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THE CLASS NUMBER OF $Q(\sqrt{-p})$ MODULO 4, FOR $p \equiv 3 \pmod{4}$ A PRIME

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If p is a prime congruent to 3 modulo 4, it is well-known that the class number $h(-p)$ of the imaginary quadratic field $Q(\sqrt{-p})$ is odd. In this paper we determine $h(-p)$ modulo 4.

The class number of $Q(\sqrt{-p})$ is odd, if p is a prime congruent to 3 modulo 4 (see for example [3: p. 413]. D.H. Lehmer [4: p. 9] has posed the problem of determining the Jacobi symbol $(-1/h(-p)) = (-1)^{(h(-p)-1)/2}$, that is, of determining $h(-p)$ modulo 4. In this paper we evaluate $h(-p)$ modulo 4 in terms of the class number $h(p)$ and the fundamental unit $\varepsilon_p = T + U\sqrt{p}$ of the corresponding real quadratic field $Q(\sqrt{p})$. It is known that T and U are positive integers which satisfy $T \equiv 0 \pmod{2}$, $U \equiv 1 \pmod{2}$, $N(\varepsilon_p) = T^2 - pU^2 = +1$. We prove

THEOREM. *If $p > 3$ is a prime congruent to 3 modulo 4 then*

$$(1) \quad h(-p) \equiv h(p) + U + 1 \pmod{4}.$$

It is easily checked that (1) does not hold for $p = 3$ ($h(-3) = h(3) = U = 1$). ($p = 3$ is a special case as this is the only value of $p \equiv 3 \pmod{4}$ for which the ring of integers of $Q(\sqrt{-p})$ has more than 2 units.) The method of proof is purely analytic in nature, it uses Dirichlet's class number formula (in various forms) for both real and imaginary quadratic fields and also some results from cyclotomy. It would be of interest to give a purely algebraic proof.

Proof. Let $p > 3$ be a prime congruent to 3 modulo 4 and set $\rho = \exp(2\pi i/p)$. For z a complex variable, we let

$$(2) \quad F_+(z) = \prod_{\substack{j=1 \\ (j,p)=+1}}^{p-1} (z - \rho^j), \quad F_-(z) = \prod_{\substack{j=1 \\ (j,p)=-1}}^{p-1} (z - \rho^j),$$

so that

$$(3) \quad F_+(z)F_-(z) = F(z),$$

where $F(z)$ is the cyclotomic polynomial of index p , that is,

$$(4) \quad F(z) = \prod_{j=1}^{p-1} (z - \rho^j) = \frac{z^p - 1}{z - 1} = z^{p-1} + z^{p-2} + \dots + z + 1.$$

F_+ and F_- are polynomials in z of degree $(p - 1)/2$ with coefficients in the ring of integers of $Q(\sqrt{-p})$ (see for example [6: p.215]). Hence we can write

$$(5) \quad F_+(z) = \frac{1}{2} (Y(z) - Z(z)\sqrt{-p}), \quad F_-(z) = \frac{1}{2} (Y(z) + Z(z)\sqrt{-p}),$$

where Y and Z are polynomials with rational integral coefficients. From (3) and (5) we have

$$(6) \quad Y(z)^2 + pZ(z)^2 = 4F(z).$$

It is also known [6: p.216] or [7: p.209] that Y and Z have the symmetry properties expressed by

$$(7) \quad Y(z) = \sum_{n=0}^{(p-3)/4} a_n (z^{(p-1)/2-n} - z^n), \quad Z(z) = \sum_{n=0}^{(p-3)/4} b_n (z^{(p-1)/2-n} + z^n),$$

where the a_n and b_n are integers with

$$a_0 = 2, a_1 = 1, a_2 = (3 - p)/4, \dots$$

and

$$b_0 = 0, b_1 = 1, b_2 = \frac{1}{2} \left(1 + \left(\frac{2}{p} \right) \right), \dots$$

(see [7] for further values of a_n and b_n ; see [6] for a table of values of Y and Z for $p \leq 29$).

Differentiating the expressions in (7) and (6) with respect to z , we obtain respectively

$$(8) \quad Y'(z) = \sum_{n=0}^{(p-3)/4} a_n \left(\left(\frac{p-1}{2} - n \right) z^{(p-3)/2-n} - n z^{n-1} \right),$$

$$Z'(z) = \sum_{n=0}^{(p-3)/4} b_n \left(\left(\frac{p-1}{2} - n \right) z^{(p-3)/2-n} + n z^{n-1} \right),$$

and

$$(9) \quad Y(z)Y'(z) + pZ(z)Z'(z) = 2F'(z).$$

Taking $z = i$ in (7) and (8) we obtain

$$(10) \quad Y(i) = \begin{cases} A_3(1 - i), & \text{if } p \equiv 3 \pmod{8}, \\ A_7(1 + i), & \text{if } p \equiv 7 \pmod{8}, \end{cases}$$

$$Z(i) = \begin{cases} -B_3(1 + i), & \text{if } p \equiv 3 \pmod{8}, \\ B_7(1 - i), & \text{if } p \equiv 7 \pmod{8}, \end{cases}$$

and

$$(11) \quad \begin{aligned} Y'(i) &= \begin{cases} C_3 + 2D_3i, & \text{if } p \equiv 3 \pmod{8}, \\ C_7 + 2D_7i, & \text{if } p \equiv 7 \pmod{8}, \end{cases} \\ Z'(i) &= \begin{cases} E_3 + 2F_3i, & \text{if } p \equiv 3 \pmod{8}, \\ E_7 + 2F_7i, & \text{if } p \equiv 7 \pmod{8}, \end{cases} \end{aligned}$$

where A_3, \dots, F_7 are rational integers (given in terms of the a_n and b_n). Using (10) and (11) in (6) and (9) with $z = i$, we obtain

$$(12) \quad \begin{cases} A_3^2 - pB_3^2 = -2, & \text{if } p \equiv 3 \pmod{8}, \\ A_7^2 - pB_7^2 = +2, & \text{if } p \equiv 7 \pmod{8}, \end{cases}$$

and

$$(13) \quad \begin{cases} A_3C_3 + 2pB_3F_3 = -1, 2A_3D_3 - pB_3E_3 = p, & \text{if } p \equiv 3 \pmod{8}, \\ A_7C_7 + 2pB_7F_7 = p, 2A_7D_7 - pB_7E_7 = 1, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Clearly from (12) and (13) we see that $A_3, B_3, C_3, E_3, A_7, B_7, C_7$ and E_7 are all odd. Now Liouville [5: p. 415] has shown that

$$(14) \quad Z(z)Y'(z) - Z'(z)Y(z) = \frac{2}{z-1} \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) z^{p-1-j}.$$

Taking $z = i$ in (14) we obtain

$$(15) \quad Z(i)Y'(i) - Z'(i)Y(i) = (L + M) + i(L - M),$$

where

$$L = \sum_{j=0}^{(p-1)/2} (-1)^j \binom{2j}{p}, \quad M = \sum_{j=0}^{(p-1)/2} (-1)^j \binom{2j+1}{p}.$$

Applying the transformation $j \rightarrow (p-1)/2 - j$ to L or M we obtain $L = M$. Also we have

$$\begin{aligned} L &= \sum_{j=1}^{(p-3)/4} \binom{4j}{p} - \sum_{j=0}^{(p-3)/4} \binom{4j+2}{p} \\ &= \sum_{j=1}^{(p-3)/4} \binom{j}{p} - \sum_{j=(p+1)/4}^{(p-1)/2} \binom{4((p-1)/2 - j) + 2}{p} \\ &= \sum_{j=1}^{(p-3)/4} \binom{j}{p} + \sum_{j=(p+1)/4}^{(p-1)/2} \binom{j}{p} = \sum_{j=1}^{(p-1)/2} \binom{j}{p}, \end{aligned}$$

so, by Dirichlet's class number formula (as $p \equiv 3 \pmod{4}$, $p < 3$) see for example [2: p. 346], we have

$$(16) \quad L = M = \left\{ 2 - \binom{2}{p} \right\} h(-p).$$

Hence from (15) and (16) we have

$$(17) \quad Z(i)Y'(i) - Z'(i)Y(i) = 2 \left\{ 2 - \left(\frac{2}{p} \right) \right\} h(-p).$$

Using (10) and (11) in (17), after equating real and imaginary parts, we obtain

$$(18) \quad \begin{cases} 3h(-p) = 2B_3D_3 - A_3E_3, & \text{if } p \equiv 3 \pmod{8}, \\ h(-p) = 2B_7D_7 - A_7E_7, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Now from (13) we have

$$(19) \quad \begin{cases} E_3 \equiv -2A_3B_3D_3 - B_3 \pmod{8}, & \text{if } p \equiv 3 \pmod{8}, \\ E_7 \equiv -2A_7B_7D_7 + B_7 \pmod{8}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Using (19) in (18) we have

$$(20) \quad h(-p) \equiv \begin{cases} -A_3B_3 \pmod{4}, & \text{if } p \equiv 3 \pmod{8}, \\ -A_7B_7 \pmod{4}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

From (4) we have $F(i) = i$, and so taking $z = i$ in (2) and (3) we obtain

$$\begin{aligned} -i\{F_-(i)\}^2 &= \frac{F_-(i)}{F_+(i)} = \prod_{j=1}^{p-1} (1 + i\rho^j)^{-(j/p)} \\ &= \exp \left(-\sum_{j=1}^{p-1} \left(\frac{j}{p} \right) \log(1 + i\rho^j) \right) \\ &= \exp \left(\sum_{n=1}^{\infty} \frac{(-i)^n}{n} \sum_{j=1}^{p-1} \left(\frac{j}{p} \right) \rho^{nj} \right) \\ &= \exp \left(i\sqrt{p} \sum_{n=1}^{\infty} \left(\frac{n}{p} \right) \frac{(-i)^n}{n} \right) \\ &= \exp \left(\sqrt{p} \sum_{m=0}^{\infty} \left(\frac{2m+1}{p} \right) \frac{(-1)^m}{2m+1} + \frac{i\sqrt{p}}{2} \left(\frac{2}{p} \right) \sum_{m=1}^{\infty} \left(\frac{m}{p} \right) \frac{(-1)^m}{m} \right) \\ &= \exp \left(h(p) \log(T + U\sqrt{p}) + \frac{\pi i}{2} \left(1 - \left(\frac{2}{p} \right) \right) h(-p) \right) \\ &= (T + U\sqrt{p})^{h(p)} i^{(1-(2/p))h(-p)} \\ &= (-1)^{(p+1)/4} (T + U\sqrt{p})^{h(p)}, \end{aligned}$$

where we have made use of the Gauss sum

$$\sum_{j=1}^{p-1} \left(\frac{j}{p} \right) \rho^{nj} = \left(\frac{n}{p} \right) i\sqrt{p},$$

and the two results

$$\sum_{m=1}^{\infty} \binom{m}{p} \frac{(-1)^m}{m} = \frac{\pi}{\sqrt{p}} \left(\left(\frac{2}{p} \right) - 1 \right) h(-p)$$

and

$$\sum_{m=0}^{\infty} \binom{2m+1}{p} \frac{(-1)^m}{2m+1} = \frac{h(p)}{\sqrt{p}} \log(T + U\sqrt{p}),$$

which follow easily by standard arguments from Dirichlet's class number formula (see for example [2: p. 343]). Hence we have (using (10))

$$\begin{aligned} (T + U\sqrt{p})^{h(p)} &= (-1)^{(p-3)/4} i F_-(i)^2 \\ &= (-1)^{(p-3)/4} i \left\{ \frac{1}{2} (Y(i) + Z(i)i\sqrt{p}) \right\}^2 \\ &= \begin{cases} \frac{1}{2} (A_3 + B_3\sqrt{p})^2, & \text{if } p \equiv 3 \pmod{8}, \\ \frac{1}{2} (A_7 + B_7\sqrt{p})^2, & \text{if } p \equiv 7 \pmod{8}. \end{cases} \end{aligned}$$

This is essentially a result of Arndt [1].

Expanding $(T + U\sqrt{p})^{h(p)}$ by the binomial theorem and equating coefficients of \sqrt{p} , we have as $h(p) \equiv 1 \pmod{2}$,

$$\begin{aligned} U^{h(p)} p^{(h(p)-1)/2} + \binom{h(p)}{2} U^{h(p)-2} T^2 p^{(h(p)-3)/2} + \dots \\ = \begin{cases} A_3 B_3, & \text{if } p \equiv 3 \pmod{8}, \\ A_7 B_7, & \text{if } p \equiv 7 \pmod{8}. \end{cases} \end{aligned}$$

As $T \equiv 0 \pmod{2}$, $U \equiv 1 \pmod{2}$, this gives

$$U(-1)^{(h(p)-1)/2} \equiv \begin{cases} A_3 B_3 \pmod{4}, & \text{if } p \equiv 3 \pmod{8}, \\ A_7 B_7 \pmod{4}, & \text{if } p \equiv 7 \pmod{8}, \end{cases}$$

so that

$$(21) \quad h(p) \equiv \begin{cases} A_3 B_3 - U + 1 \pmod{4}, & \text{if } p \equiv 3 \pmod{8}, \\ A_7 B_7 - U + 1 \pmod{4}, & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Putting (20) and (21) together, we obtain (1) as required.

From (1) we have $(-1/h(-p)) = (-1)^{(h(-p)-1)/2} = (-1)^{(h(p)+U)/2}$. In particular whenever $h(p) = 1$ (a common occurrence) we have $(-1/h(-p)) = (-1)^{(U+1)/2}$.

In [8] the author has treated, in a similar way, Lehmer's question [4: p. 10] regarding $h(-2p)$ modulo 8, when p is a prime congruent to 5 modulo 8.

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Patrick Robert Ahern, <i>On a theorem of Hayman concerning the derivative of a function of bounded characteristic</i>	297
Walter Allegretto, <i>Finiteness of lower spectra of a class of higher order elliptic operators</i>	303
Leonard Asimow, <i>Superharmonic interpolation in subspaces of $C_c(X)$</i>	311
Steven F. Bellenot, <i>An anti-open mapping theorem for Fréchet spaces</i>	325
B. J. Day, <i>Locale geometry</i>	333
John Erik Fornæss and Steven Krantz, <i>Continuously varying peaking functions</i>	341
Joseph Leonide Gerver, <i>Long walks in the plane with few collinear points</i>	349
Joseph Leonide Gerver and Lawrence Thom Ramsey, <i>On certain sequences of lattice points</i>	357
John R. Graef, Yuichi Kitamura, Takaši Kusano, Hiroshi Onose and Paul Winton Spikes, <i>On the nonoscillation of perturbed functional-differential equations</i>	365
James A. Huckaba and James M. Keller, <i>Annihilation of ideals in commutative rings</i>	375
Anzelm Iwanik, <i>Norm attaining operators on Lebesgue spaces</i>	381
Surjit Singh Khurana, <i>Pointwise compactness and measurability</i>	387
Charles Philip Lanski, <i>Commutation with skew elements in rings with involution</i>	393
Hugh Bardeen Maynard, <i>A Radon-Nikodým theorem for finitely additive bounded measures</i>	401
Kevin Mor McCrimmon, <i>Peirce ideals in Jordan triple systems</i>	415
Sam Bernard Nadler, Jr., Joseph E. Quinn and N. Stavrakas, <i>Hyperspaces of compact convex sets</i>	441
Ken Nakamura, <i>An explicit formula for the fundamental units of a real pure sextic number field and its Galois closure</i>	463
Vassili Nestoridis, <i>Inner functions invariant connected components</i>	473
Vladimir I. Olikier, <i>On compact submanifolds with nondegenerate parallel normal vector fields</i>	481
Lex Gerard Oversteegen, <i>Fans and embeddings in the plane</i>	495
Shlomo Reisner, <i>On Banach spaces having the property G.L.</i>	505
Gideon Schechtman, <i>A tree-like Tsirelson space</i>	523
Helga Schirmer, <i>Fix-finite homotopies</i>	531
Jeffrey D. Vaaler, <i>A geometric inequality with applications to linear forms</i>	543
William Jennings Wickless, <i>T as an \mathcal{Q} submodule of G</i>	555
Kenneth S. Williams, <i>The class number of $Q(\sqrt{-p})$ modulo 4, for $p \equiv 3 \pmod{4}$ a prime</i>	565
James Chin-Sze Wong, <i>On topological analogues of left thick subsets in semigroups</i>	571