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GENERALIZATION OF A THEOREM OF LANDAU

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A well known theorem of Landau asserts that

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{\phi(n) \log \log n}{n} = e^{-\gamma}$$

where $\gamma =$ Euler's constant. In this paper a generalization is obtained by focusing on

$$(1.2) \quad G(k) = \lim_{n \rightarrow \infty} (\log \log n)^{1/k} \max \left(\frac{\phi(n+1)}{n+1}, \dots, \frac{\phi(n+k)}{n+k} \right).$$

Clearly, the assertion $G(1) = e^{-\gamma}$ is precisely Landau's theorem. It is proved that

$$(1.3) \quad G(k) = e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p} \right)^{-1/k} \psi(k)$$

where

$$(1.4) \quad \psi(k) = \prod_{p | k} \left(1 - \frac{1}{p} \right)^{1/p} \prod_{p < k} \left(1 - \frac{1}{p} \right)^{(1/k)[k/p] + 1/k}.$$

The function $\psi(k)$ satisfies $0 < \psi(k) \leq 1$ and it is easily seen from (1.4) that

$$(1.5) \quad \lim_{k \rightarrow \infty} \psi(k) = \prod_p \left(1 - \frac{1}{p} \right)^{1/p}.$$

2. Preliminary lemmas. The results obtained in this paper depend on the following well known theorems [1], [2], and [3].

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{\phi(n) \log \log n}{n} = e^{-\gamma} \quad (\text{Landau's theorem})$$

$$(2.2) \quad \sum_{p \leq x} \frac{1}{p} = \log \log x + c + O\left(\frac{1}{\log x}\right) \quad (\text{Mertens'})$$

$$(2.3) \quad \prod_{p \leq x} \left(1 - \frac{1}{p} \right) = \frac{e^{-\gamma}}{\log x} + O\left(\frac{1}{\log^2 x}\right) \quad (\text{Mertens'})$$

3. Proof of (1.3). We introduce

$$(3.1) \quad \left(\frac{\phi(n)}{n} \right)_k = \prod_{\substack{p | n \\ p \geq k}} \left(1 - \frac{1}{p} \right)$$

and

$$(3.2) \quad f_k(n) = \prod_{\substack{p | n \\ p < k}} \left(1 - \frac{1}{p} \right)$$

and note that $f_k(n)$ is periodic with period $\Delta_k = \prod_{p < k} p$.

We also observe that (1.2) is clearly equivalent to

$$(3.3) \quad G(k) = \min_{1 \leq J \leq \Delta_k} \lim_{\substack{n \rightarrow \infty \\ n \equiv J \pmod{k}}} (\log \log n)^{1/k} \max \left(\frac{\phi(n+1)}{n+1}, \dots, \frac{\phi(n+k)}{n+k} \right).$$

On the sequence $n \equiv J \pmod{\Delta_k}$

$$(3.4) \quad (\log \log n) \prod_{i=1}^k \frac{\phi(n+i)}{n+i} = (\log \log n) \prod_{i=1}^k \left(\frac{\phi(n+i)}{n+i} \right)_k f_k(J+i).$$

Since a prime p divides $n+i$ and $n+j$ only if p divides $i-j$, $1 \leq j < i \leq k$; and the primes involved in $(\phi(n)/n)_k$ are $p \geq k$, we have

$$\prod_{i=1}^k \left(\frac{\phi(n+i)}{n+i} \right)_k = \left(\frac{\phi \left[\prod_{i=1}^k (n+i) \right]}{\prod_{i=1}^k (n+i)} \right)_k.$$

This together with the result

$$\lim_{n \rightarrow \infty} (\log \log n) \left(\frac{\phi(n)}{n} \right)_k = e^{-\gamma} \prod_{p < k} \left(1 - \frac{1}{p} \right)^{-1}$$

(which follows from Landau's theorem) yields

$$(\log \log n) \prod_{i=1}^k \left[\frac{\phi(n+i)}{n+i} \right] \geq (1 + o(1)) e^{-\gamma} \prod_{p < k} \left(1 - \frac{1}{p} \right)^{-1} \prod_{i=1}^k f_k(J+i),$$

which implies

$$(3.5) \quad \lim_{\substack{n \rightarrow \infty \\ n \equiv J \pmod{k}}} (\log \log n)^{1/k} \max_{i=1, \dots, k} \left(\frac{\phi(n+i)}{n+i} \right) \\ \geq e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p} \right)^{-1/k} \left[\prod_{i=1}^k f_k(J+i) \right]^{1/k}.$$

In (3.5), taking the minimum over J , $1 \leq J \leq \Delta_k$, and using (3.3) yields

$$(3.6) \quad G(k) \geq e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p} \right)^{-1/k} \left[\min_{1 \leq J \leq \Delta_k} \prod_{i=1}^k f_k(J+i) \right]^{1/k}.$$

Choose J^* such that

$$\left[\min_{1 \leq J \leq \Delta_k} \prod_{i=1}^k f_k(J+i) \right]^{1/k} = \left[\prod_{i=1}^k f_k(J^*+i) \right]^{1/k}.$$

We next observe that for the $\psi(k)$ given in (1.4) we have

$$(3.7) \quad \left[\prod_{i=1}^k f_k(J^*+i) \right]^{1/k} = \psi(k).$$

To see this note first that the left side of (3.7) equals

$$(3.8) \quad \min_{1 \leq J \leq \Delta_k} \left[\prod_{\substack{p|J+1 \\ p < k}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|J+2 \\ p < k}} \left(1 - \frac{1}{p}\right) \cdots \prod_{\substack{p|J+k \\ p < k}} \left(1 - \frac{1}{p}\right) \right]^{1/k}.$$

Since each of the factors $(1 - 1/p) < 1$, the minimum of the product in (3.8) is achieved for that value of J for which each prime $p < k$ divides as many of the k integers $J + 1, \dots, J + k$ as possible. Since $p < k, k = pt + r, t = [k/p], 0 \leq r < p$. If $r = 0$, i.e., $p|k$, then the k integers $J + 1, \dots, J + k$ can be broken up into exactly t complete residue systems modulo p and in each system we have one integer $\equiv 0(\text{mod } p)$; this situation is independent of the choice of J . If $r > 0$ then the k integers $J + 1, \dots, J + k$ form t complete residue classes modulo p together with $r < p$ remaining integers. In each of the complete residue classes there is one integer $\equiv 0(\text{mod } p)$. We would like to show that it can be arranged that for each $p < k, p \nmid k$, one of the r remaining integers is $\equiv 0(\text{mod } p)$, and thus we have $[k/p] + 1$ integers divisible by p . Since $1 \leq J \leq \Delta_k$ where $\Delta_k = \prod_{p < k} p$, we can choose $J = \Delta_k - 1$; then every $p < k$ divides $J + 1$. Hence for $p \nmid k$, the $[k/p] + 1$ integers $J + 1 + \tau p, 0 \leq \tau \leq t$ are divisible by p as desired, and (3.7) follows.

From (3.6) and (3.7) we see that

$$(3.9) \quad G(k) \geq e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p}\right)^{-1/k} \psi(k);$$

and it remains to prove the reverse inequality. This is achieved by showing that there exists an infinite sequence $n \equiv J^*(\text{mod } \Delta_k)$ on which

$$(3.10) \quad \lim_{\substack{n \rightarrow \infty \\ n \equiv J^*(\text{mod } \Delta_k)}} (\log \log n)^{1/k} \max_{i=1, \dots, k} \left(\frac{\phi(n+i)}{n+i} \right) \leq e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p}\right)^{-1/k} \psi(k).$$

This is done by producing a sequence $n \equiv J^*(\text{mod } \Delta_k)$ for which

$$(3.11) \quad (\log \log n)^{1/k} \max_{i=1, \dots, k} \left(\frac{\phi(n+i)}{n+i} \right)_k \sim e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p}\right)^{-1/k} \lambda_i$$

where for all $i = 1, \dots, k$,

$$\lambda_i = \frac{\psi(k)}{f_k(J^* + i)}.$$

On this sequence

$$(\log \log n)^{1/k} \max_{i=1, \dots, k} \left(\frac{\phi(n+i)}{n+i} \right) \sim e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p}\right)^{-1/k} \max_{i=1, \dots, k} (\lambda_i f_k(J^* + i))$$

$$\sim e^{-i/k} \prod_{p < k} \left(1 - \frac{1}{p}\right)^{-1/k} \psi(k),$$

which gives the reverse inequality to (3.9) and establishes (1.3).

To construct the sequence $n \equiv J^*(\text{mod } \Delta_k)$ which satisfies (3.8) let

$$B_1 = \prod_{k \leq p < \exp(c_1 \log x)} p,$$

$$B_i = \prod_{\exp((c_{i-1}(\log x))^{i-1}) \leq p < \exp(c_i(\log x)^i)} p, \quad i = 2, \dots, k;$$

where $c_k = 1$, and for $i = 0, \dots, k-1$, c_i is determined by

$$\frac{c_{i-1}}{c_i} = e^{-i/k} \prod_{p < k} \left(1 - \frac{1}{p}\right)^{-1/k} \lambda_i.$$

Since $\prod_{i=1}^k \lambda_i = 1$ it follows that $c_0 = e^{-i/k} \prod_{p < k} (1 - 1/p)$. As the B_i , $i = 1, \dots, k$ are k integers made up of primes $p \geq k$ and are relatively prime in pairs, as well as each relatively prime to Δ_k , by the Chinese Remainder Theorem the system

$$(3.13) \quad \begin{aligned} y + 1 &\equiv O(\text{mod } B_1) \\ y + 2 &\equiv O(\text{mod } B_2) \\ &\vdots \\ y + k &\equiv O(\text{mod } B_k) \\ y &\equiv J^*(\text{mod } \Delta_k) \end{aligned}$$

has a solution $y = n^*$, $0 < n^* < \Delta_k \prod_{i=1}^k B_i$ which is unique modulo $\Delta_k \prod_{i=1}^k B_i$.

For this integer $n^* \equiv J^*(\text{mod } \Delta_k)$ we have for $i = 1, \dots, k$

$$\begin{aligned} \left(\frac{\phi(n^* + i)}{n^* + i}\right)_k &= \prod_{\substack{p | n^* + i \\ p \geq k}} \left(1 - \frac{1}{p}\right) \leq \prod_{p | B_i} \left(1 - \frac{1}{p}\right) \\ &\leq \frac{c_{i-1}}{c_i} \left(\frac{1}{\log x}\right) + O\left(\frac{1}{\log^2 x}\right), \end{aligned}$$

(note that the value obtained for c_0 validates this for $i = 1$). Then

$$(3.14) \quad \begin{aligned} &\left(\frac{\phi(n^* + i)}{n^* + i}\right)_k f_k(J^* + i) \\ &\leq \frac{\lambda_i e^{-i/k} \prod_{p < k} \left(1 - \frac{1}{p}\right)^{-1/k}}{\log x} f_k(J^* + i)(1 + o(1)). \end{aligned}$$

But from the Prime Number Theorem since

$$n^* < \Delta_k \prod_{i=1}^k B_i = \prod_{p < \exp((\log x)^k)} p, \quad (c_k = 1),$$

it follows that

$$\log n^* \leq \sum_{p < \alpha x p^{(\log x)^k}} \log p = O(e^{(\log x)^k})$$

so that

$$(3.15) \quad \log \log n^* \leq (\log x)^k + O(1).$$

Since (3.14) holds for all $i = 1, \dots, k$, it certainly holds for the maximum of these functions. Thus inserting (3.15) in (3.14) yields

$$(3.16) \quad (\log \log n^*)^{1/k} \max_{i=1, \dots, k} \left(\frac{\phi(n^* + i)}{n^* + i} \right)_k f_k(J^* + i) \\ \leq (1 + o(1)) e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p} \right)^{1/k} \psi(k).$$

Clearly as x tends to infinity the n^* (which depends on x) also tends to infinity, so that (3.16) yields

$$(3.17) \quad G(k) \leq e^{-\gamma/k} \prod_{p < k} \left(1 - \frac{1}{p} \right)^{-1/k} \psi(k)$$

which completes the proof of (1.3).

REFERENCES

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, (1968), 349-354.
2. E. Landau, *Über den Verlauf der zahlentheoretischen Funktion $\phi(x)$* , Archiv der Mathematik und Physik, (3) 5 (1903), 86-91.
3. F. Mertens, *Ein Beitrag zur analytischen Zahlentheorie*, Journal für die reine und angewandte Mathematik, Bd. 78 (1874), 46-62.

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