Pacific Journal of Mathematics

T AS AN $\mathcal{G}$ SUBMODULE OF G

WILLIAM JENNINGS WICKLESS

Vol. 83, No. 2 April 1979

PACIFIC JOURNAL OF MATHEMATICS
Vol. 83, No. 2, 1979

T AS AN $\mathscr{E}$ SUBMODULE OF G

W. J. WICKLESS

Let G be a mixed abelian group with torsion subgroup T. T is viewed as an $\mathscr{E}$ submodule of G, where $\mathscr{E} = \operatorname{End} G$. It is shown that T is superfluous in G if and only if, $\forall_p$, either T_p is divisible or G/T_p is not p divisible. If G is not reduced, T is essential in G if and only if T contains a $Z(p^\infty)$. Let $I(G)$ $[I(T)]$ be the $\mathscr{E}$ injective hull of G $[T]$. Then $I(G) = I(T) \oplus X$ with X torsion free divisible and T is a pure subgroup of $I(G)$. This can be used to obtain several results; for example, if $Q \not\subseteq I(T)$, TFAE: 1. $T \operatorname{ess} G$, 2. $I(G) \cong I(T)$ as abelian groups, 3. $Q \not\subseteq I(G)$. The condition $T \operatorname{ess} G$ is characterized if T is a summand or if G is algebraically compact. If T is bounded or if T is a p-group, $T^1 = (0)$ and G is reduced cotorsion, T is not essential. In fact, for bounded T there is an $\mathscr{E}$ isomorphism $I(G) \cong I(T) \oplus I(G/T)$. Some information is obtained on the p-basic subgroups of $I(T)$ as a function of those of T. A condition is given for $I(T) \supseteq \oplus_c Q$. These last theorems specialize to $I({}_E T)$, where $E = \operatorname{End} T$.

Preliminaries. In the last fifteen years several authors have written papers concerning an abelian group G viewed as a module over $\mathscr{E}$, its ring of endomorphisms.

Let G be a mixed abelian group with maximal torsion subgroup T. In this paper we consider T as an $\mathscr{E}$ submodule of G. We determine when T is superfluous in G and then study the more difficult question of determining when T is essential in G. (If $(0) \neq T \neq G$, it is easy to prove that T is neither essential nor superfluous as a Z submodule of G.)

The latter question leads to consideration of the injective hulls $I(T)$, $I(G)$—taken with respect to $\mathscr{E}$.

Our notation, with minor exceptions, is that of [1].

1. T as a superfluous submodule of G.

Henceforth, let G be a mixed abelian group, $T = t(G)$ its torsion subgroup and $\mathscr{E} = \operatorname{End} G$. To avoid stating the trivial cases of our results we always assume $(0) \neq T \neq G$. We begin by characterizing those mixed G for which ${}_{\mathscr{E}}T$ is superfluous in ${}_{\mathscr{E}}G$ $(T \ll G)$. In our context $T \ll G$ if and only if whenever K is a fully invariant subgroup of G with $K + T = G$, then $K = G$.

LEMMA 1. *Let $T = \oplus T_p$ be a decomposition of T into its p components. Then $T \ll G$ if and only if $T_p \ll G$, $\forall p$.*

Proof. The only if part of the implication is immediate since submodules of superfluous submodules are superfluous.

Suppose $T_p \ll G$, $\forall p$, and $T \not\ll G$. Then we must have $T + K = G$ for some fully invariant $K \neq G$. Clearly, $K \not\supseteq T_p$ for some p. Let $K' = K + \sum_{q \neq p} T_q$. Since K' is fully invariant with $K' + T_p = G$, $K' = G$.

Let $t \in T_p$ and suppose that t has order $o(t) = p^l$. Write $t = x + y$ with $x \in K$, $o(y) = n$, $(n, p) = 1$. If $a, b \in Z$ with $ap^l + bn = 1$, then $t = (ap^l + bn)t = bnt = bnx \in K$. Thus, $T_p \subseteq K$, a contradiction.

THEOREM 1. *$T \ll G$ if and only if, $\forall p$, either T_p is divisible or G/T_p is not p divisible.*

We prove the contrapositive in both directions.

Proof. Suppose $\exists p$ with T_p not divisible and G/T_p p divisible. Then $T_p \not\subseteq pG$ and $G = pG + T_p$. Thus, $T_p \not\ll G$ and, by Lemma 1, $T \not\ll G$.

Conversely, suppose $T \not\ll G$. Then $\exists p$ with $T_p \not\ll G$. Let K be a proper fully invariant subgroup with $K + T_p = G$. We cannot have T_p divisible, for then $K \supseteq \mathrm{Hom}(G, T_p)K = T_p$. (If $x \in K$, $o(x) = \infty$, and $t \in T_p$, the map $Zx \to Zt$ extends to G.)

G/T_p is p divisible if and only if $K \subseteq pG + T_p$. Assume that G/T_p is not p divisible. Then there is an $x \in K \backslash pG + T_p$. Therefore, $\forall t \in T_p$, the p-height of $x + t$ in G, $h_p^G(x + t)$, is zero.

Thus, for every positive integer l, $\bar{x} = x + p^l G$ must have order exactly p^l in $G/p^l G$. But then, $\forall t \in T_p$, we can construct an endomorphism of G mapping $x \to \bar{x} \to t$. This implies $K \supseteq T_p$, a contradiction. The theorem follows.

2. T as an essential submodule of G-basic results. We next consider the more difficult problem of deciding when ${}_{\mathscr{E}}T$ is essential in ${}_{\mathscr{E}}G(T \operatorname{ess} G)$. We first dispose of the nonreduced case.

THEOREM 2. *Let G be a nonreduced group. Then $T \operatorname{ess} G$ if and only if T contains a $Z(p^\infty)$.*

Proof. If $T \supseteq Z(p^\infty)$ then, $\forall x \in G$ with $o(x) = \infty$, $\exists \alpha \in \mathscr{E}$ with $0 \neq \alpha(x) \in Z(p^\infty)$. This, clearly, is enough to imply $T \operatorname{ess} G$.

Conversely, suppose T contains no $Z(p^\infty)$. Then, since G is not reduced, the maximum divisible subgroup D of G is nontrivial and torsion free. Hence $T \cap D = 0$, so T is not essential in G.

From now on we assume G is reduced.

To investigate the question of when $T \operatorname{ess} G$, it is natural to consider the $\mathscr{E}$ injective hulls. Let $I(G)$ be the injective hull of the module ${}_{\mathscr{E}}G$. Since ${}_{\mathscr{E}}T \leqq {}_{\mathscr{E}}G$ we can regard $I(T)$, the injective hull of ${}_{\mathscr{E}}T$, as a maximal $\mathscr{E}$ essential extension of T in $I(G)$. If $I(T)$ is constructed in this way we have an $\mathscr{E}$ decomposition: $I(G) = I(T) \oplus X$. Clearly, $T \operatorname{ess} G$ if and only if $X = (0)$.

THEOREM 3. *Let X be as above. Then X is torsion free divisible as an abelian group.*

Proof. If $t(X)$, the torsion subgroup of X, were nonzero, then $I(T) \oplus t(X)$ would be an $\mathscr{E}$ essential extension of T in $I(G)$ properly containing $I(T)$—a contradiction. Thus, X is torsion free. Since X is an injective module, X must also be divisible.

COROLLARY. *$T \operatorname{ess} G$ if and only if $I(T)$ and $I(G)$ are isomorphic $\mathscr{E}$ modules.*

Proof. Suppose $\theta\colon I(T) \to I(G)$ is an $\mathscr{E}$ isomorphism. Then $\theta(T) \operatorname{ess} I(G)$. By Theorem 3, $\theta(T) \cap X = (0)$. Thus, $X = (0)$ and $T \operatorname{ess} G$.

The next theorem is central for our results.

THEOREM 4. *T is a pure subgroup of $I(G)$ ($T \lhd I(G)$).*

Proof. Let $D(G)$ be the Z injective hull of G and let A be the injective left $\mathscr{E}$ module $\operatorname{Hom}_Z(\mathscr{E}, D(G))$. Regard $G \subseteqq A$ via $G \cong \operatorname{Hom}_{\mathscr{E}}(\mathscr{E}, G)$ and take $I(G)$ to be a maximal $\mathscr{E}$ essential extension of G in A. It suffices to show $T \lhd A$. Let $\delta \in T$ with $p\delta = 0$. Suppose $h_p^T(\delta) = m < \infty$, but $\delta = p^{m+1}\alpha$, $\alpha \in A$.

Write $\delta = p^m\delta'$, $\delta' \in T$. Then $T = \langle \delta' \rangle \oplus T'$ ([1], Corollary 27.2). Let $\pi \in \mathscr{E}$ be projection onto $\langle \delta' \rangle$. Then $\delta(\pi) = \pi(\delta) = \delta = p^{m+1}\alpha(\pi) = \alpha(p^{m+1}\pi) = 0$—a contradiction. Thus, we have proved: $\delta \in T[p] \to h_p^T(\delta) = h_p^A(\delta)$. This shows $T \lhd A$ ([1], (h), p. 114).

COROLLARY 1. *If T is a torsion group, $E = \operatorname{End} T$, then $T \lhd I({}_E T)$.*

This is proved by putting $G = T$ in the above.

COROLLARY 2. *Suppose $T \subset G$ with $T^1 = G^1$, G/T divisible. Then $T \operatorname{ess} G$. (Here T^1 $[G^1]$ denotes the first Ulm subgroup of T $[G]$.)*

Proof. Since $T \lhd I(G)$, G/T divisible, we have $G \lhd I(G)$. If

$G^1 = T^1$ and X is as in Theorem 3, $X \cap G = (0)$, so $X = (0)$. Thus, $T \operatorname{ess} G$.

COROLLARY 3. *Let $T \subset G$ with $T^1 = (0)$. Then $I(T)^1 = (0)$.*

Proof. $I(T)^1$ is an $\mathscr{E}$ submodule of $I(T)$. Since $T^1 = (0)$ and $T \lhd I(T)$, $I(T)^1 \cap T = (0)$. Thus, $I(T)^1 = (0)$.

THEOREM 5. *Let $T \subset G$ with $Q \not\subseteq I(T)$. Then TFAE*: 1. *$T \operatorname{ess} G$*; 2. *$I(T) \cong I(G)$ as abelian groups*; 3. *$Q \not\subseteq I(G)$. Moreover, if* 1—3 *hold, then $T^1 = G^1$.*

Proof. The implications $1 \rightarrow 2$, $2 \rightarrow 3$ are obvious. If $Q \not\subseteq I(G)$, then the X of Theorem 3 is zero, so $T \operatorname{ess} G$.

To prove the additional statement, note that $I(T)$ is an algebraically compact group ([1], p. 178) which, by assumption, contains no Q's. Thus, there can be no elements of infinite order in $I(T)^1$. If 1—3 hold, the same is true for $I(G)^1$. Thus, in this case, $G^1 = T^1$.

COROLLARY. *Let $T \subset G$ with $T^1 = (0)$. Then conditions* 1—3 *are equivalent. Moreover, if* 1—3 *hold, then $G^1 = (0)$.*

Proof. If $T^1 = (0)$, then $I(T)^1 = (0)$, so $Q \not\subseteq I(T)$.

Theorem 5 raises the questions: When are $I(T)$ and $I(G)$ isomorphic as abelian groups? Is this sufficient for $T \operatorname{ess} G$? Here is a partial result.

THEOREM 6. *Let $\bar{I}$ be the $\mathscr{E}$ injective hull of the factor module G/T. Write $I(T) = H \oplus K$, where H is the maximal torsion free divisible subgroup of $I(T)$. Let $r = \operatorname{rank} H$, $\bar{r} = \operatorname{rank} \bar{I}$. If r is infinite and $r \geqq \bar{r}$, then $I(G) \overset{+}{\cong} I(T)$.*

Proof. Embed $I(G)$ into $I(T) \oplus \bar{I}$ in the standard way (via $\alpha \oplus \beta$ where α and β are the extensions to $I(G)$ of $T \subset I(T)$ and $G \rightarrow G/T \subset \bar{I}$ respectively). Then, as $\mathscr{E}$ modules, $I(G) \oplus Y \cong I(T) \oplus \bar{I}$. Since $I(G) = I(T) \oplus X$, we have:

$$(*) \qquad I(T) \oplus X \oplus Y \cong I(T) \oplus \bar{I}\,.$$

The additive group of $\bar{I}$ is torsion free divisible, since $\bar{I}$ is the injective hull of a module whose additive group is torsion free. Thus, the number of Q's on the right-hand side of $(*)$ is $r + \bar{r} = r$, so $\operatorname{rank} X \leqq r$. But then, $I(G) = I(T) \oplus X \overset{+}{\cong} I(T)$.

EXAMPLE. For each prime p, let T_p be the group generated by $\{a_i \mid i = 0, 1, 2, 3, \cdots\}$ with relations $\{pa_0 = 0,\ p^n a_n = a_0,\ n = 1, 2, 3, \cdots\}$. Let $T = \bigoplus_p T_p$ and let $G = Q \oplus T$. Then $\bar{r} = 1$ and (as we will see in Theorem 13) $r \geqq c$. Thus, $I(G) \overset{+}{\cong} I(T)$. Since T is reduced, T is not essential in G.

3. T as an essential submodule of G—some special cases.

In this section we consider the essentiality of T in G in some special cases. First we consider the situation for bounded T. The following theorem shows if T is bounded, then T is never essential in G.

THEOREM 7. *Let $T \subset G$ with $nT = (0)$ and let $\bar{I} = I(G/T)$. Then:*
1. *$nI(T) = (0)$;*
2. *$I(G)$ is $\mathscr{E}$ isomorphic to $I(T) \oplus \bar{I}$.*

Proof. Let $D(G)$, $D(T)$, $D(G/T)$ be the Z injective hulls of G, T, G/T and let A, B, C be the injective left $\mathscr{E}$ modules $\mathrm{Hom}_Z(\mathscr{E}, D(M))$ where $M = G, T, G/T$, respectively. As in Theorem 4, regard $T \subseteqq G \subseteqq I(G) \subseteqq A$. Suppressing the obvious isomorphism, write $A = B \oplus C$—an $\mathscr{E}$ direct sum. Under these identifications $T = B \cap G$.

To prove (1), recall $T \lhd A$, so in this case, $T \cap nA = nT = (0)$. Thus, if $x \in I(T)$ with $nx \neq 0$, then, for some $\lambda \in \mathscr{E}$, $0 \neq \lambda(nx) \in T \cap nA$—a contradiction.

To prove (2), first note that $B \cap I(G)$ is an essential extension of $T = B \cap G$. Choose $I(T) \subseteqq I(G)$ as before—with the additional requirement $I(T) \supseteqq B \cap I(G)$.

Let $x \in I(T)$, say $x = b + c$, $b \in B$, $c \in C$. Since C is torsion free and $nx = 0$, we must have $c = 0$. Thus, $I(T) \subseteqq B$. It follows that $I(T) = B \cap I(G)$.

Let $\pi \in \mathrm{Hom}_{\mathscr{E}}(A, C)$ be projection onto C and let $\pi' = \pi|_{I(G)}$. Clearly, $\mathrm{Ker}\,\pi' = B \cap I(G) = I(T)$, so write $I(G) = I(T) \oplus Y$ with π' a monomorphism on Y.

To finish the proof of (2), we claim $\pi'(Y)$ is an $\mathscr{E}$ injective hull of G/T. To see this, first note that if G/T is embedded in C via $e\colon g + T \rightarrow$ evaluation at $g + T$, we have $e(G/T) = \pi'(G) \subseteqq \pi'(Y)$, so $\pi'(Y)$ is an injective containing $e(G/T) \cong G/T$. Furthermore, if $0 \neq \pi'(y) \in \pi'(Y)$, then $\exists \lambda \in \mathscr{E}$ with $0 \neq \lambda(y) \in G \cap Y$. Thus, $0 \neq \pi'\lambda(y) = \lambda\pi'(y) \in \pi'(G) = e(G/T)$. This proves that $e(G/T) \operatorname{ess} \pi'(Y)$. The theorem follows.

EXAMPLE. Let $T = \bigoplus_{p \in P} Z(p)$, where P is an infinite set of primes, and let $G = Z \oplus T$. Then $T \operatorname{ess} G$, so $I(G) = I(T)$ and, in view of Theorem 4, $I(T)^1 = (0)$. Moreover, it is easy to see that $\bar{I} \cong_Z Q$. Thus, if T is an unbounded group direct summand of G, we need not have the decomposition of $I(G)$ given in (2).

The following gives one characterization of $T \operatorname{ess} G$ in the splitting case.

THEOREM 8. *Let* $T = \bigoplus T_p \subset G$. *Let* $k_p = \text{l.u.b.}\{l \mid G$ *has a* $Z(p^l)$ *summand*$\}$ *and let* $H = \{x \in G \mid o(x) = \infty, h_p^G(x) \geqq k_p \forall p\}$. *Then*:
(1) *If* $H = (0)$, $T \operatorname{ess} G$;
(2) *If* $G = T \bigoplus F$ *and* $T \operatorname{ess} G$, *then* $H = (0)$.

Proof. (1) is clear. To prove (2) suppose $G = T \bigoplus F$ and $0 \neq x \in H$. Then, for some positive integer n, $0 \neq nx \in H \cap F$. Clearly, nx cannot be mapped by an endomorphism of G onto any nonzero element of a bounded T_p.

If T_p is unbounded, then G has an unbounded p-basic subgroup, so $k_p = \infty$. Thus, $h_p^G(nx) = h_p^F(nx) = \infty$. If $\lambda \in \mathscr{E}$ with $0 \neq \lambda(nx) \in T_p$, then λ restricts to a nonzero map of the subgroup $\{m/p^k(nx) \mid m, k \in Z\} \subseteqq F$ into T_p. This is impossible since T_p is reduced. Thus, nx cannot be mapped by an endomorphism of G onto a nonzero element of any T_p. The result follows.

It is easy to describe when $T \operatorname{ess} G$ for algebraically compact G.

THEOREM 9. *Let* $T = \bigoplus T_p \subset G$ *with* G *(reduced) algebraically compact. Write* G *as a product of* p*-adic modules,* $G = \Pi G_p$. *Then* $T \operatorname{ess} G$ *if and only if,* $\forall p$, *either* $T_p = G_p$ *or* T_p *is unbounded.*

Proof. It is immediate that $T \operatorname{ess} G$ if and only if, $\forall p$, $T_p \operatorname{ess} G_p$. If $\exists p$ with $T_p \neq G_p$ and T_p bounded, then T_p is not essential in G_p.

Conversely, by considering projections onto summands of a p-adic basis for G_p, it is easy to see that T_p unbounded implies $T_p \operatorname{ess} G_p$.

We close this section with:

THEOREM 10. *Let* $T \subset G$ *with* G *(reduced) cotorsion,* T *a* p*-group,* $T^1 = (0)$. *Then* T *is not essential in* G.

Proof. If T is bounded, T is not essential. If T is an unbounded p-group, $(0) \neq P\text{ext}\,(Q/Z, T) = [\text{Ext}\,(Q/Z, T)]^1$. Since G is reduced cotorsion, $G \cong \text{Ext}\,(Q/Z, G) \cong \text{Ext}\,(Q/Z, T) \bigoplus \text{Ext}\,(Q/Z, G/T)$ ([1] H, p. 234 and Lemma 55.2). Thus $G^1 \neq (0)$, $T^1 = (0)$ and T cannot be essential in G.

4. **The structure of** $I(T)$. In this section we prove three

theorems concerning the structure of $I(T)$. With trivial modification, each of these theorems can be rewritten to give the same result for the injective hull of a torsion group over its own endomorphism ring.

Since $I(T)$ is algebraically compact, it is natural to try to find out what its p-basic subgroups look like as a function of the p-basic subgroups of T. In the case $T^1 = (0)$, this information would characterize $I(T)$ as an abelian group. The next result shows that $I(T)$ is generally large with respect to T.

THEOREM 11. *Let B $[B']$ be a p-basic subgroup of T $[I(T)]$. Let $f =$ final rank B. If $Z(p^k)$ occurs in B, then B' contains $\bigoplus_{\gamma \in \bar{\mathscr{A}}} \langle z_\gamma \rangle$ with $|\bar{\mathscr{A}}| = 2^{2^f}$, $o(z_\gamma) \geqq p^k$, $\forall \gamma$.*

Proof. Suppose B contains a $Z(p^k)$. Write $G = \langle b \rangle \oplus Y$, $o(b) = p^k$, and let $\bigoplus_{\alpha \in A} \langle b_\alpha \rangle \subseteqq B$ with $|A| = f$, $o(b_\alpha) \geqq p^k \forall \alpha$.

Choose $\{A_\beta \mid \beta \in \mathscr{A}\}$ a collection of subsets of A such that: $|\mathscr{A}| = 2^f$, if F is any finite subset of $\mathscr{A}$ and $\beta_0 \in F$ then $[A_{\beta_0} \backslash \bigcup_{\beta \neq \beta_0, \beta \in F} A_\beta] \neq \varnothing$. (See [1[, Lemma 46.2.)

For $\beta \in \mathscr{A}$ define $\delta_\beta \in \operatorname{Hom}(\bigoplus \langle b_\alpha \rangle, \langle b \rangle)$ by $\delta_\beta(b_\alpha) = X_\beta(\alpha) b - X_\beta$ the characteristic function of A_β. Extend each δ_β to $\mathscr{E}$.

It is clear that the left ideals $\mathscr{E}\delta_\beta$ form a direct sum s in $\mathscr{E}$.

Let $\{C_\gamma \mid \gamma \in \bar{\mathscr{A}}\}$ be a family of subsets of $\mathscr{A}$ with the above independence property, $|\bar{\mathscr{A}}| = 2^{2^f}$. Consider:

$$\begin{array}{ccccc} 0 & \longrightarrow & S & \longrightarrow & \mathscr{E} \\ & & \downarrow{\scriptstyle \lambda_\gamma} & \swarrow{\scriptstyle \lambda'_\gamma} & \\ & & I(T) & & \end{array}$$

Here λ_γ is the $\mathscr{E}$ map defined by $\lambda_\gamma(\delta_\beta) = X_{C_\gamma}(\beta) b$, X_{C_γ} the characteristic function of the subset C_γ, and λ'_γ is the map obtained by injectively.

Let $z_\gamma = \lambda'_\gamma(1)$. We have $\delta_\beta(z_\gamma) = X_{C_\gamma}(\beta) b$. It is easy to see from this equation that $\{z_X \mid X \in \bar{\mathscr{A}}\}$ is a p independent set of elements of order $\geqq p^k$. This can be included as a summand of B'. The result follows.

Continuing with the same notation we have:

THEOREM 12. *If B' contains a $Z(p^k)$ so does B.*

Proof. If B' contains $Z(p^k)$ then $I(T)$ has a $Z(p^k)$ summand.

Therefore, so does Hom $(\mathscr{E}, D(T))$. ($I(T)$ can be regarded as a direct summand of Hom $(\mathscr{E}, D(T))$. Therefore, so does Hom $(\mathscr{E}, D(T)_p)$.

The pure exact sequence $0 \to t(\mathscr{E}) \to \mathscr{E} \to \mathscr{E}/t(\mathscr{E}) \to 0$ yields $0 \to [\mathscr{E}/t(\mathscr{E})]^* \to \mathscr{E}^* \to t(\mathscr{E})^* \to 0$, where $M^* = \mathrm{Hom}_Z(M, D(T)_p)$. This sequence is pure exact, so splits, since all its terms are algebraically compact. (In this proof "splits" means splits as an exact sequence of abelian groups.) Since $[\mathscr{E}/t(\mathscr{E})]^*$ is torsion free, $t(\mathscr{E})^*$ must have a $Z(p^k)$ summand.

Now $t(\mathscr{E})^* = [t(\mathscr{E})_p]^*$. Let B_0 be a basic subgroup for $t(\mathscr{E})_p$. Repeat the above procedure with $0 \to B_0 \to t(\mathscr{E})_p \to t(\mathscr{E})_p/B_0 \to 0$ to conclude that B_0^* must have a $Z(p^k)$ summand.

Since B_0 is a direct sum of cyclics, B_0 itself must have a $Z(p^k)$ summand. Thus, $\mathscr{E}$ and, therefore, Hom (G, T_p) have $Z(p^k)$ summands.

Let $\bar{B}$ be a p-basic subgroup for G. The p-pure exact sequence $0 \to \bar{B} \to G \to G/\bar{B} \to 0$ yields the p-pure exact sequence $0 \to (G/\bar{B})^{\nabla} \to G^{\Delta} \to (\bar{B})^{\Delta}$ where $M^{\Delta} = \mathrm{Hom}_Z(M, T_p)$. Since $(G/\bar{B})^{\Delta} \cong W \oplus \bigoplus_r Q_r$, where W is the p-adic completion of a direct sum of copies of the p-adic integers, this sequence also splits. It's not hard to show that $(\bar{B})^{\Delta}$ must have a $Z(p^k)$ summand.

Say $\bar{B} = \bar{B}_1 \oplus \bar{B}_2$, where $\bar{B}_1 = \bigoplus_\alpha Z(p^{l_\alpha})$ is a direct sum of finite p-power cyclics and $\bar{B}_2 = \bigoplus_\beta Z_\beta$ is free. Then $\bar{B}^{\Delta} = (\bar{B}_1)^{\Delta} \oplus (\bar{B}_2)^{\Delta}$, so one of these groups must contain a $Z(p^k)$ summand.

If $(\bar{B}_1)^{\Delta} \cong \prod_\alpha T_p[p^{l_\alpha}]$ has a $Z(p^k)$ summand, then $\bar{B}_1$ itself must, so T does.

If $(\bar{B}_2)^{\Delta} \cong \prod = \prod_\beta (T_p)_\beta$ has a $Z(p^k)$ summand, again T does. (If $\prod = \langle y \rangle \oplus Y$, $o(y) = p^k$, then $h_p^{\Pi}(p^{k-1}y) = k-1$. If $y = [y_\beta]$, $y_\beta \in (T_p)_\beta$, then, for some β_0, $h_p^{(T_p)\beta_0}(p^{k-1}y_{\beta_0}) = k-1$ and, therefore, $o(p^{k-1}y_{\beta_0}) = p$. Thus, y_{β_0} is contained in a $Z(p^k)$ summand of $(T_p)_{\beta_0}$.)

Thus, in either of the above cases, B contains a $Z(p^k)$.

In view of Theorem 5, it is of interest to discover when $Q \subseteqq I(T)$. (Obviously, we must have $T^1 \neq (0)$.) We are unable to decide if $T^1 \neq (0)$ is also sufficient for $Q \subseteqq I(T)$. We close the paper with a result in this direction. First, we need two lemmas.

LEMMA 2. *Let* $T = \bigoplus T_p \subset G$ *and suppose* $T_p^1 \neq (0)$ *whenever* $T_p \neq (0)$. *Then* ${}_{\mathscr{E}}T^1 \operatorname{ess} {}_{\mathscr{E}}T$.

Proof. If $t \in T \backslash T^1$, then $\Pi(t) \neq 0$, Π the projection onto $\langle a \rangle$, some $Z(p^k)$ summand of G. It is easy to construct $\theta \in \mathrm{Hom}_Z(\langle a \rangle, T_p^1)$ with $\theta\Pi(t) \neq 0$. Thus, ${}_{\mathscr{E}}T^1 \operatorname{ess} {}_{\mathscr{E}}T$.

Let $\bar{\mathscr{E}} = \mathscr{E}/t(\mathscr{E})$. Since $t(\mathscr{E})T^1 = (0)$ we can regard T^1 as an $\bar{\mathscr{E}}$ module.

LEMMA 3. *Let* $\mathscr{I}$ *be the* $\bar{\mathscr{E}}$ *injective hull of* T^1 *and let* D *be* *the maximal divisible subgroup of* $I(T)$. *Then, under the assumption of Lemma* 2, $\mathscr{I} \cong D$.

Proof. By Lemma 2, ${}_{\mathscr{E}}T^1 \operatorname{ess} {}_{\mathscr{E}}T$, so $I_{\mathscr{E}}(T^1) = I(T)$.

Now $\mathscr{I}$ is an $\mathscr{E}$ essential extension of T^1, so we can regard $\mathscr{I} \subset I_{\mathscr{E}}(T^1) = I(T)$. Since $\mathscr{I}$ is an injective module over a ring with torsion free additive group, $\mathscr{I} \subseteq D$. But D is an $\bar{\mathscr{E}}$ essential extension of T^1. Thus, $\mathscr{I} = D$.

THEOREM 13. *Let* $E = \operatorname{End} T$, $\bar{E} = E/t(E)$ *and suppose* $R\colon \bar{\mathscr{E}} \to \bar{E}$ *is onto, where* R *is the restriction map. Then, if* T^1 *is unbounded,* $I(T) \supseteq \bigoplus_c Q$.

Proof. Let $T_1 = \{\bigoplus T_p \mid T_p^1 \neq 0\}$, $T_2 = \{\bigoplus T_p \mid T_p^1 = (0)\}$. Clearly, T_1 and T_2 are $\mathscr{E}$ submodules and $I(T) \cong I(T_1) \bigoplus I(T_2)$. It suffices to show $I(T_1) \supseteq \bigoplus_c Q$, so, without loss of generality, assume $T = T_1$. Then Lemma 3 applies, so it is enough to construct c independent elements of infinite order in $\mathscr{I} \cong D$.

Choose $\{x_i \mid i = 1, 2, 3, \cdots\} \subseteq T^1$ with $\{o(x_i) = p_i^{s_i}\}$ unbounded. For each fixed i, choose distinct $\bigoplus_{j=1}^{\infty} \langle b_{ij} \rangle$ part of a p_i-basic subgroup of T such that $\sum_{i,j} \langle b_{ij} \rangle$ is direct and such that $o(b_{ij}) \geq p_i^{j^2}$. (Each T_p is reduced with $T_p^1 \neq (0)$, thus has an unbounded basic.) Finally, choose $\{x_{ij}\} \subseteq T$ with $p_i^j x_{ij} = x_i$.

Now define $\delta_i \in \operatorname{Hom}_Z (\bigoplus_j \langle b_{ij} \rangle, T_{p_i})$ by $\delta_i(b_{ij}) = x_{ij}$. Each δ_i is a small homomorphism (see [1], Lemma 46.3) so each δ_i extends to an endomorphism of T_{p_i} and, thus, to an endomorphism of T. Still call this extension δ_i.

LEMMA 4. $\sum_i \bar{\mathscr{E}} \bar{\delta}_i$ *is an* $\bar{\mathscr{E}}$ *direct sum in* $\bar{E}$. *Here* $\bar{\delta}_i = \delta_i + t(E)$ *and* $\bar{E}$ *is regarded as a left* $\bar{\mathscr{E}}$ *module in the natural way.*

The proof of Lemma 4 is not difficult and is left to the reader.

Let $\{N_\alpha \mid \alpha \in A\}$ be a family of subsets of the natural numbers with $|A| = c$ such that if $F \subseteq A$ is finite and $\alpha_0 \in F$ then $[N_{\alpha_0} \backslash \bigcup_{\alpha \in F, \alpha \neq \alpha_0} N_\alpha]$ is countable.

For all $\alpha \in A$, consider the diagram of $\bar{E}$ modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & \bigoplus_i \bar{\mathscr{E}} \bar{\delta}_i & \longrightarrow & \bar{E} \\ & & \Big\downarrow \lambda_\alpha & \swarrow \lambda'_\alpha & \\ & & \mathscr{I} & & \end{array}$$

Here λ_α is the $\bar{\mathscr{E}}$ map defined by $\lambda_\alpha(\bar{\delta}_i) = X_{N_\alpha}(i) x_i$, X_{N_α} the characteristic function of N_α, and λ'_α the $\bar{\mathscr{E}}$ map obtained by injectivity.

Set $z_\alpha = \lambda'_\alpha(\bar{1})$, $\bar{1}$ the identity of the ring $\bar{E}$. Since $R\colon \bar{\mathscr{E}} \to \bar{E}$

is onto, choose $\bar{\sigma}_i \in \bar{\mathscr{E}}$ with $R(\bar{\sigma}_i) = \bar{\delta}_i$.

Then $\bar{\sigma}_i(z_\alpha) = \lambda'_\alpha(\bar{\sigma}_i\bar{1}) = \lambda'_\alpha(\bar{\delta}_i) = X_{N_\alpha}(i)x_i$. This equation, together with $\{o(x_i)\}$ unbounded, easily implies that $\{z_\alpha \mid \alpha \in A\}$ is an independent set of elements of infinite order. Thus, $I(T) \supseteqq \bigoplus_c Q$.

COROLLARY. *Let T be a torsion group with T^1 unbounded and* $E = \operatorname{End} T$. *Then* $I_E(T) \supseteqq \bigoplus_c Q$.

Added in proof. The proof of Theorem 13 can be modified, using a procedure similar to that of Theorem 11, to construct $\bigoplus_{2^c} Q \subseteqq I(T)$.

REFERENCES

1. L. Fuchs, *Infinite Abelian Groups*, Vol. 1, Academic Press, New York, (1970).

Received July 24, 1978 and in revised form November 6, 1978. This paper was written while I was visiting Tulane University. I would like to thank Professor Laszlo Fuchs for his generous advice and assistance during my visit.

THE UNIVERSITY OF CONNECTICUT
STORRS, CT 06268

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $84.00 a year (6 Vols., 12 issues). Special rate: $42.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 83, No. 2 April, 1979