GENERALIZATION OF A THEOREM OF LANDAU

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A well known theorem of Landau asserts that

\[
\lim_{n \to \infty} \frac{\phi(n) \log \log n}{n} = e^{-\gamma}
\]

where \( \gamma \) = Euler’s constant. In this paper a generalization is obtained by focusing on

\[
G(k) = \lim_{n \to \infty} (\log \log n)^{1/k} \max \left( \frac{\phi(n+1)}{n+1}, \ldots, \frac{\phi(n+k)}{n+1} \right).
\]

Clearly, the assertion \( G(1) = e^{-\gamma} \) is precisely Landau’s theorem. It is proved that

\[
G(k) = e^{-\gamma/k} \prod_{p \leq k} \left( 1 - \frac{1}{p} \right)^{1/k} \psi(k)
\]

where

\[
\psi(k) = \prod_{p \leq k} \left( 1 - \frac{1}{p} \right)^{1/p} \prod_{\substack{p | k \mid k \leq k}} \left( 1 - \frac{1}{p} \right)^{(1/k)[k/p]+1/k}.
\]

The function \( \psi(k) \) satisfies \( 0 < \psi(k) \leq 1 \) and it is easily seen from (1.4) that

\[
\lim_{k \to \infty} \psi(k) = \prod_{p} \left( 1 - \frac{1}{p} \right)^{1/p}.
\]

2. Preliminary lemmas. The results obtained in this paper depend on the following well known theorems [1], [2], and [3].

\[
\lim_{n \to \infty} \frac{\phi(n) \log \log n}{n} = e^{-\gamma} \quad \text{(Landau’s theorem)}
\]
\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + c + O\left( \frac{1}{\log x} \right) \quad \text{(Mertens’)}
\]
\[
\prod_{p \leq x} \left( 1 - \frac{1}{p} \right) = \frac{e^{-\gamma}}{\log x} + O\left( \frac{1}{\log^2 x} \right) \quad \text{(Mertens’)}
\]

3. Proof of (1.3). We introduce

\[
\left( \frac{\phi(n)}{n} \right)_k = \prod_{p \mid n \atop p \leq k} \left( 1 - \frac{1}{p} \right)
\]

and

\[
f_k(n) = \prod_{p \leq k} \left( 1 - \frac{1}{p} \right)
\]

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and note that \( f_k(n) \) is periodic with period \( \Delta_k = \prod_{p < k} p \).

We also observe that (1.2) is clearly equivalent to

\[
(3.3) \quad G(k) = \min_{1 \leq J \leq \Delta_k} \lim_{n \to \infty} \left( \log \log n \right)^{1/k} \max_{1 \leq i \leq k} \left( \frac{\phi(n + i)}{n + i} \right).
\]

On the sequence \( n = J \pmod{\Delta_k} \)

\[
(3.4) \quad \left( \log \log n \right) \prod_{i=1}^{k} \left( \frac{\phi(n + i)}{n + i} \right) = \left( \log \log n \right) \prod_{i=1}^{k} \left( \frac{\phi(n + i)}{n + i} \right) f_k(J + i).
\]

Since a prime \( p \) divides \( n + i \) and \( n + j \) only if \( p \) divides \( i - j \), \( 1 \leq j < i \leq k \); and the primes involved in \( (\phi(n)/n)_k \) are \( p \geq k \), we have

\[
\prod_{i=1}^{k} \left( \frac{\phi(n + i)}{n + i} \right) = \left( \frac{\prod_{i=1}^{k} (n + i)}{\prod_{i=1}^{k} (n + i)} \right)^{1/k}.
\]

This together with the result

\[
\lim_{n \to \infty} \left( \log \log n \right) \left( \frac{\phi(n)}{n} \right)_k = e^{-\gamma} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{-1}
\]

(which follows from Landau's theorem) yields

\[
\left( \log \log n \right) \prod_{i=1}^{k} \left( \frac{\phi(n + i)}{n + i} \right) \geq (1 + o(1)) e^{-\gamma} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{i=1}^{k} f_k(J + i),
\]

which implies

\[
(3.5) \quad \lim_{n \to \infty} \left( \log \log n \right)^{1/k} \max_{i=1,\ldots,k} \left( \frac{\phi(n + i)}{n + i} \right)
\geq e^{-\gamma/k} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{-1/k} \left[ \min_{1 \leq J \leq \Delta_k} \prod_{i=1}^{k} f_k(J + i) \right]^{1/k}.
\]

In (3.5), taking the minimum over \( J, 1 \leq J \leq \Delta_k \), and using (3.3) yields

\[
(3.6) \quad G(k) \geq e^{-\gamma/k} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{-1/k} \left[ \min_{1 \leq J \leq \Delta_k} \prod_{i=1}^{k} f_k(J + i) \right]^{1/k}.
\]

Choose \( J^* \) such that

\[
\left[ \min_{1 \leq J \leq \Delta_k} \prod_{i=1}^{k} f_k(J + i) \right]^{1/k} = \left[ \prod_{i=1}^{k} f_k(J^* + i) \right]^{1/k}.
\]

We next observe that for the \( \psi(k) \) given in (1.4) we have

\[
(3.7) \quad \left[ \prod_{i=1}^{k} f_k(J^* + i) \right]^{1/k} = \psi(k).
\]

To see this note first that the left side of (3.7) equals
\[(3.8) \quad \min_{1 \leq j \leq d_k} \left[ \prod_{p < k} \left( 1 - \frac{1}{p} \right) \prod_{p | j + 1} \left( 1 - \frac{1}{p} \right) \cdots \prod_{p | j + k} \left( 1 - \frac{1}{p} \right) \right]^{1/k}. \]

Since each of the factors \((1 - 1/p) < 1\), the minimum of the product in (3.8) is achieved for that value of \(J\) for which each prime \(p < k\) divides as many of the \(k\) integers \(J + 1, \ldots, J + k\) as possible. Since \(p < k, k = pt + r, t = [k/p], 0 \leq r < p\). If \(r = 0\), i.e., \(p | k\), then the \(k\) integers \(J + 1, \ldots, J + k\) can be broken up into exactly \(t\) complete residue systems modulo \(p\) and in each system we have one integer \(\equiv 0(\text{mod } p)\); this situation is independent of the choice of \(J\). If \(r > 0\) then the \(k\) integers \(J + 1, \ldots, J + k\) form \(t\) complete residue classes modulo \(p\) together with \(r < p\) remaining integers. In each of the complete residue classes there is one integer \(\equiv 0(\text{mod } p)\). We would like to show that it can be arranged that for each \(p < k, p \perp k\), one of the \(r\) remaining integers is \(\equiv 0(\text{mod } p)\), and thus we have \([k/p] + 1\) integers divisible by \(p\). Since \(1 \leq J \leq \Delta_k\) where \(\Delta_k = \prod_{p < k} p\), we can choose \(J = \Delta_k - 1\); then every \(p < k\) divides \(J + 1\). Hence for \(p \perp k\), the \([k/p] + 1\) integers \(J + 1 + \tau p, 0 \leq \tau \leq t\) are divisible by \(p\) as desired, and (3.7) follows.

From (3.6) and (3.7) we see that
\[(3.9) \quad G(k) \geq e^{-\gamma/k} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{-1/k} \psi(k); \]
and it remains to prove the reverse inequality. This is achieved by showing that there exists an infinite sequence \(n \equiv J^* (\text{mod } \Delta_k)\) on which
\[(3.10) \quad \lim_{n \to \infty} \frac{\log \log n}{\log log n} \max_{1 \leq j \leq d_k} \left( \frac{n + j}{n} \right)^{\psi(k)} \leq e^{-\gamma/k} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{-1/k} \psi(k). \]

This is done by producing a sequence \(n \equiv J^* (\text{mod } \Delta_k)\) for which
\[(3.11) \quad (\log n)^{1/k} \max_{1 \leq j \leq d_k} \left( \frac{n + j}{n} \right)^{\psi(k)} \sim e^{-\gamma/k} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{-1/k} \lambda_i \]
where for all \(i = 1, \ldots, k, \lambda_i = \frac{\psi(k)}{f_k(J^* + i)}. \]

On this sequence
\[(\log n)^{1/k} \max_{1 \leq j \leq d_k} \left( \frac{n + j}{n} \right)^{\psi(k)} \sim e^{-\gamma/k} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{-1/k} \max_{1 \leq j \leq d_k} (\lambda_i f_k(J^* + i)) \]
which gives the reverse inequality to (3.9) and establishes (1.3).

To construct the sequence \( n = J^*(\text{mod } \Delta_k) \) which satisfies (3.8) let

\[
B_i = \prod_{\substack{p \leq \exp(e^{1/4}k) \\ p \neq k}} p, \quad B_k = \prod_{\exp(e^{1/4}k) \leq p < \exp(e^{1/4}k)^{1/2}}} p, \quad \hat{B}_1 = \prod_{p \leq \exp(e^{1/4}k)} \alpha(p),
\]

where \( c_k = 1 \), and for \( i = 0, \ldots, k-1 \), \( c_i \) is determined by

\[
\frac{c_{i-1}}{c_i} = e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p}\right)^{-1/k} \lambda_i.
\]

Since \( \prod_{i=1}^k \lambda_i = 1 \) it follows that \( c_0 = e^{-\gamma} \prod_{p < k} (1 - 1/p) \). As the \( B_i, \ i = 1, \ldots, k \) are \( k \) integers made up of primes \( p \geq k \) and are relatively prime in pairs, as well as each relatively prime to \( \Delta_k \), by the Chinese Remainder Theorem the system

\[
y + 1 \equiv O(\text{mod } B_i) \\
y + 2 \equiv O(\text{mod } B_2) \\
\vdots \\
y + k \equiv O(\text{mod } B_k) \\
y \equiv J^*(\text{mod } \Delta_k)
\]

has a solution \( y = n^*, 0 < n^* < \Delta_k \prod_{i=1}^k B_i \) which is unique modulo \( \Delta_k \prod_{i=1}^k B_i \).

For this integer \( n^* = J^*(\text{mod } \Delta_k) \) we have for \( i = 1, \ldots, k \)

\[
\left(\frac{\phi(n^* + i)}{n^* + i}\right)_k = \prod_{p, \lambda_i \leq \log^a x} \left(1 - \frac{1}{p}\right) \leq \prod_{p, \lambda_i} \left(1 - \frac{1}{p}\right)
\]

\[
\leq \frac{c_{i-1}}{c_i} \left(\frac{1}{\log x}\right) + O\left(\frac{1}{\log^a x}\right),
\]

(note that the value obtained for \( c_0 \) validates this for \( i = 1 \)). Then

\[
\left(\frac{\phi(n^* + i)}{n^* + i}\right) f_k(J^* + i) \\
\leq \frac{\lambda_i e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p}\right)^{-1/k}}{\log x} f_k(J^* + i)(1 + o(1))
\]

But from the Prime Number Theorem since

\[
n^* < \Delta_k \prod_{i=1}^k B_i = \prod_{p < \exp(\log^a x)} p, \quad (c_k = 1),
\]
it follows that
\[ \log n^* \leq \sum_{p < n \exp(\log x)^k} \log p = O(e^{(\log x)^k}) \]
so that
\[ (3.15) \quad \log \log n^* \leq (\log x)^k + O(1) . \]
Since (3.14) holds for all \( i = 1, \ldots, k \), it certainly holds for the maximum of these functions. Thus inserting (3.15) in (3.14) yields
\[ (\log \log n^*)^{1/k} \max_{i=1,\ldots,k} \left( \frac{\phi(n^* + i)}{n^* + i} \right)^{1/k} f_k(J^* + i) \]
\[ \leq (1 + o(1)) e^{-\gamma/k} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{1/k} \psi(k) . \]
Clearly as \( x \) tends to infinity the \( n^* \) (which depends on \( x \)) also tends to infinity, so that (3.16) yields
\[ (3.17) \quad G(k) \leq e^{-\gamma/k} \prod_{p < k} \left( 1 - \frac{1}{p} \right)^{-1/k} \psi(k) \]
which completes the proof of (1.3).
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