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Let O_{AL}(resp. O_{AS}) be the class of open Riemann surfaces on which there exists no nonconstant analytic functions f such that $\log^+|f|$ have harmonic (resp. quasi-bounded harmonic) majorant. It is shown that $O_{AL} = O_{AS}$ for surfaces of finite genus.

1. An analytic function f on an open Riemann surface R is said to be *Lindelöfian* if $\log^+|f|$ has a harmonic majorant ([2]). Denote by $AL(R)$ the class of Lindelöfian analytic functions on R. Relating to the class $AL(R)$, consider the class $AS(R)$ which consists of analytic functions f on R such that $\log^+|f|$ has a *quasi-bounded* harmonic majorant. The class $AS(R)$ is referred to as the *Smirnov* class ([4] and [4]). Denote by O_{AL}(resp. O_{AS}) the class of open Riemann surfaces R such that $AL(R)$(resp. $AS(R)$) consists of only constant functions. It is known that $O_G < O_{AL} < O_{AS}$ (strict inclusions) in general and that $O_G = O_{AL}$ for surfaces of finite genus ([2] and [5]). In this paper, it is shown that $O_G = O_{AS}$, and therefore $O_G = O_{AL} = O_{AS}$, for surfaces of *finite* genus (cf. [3]).

2. Let s be a superharmonic function on a hyperbolic Riemann surface R and e be a compact subset of R such that $R - e$ is connected. Denote by $\Phi(s, e)$ the class of superharmonic functions v on R such that $v \geqq s$ on e except for a polar set. Consider the function $(s, e)(p) = \inf_{v \in \Phi(s,e)} v(p)$ on R. Then (s, e) has following properties (see [1]):

LEMMA. *(s, e) is superharmonic on R, $(s, e) = H_s^{R-e}$ (the solution of the Dirichlet problem with boundary values s on ∂e and 0 on ∂R) on $R - e$, and $(s, e) = s$ on e except for a polar set.*

3. THEOREM. *The relation $O_G = O_{AS}$ is valid for surfaces of finite genus.*

Proof. We only have to show that $O_G \supset O_{AS}$. Let F be of finite genus not belonging to O_G and S be a compact surface such that $F \subset S$. In order to show that $F \notin O_{AS}$, we may assume that $K = F^c = S - F$ is totally disconnected. Hence we can decompose K into two compact sets E and e such that E and e have positive capacity. Set $R = E^c = S - E$ and choose a point $x \in e$ which is a regular boundary point for $R - e$. Let $e_n = e \cap \{z \in R; G_R(z, x) \leqq n\} (n \in N)$, where $G_R(\cdot, x)$ is the Green's function on R with pole at x. Set $h_n =$

$(G_R(\cdot, x), e_n)$ for $n \in N$. Then it is easily seen that $\{h_n\}$ is increasing and $h_n \in HB(R - e)$ (the class of bounded harmonic functions on $R - e$). Here and hereafter, the lemma in no. 2 will be used repeatedly without referring to it. Let y be an arbitrarily fixed point in $R - e$. Again, we set $u_n = (G_R(\cdot, y), e_n)(n \in N)$ and $u = (G_R(\cdot, y), e)$. Then, since $\{u_n\}$ is increasing and $u_n \leqq u$, the limit function U of $\{u_n\}$ exists, is superharmonic on R, and $U \leqq u$. On the other hand, since $u_n \leqq U \leqq G_R(\cdot, y)$ and $u_n = G_R(\cdot, y)$ on e_n except for a polar set for every $n \in N$, $U = G_R(\cdot, y)$ on e except for a polar set by the fact that the union of countably many polar sets is also polar, and a fortiori $U \geqq u$, which implies that $U = u$. Observe that

$$\begin{aligned} h_n(y) &= H^{R-e_n}_{G_R(\cdot, x)}(y) = G_R(y, x) - G_{R-e_n}(y, x) \\ &= G_R(x, y) - G_{R-e_n}(x, y) = H^{R-e_n}_{G_R(\cdot, y)}(x) \\ &= u_n(x) \uparrow u(x) = (G_R(\cdot, y), e)(x) \qquad (n \longrightarrow \infty) \\ &= G_R(x, y) . \end{aligned}$$

Here the regularity of x is used in the last equality. Consequently we see that the increasing sequence $\{h_n\}$ with $h_n \in HB(R - e)$ converges to $G_R(\cdot, x)$, i.e., $G_R(\cdot, x)$ is quasi-bounded on $R - e$.

Consider a meromorphic function f on S with a single pole of order k at x. Then $\log^+ |f| \leqq kG_R(\cdot, x) + C$ for a sufficiently large constant C. Therefore $f \in AS(R - e) = AS(F)$, i.e., $F \notin O_{AS}$. This completes the proof.

References

1. C. Constantinescu and A. Cornea, *Ideale Ränder Riemannscher Flächen*, Springer, 1963.
2. M. Heins, *Lindelöfian maps*, Ann. Math., **62** (1955), 418-446.
3. J. L. Schiff, *On a class of analytic functions of Smirnov*, Canad. J. Math., **31** (1979), 181-183.
4. V. I. Smirnov, *Sur les formules de Cauchy et de Green et quelques problèmes qui s'y rattachent*, Izv. AN SSSR, ser. fiz.-mat., **3** (1932), 337-372.
5. S. Yamashita, *On some families of analytic functions on Riemann surfaces*, Nagoya Math. J., **31** (1968), 57-68.

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