} \right\}$$

uniformly for $x \in \mathbb{R}^N$. It follows that

$$\lim_{m \to \infty} \frac{\lambda_m}{\sqrt{m}} = (2\pi)^{1/2}(\det \Omega)^{1/2N}$$

and hence

$$\lim_{m \to \infty} D_{\lambda_m}(\mathcal{L}_Q^{(m)})(\bar{x}) = \exp\{-\pi(\det \Omega)^{1/N} \bar{x}^T \Omega^{-1} \bar{x}\}$$

uniformly for $x \in \mathbb{R}^N$. If the Conjecture 6 is true then for each
Letting \( m \to \infty \) on the left hand side of (5.2) and we have proved that \( \mathcal{X}_Q(\mathbf{x}) \) is more peaked than \( \exp\{-\pi (\det \Omega)^{1/N} \mathbf{x}^T \Omega^{-1} \mathbf{x} \} \) on \( \mathbb{R}^N \). By the same method used to prove Theorem 1 we obtain

**Theorem 7.** Assume that the Conjecture 6 holds and let \( A \) be a real \( N \times K \) matrix, \( \text{rank}(A) = K \). Then

\[
(\det \Omega)^{-K/N} |\det A^T \Omega^{-1} A|^{-1/2} \leq \int_{R^K} \mathcal{X}_Q(A \mathbf{x}) d\mu_K(\mathbf{x}).
\]

If the set \( Q \) in Theorem 7 is such that \( \Omega \) is a constant multiple of the identity matrix then the left hand side of (5.3) is simply \( |\det A^T A|^{-1/2} \). Just as in our proof of the corollary to Theorem 1, we deduce that in this case \( \mu_K(Q \cap P_K) \geq 1 \), where \( P_K \) is a \( K \)-dimensional subspace of \( \mathbb{R}^N \). There is also an application of Theorem 7 to linear forms. If \( L_j(\mathbf{x}), j = 1, 2, \ldots, N \), are \( N \) linear forms in \( K \)-variables we could determine precise conditions under which

\[
\left( \sum_{j=1}^{N} |L_j(\mathbf{v})|^p \right)^{1/p} \leq \varepsilon
\]

at a nonzero lattice point \( \mathbf{v} \) for any \( p \geq 1 \) and \( \varepsilon > 0 \). At present, however, these results remain hypothetical since they depend on the open problem stated in Conjecture 6.

**References**

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