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THEOREM 0.1. Let P^n and Q^n be simply connected Poincare complexes such that $P_{(2)} \cong Q_{(2)}$. Assume $n \leqq 2k-2$. Then P^n Poincare embeds in S^{n+k} *if* and only if Q^n Poincare embeds in S^{n+k}.

The Browder-Sullivan-Casson-Wall embedding theorem [see [23] Chap. 12] then implies the analogous result for manifolds which has also been proven by Rigdon [18] using entirely different methods.

The proof of (0.1) relies upon the following:

THEOREM 0.2. (*Localize at odd primes.*) *Let X be a (q-1)-connected space, and suppose $X \cong \sum \bar{X}$. Then for $m \leqq 3q-2$, $\sum^\infty$: $\pi_m(X) \to \pi_m^s(X)$ has a right inverse.*

This result is false if we do not localize at odd primes. For example, Mahowald's $\eta_j \in \pi_{2^j}^s$ do not desuspend to $\pi_{2\cdot 2^j-3}(S^{2^j-3})$ (see [14]). The result is also false if X is not a suspension, e.g., $X = S^i \times S^i$ and $m = 2i$. Since $\pi_3^s = \boldsymbol{Z}/24$ and $\pi_5(S^2) = \boldsymbol{Z}/2$, $m \leqq 3q-2$ is best possible.

COROLLARY 0.3. (*Localize at odd primes.*) *Let X be a (q-1)-connected space. Then for $i \geqq 1$ and $m \leqq 3q+2i-2$.*

$\pi_{m+i}(\sum^i X) \cong \pi_m^s(X) \oplus \pi_{m+i+1}^s(\sum^i X \wedge \sum^i X)^{Z_2}$ where $\boldsymbol{Z}_2$ acts on $\sum^i X \wedge \sum^i X$ by switching factors. The nonzero elements in the $\pi_m^s(X)$ term are permanent in the sense that they desuspend to ΣX and remain nonzero under the suspension homomorphism. The nonzero elements in the $\pi_{m+i+1}^s(\Sigma^i X \wedge \Sigma^i X)^{Z_2}$ term are just flashes in the sense that they do not desuspend and die under a single suspension.

If X is a sphere, then this corollary implies the well known result that for $r \leqq 2n-2$

$$\pi_{n+r}(S^n) = \begin{cases} \pi_r^s & n \text{ odd} \\ \pi_r^s \oplus \pi_{r-n+1}^s & n \text{ even} \end{cases}$$

(see [16], [22], [21], and [7] Appendix 2).

Elsewhere [13] in joint works with Ib Madsen and Larry Taylor (0.2) is applied to the classification of P.L. manifolds.

I.

$$Q(\ ) = \Omega^\infty \Sigma^\infty(\ )\,.$$

Proof of (0.2). Consider the following commutative diagram

$$\begin{array}{ccc} \Omega\Sigma\bar{X} & \xrightarrow{h_2} & Q\bar{X}\wedge\bar{X} \\ \downarrow{\scriptstyle \Sigma_1^\infty} & & \downarrow{\scriptstyle Q(i)} \\ \Omega Q\Sigma\bar{X} = Q\bar{X} & \xrightarrow{h_\infty} & QS^\infty \ltimes_{Z_2}\bar{X}\wedge\bar{X} \\ \downarrow{\scriptstyle \Omega h'_\infty} & & \downarrow \\ \Omega QS^\infty\ltimes_{Z_2}\Sigma\bar{X}\wedge\Sigma\bar{X} & \xrightarrow[\sim]{j} & Q(S^\infty\ltimes_{Z_2}\bar{X}\wedge\bar{X}/\bar{X}\wedge\bar{X}) \end{array} \tag{1.1}$$

where h_2, h_∞, and h'_∞ are Hopf invariant maps coming from stable decompositions of $\Omega\Sigma\bar{X}$, $Q\bar{X}$, and $Q\Sigma\bar{X}$. (See [15] and [5].) $i\colon \bar{X}\wedge\bar{X}\to S^\infty\ltimes_{Z_2}\bar{X}\wedge\bar{X}$ is the inclusion map, and j comes from the homotopy equivalence

$$\Sigma(S^\infty\ltimes_{Z_2}\bar{X}\wedge\bar{X}/\bar{X}\wedge\bar{X}) \xrightarrow{\sim} S^\infty\ltimes_{Z_2}\Sigma\bar{X}\wedge\Sigma\bar{X} \text{ (see 2.3 of [15])}\,.$$

Since Q sends cofibrations to fibrations, the right vertical edge of (1.1) is a fibration sequence. Milgram's *EHP* sequence (see [15]) implies that $\Omega\Sigma\bar{X}$ is $(3q-3)$-equivalent to the fibre of $\Omega h'_\infty$. Since $\Sigma^\infty\colon \pi_m(\Sigma\bar{X})\to\pi^s_m(\Sigma\bar{X})$ is induced by Σ_1^∞, we are done if we can show $Q(i)$ has a right inverse when we localize at odd primes.

Consider the following commutative diagram

$$\begin{array}{ccc} \bar{X}\wedge\bar{X} \cong & & S^\infty\times\bar{X}\wedge\bar{X} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \pi_{\text{double cover}}} \\ S^\infty\ltimes_{Z_2}\bar{X}\wedge\bar{X} & \xleftarrow[p]{} & S^\infty\times_{Z_2}\bar{X}\wedge\bar{X} \end{array}$$

where p pinches $S^\infty/Z_2\times *$ to a point. Notice that $Q(p)_{(\text{odd})}$ is a homotopy equivalence. Let

$$t\colon Q(S^\infty\times_{Z_2}\bar{X}\wedge\bar{X}) \longrightarrow Q(S^\infty\times\bar{X}\wedge\bar{X})$$

be the transfer for the double cover π. Then $(Q(\pi)\circ t)^{-1}_{(\text{odd})}$ is a homotopy equivalence, and $t\circ(Q(\pi)\circ t)^{-1}_{(\text{odd})}\circ Q(p)^{-1}_{(\text{odd})}$ is a right inverse for $Q(i)_{(\text{odd})}$.

REMARK. If $\bar{X}\cong\Sigma\bar{\bar{X}}$, $m\leqq 3q-4$, and we localize at odd primes; then a right inverse to Σ^∞ can be derived from the following left inverse to Milgram's map $\partial\colon \pi_m(S^\infty \ltimes_{Z_2} X \wedge X) \to \pi_{m-1}(X)$:

$$\pi_{m-1}(X) \xrightarrow{H_X} \pi_m^s(X \wedge X)^{Z_2} \cong \pi_m(S^\infty \ltimes_{Z_2} X \wedge X) .$$

Proof of (0.3). (Localize at odd primes.) By considering diagram (1.1) with X replaced by $\Sigma^{i-1}X$, one gets that when $m + i \leqq 3(q + i) - 2$

$$\begin{aligned}\pi_{m+i}(\Sigma^i X) &\cong \pi_{m+i-1}(\Omega\Sigma\Sigma^{i-1}X)\\ &\cong \pi_{m+i-1}(\Omega Q\Sigma^i X) \oplus \pi_{m+i}(\Omega Q S^\infty \ltimes_{Z_2} \Sigma^i X \wedge \Sigma^i X)\\ &\cong \pi_m^s(X) \oplus \pi_{m+i+1}^s(S^\infty \ltimes_{Z_2} \Sigma^i X \wedge \Sigma^i X) ,\end{aligned}$$

where $h_2\colon \pi_{m+i}(\Sigma^i X) \to \pi_{m+i-1}(Q\Sigma^{i-1}X \wedge \Sigma^{i-1}X)$ is 1-1 on $\pi_{m+i+1}^s(S^\infty \ltimes_{Z_2} \Sigma^i X \wedge \Sigma^i X)$. Thus the nonzero elements in the $\pi_{m+i+1}^s(S^\infty \ltimes_{Z_2} \Sigma^i X \wedge \Sigma^i X)$ term do not desuspend.

The double cover $\pi\colon S^\infty \times \Sigma^i X \wedge \Sigma^i X \to S^\infty \times_{Z_2} \Sigma^i X \wedge \Sigma^i X$ induces an isomorphism

$$\pi_{m+i+1}^s(\Sigma^i X \wedge \Sigma^i X)^{Z_2} \cong \pi_{m+i+1}^s(S^\infty \ltimes_{Z_2} \Sigma^i X \wedge \Sigma^i X) .$$

Furthermore, the commutativity of the following diagram

$$\begin{array}{ccccc}\pi_{m+i}(\Sigma\Sigma^{i-1}X \wedge \Sigma^{i-1}X) & \xrightarrow{\Sigma} & \pi_{m+i+1}(\Sigma^i X \wedge \Sigma^i X) & \longrightarrow & \pi_{m+i+1}(S^\infty \ltimes_{Z_2} \Sigma^i X \wedge \Sigma^i X)\\ & {}_{[\iota,\iota]\cdot(\ )}\searrow & & \swarrow{}_{\partial} & \\ & & \pi_{m+i}(\Sigma^i X) & & \end{array}$$

implies that the elements in the $\pi_{m+i+1}(S^\infty \ltimes_{Z_2} \Sigma^i X \wedge \Sigma^i X)$-term die after a single suspension.

Open Problems.

1. *Conjecture.* If $\alpha \in \pi_n Y$ and $\Sigma^\infty a = 0$, then $\Sigma^k a = 0$ for $k \geqq [n + 2/2]$.

Surgery theory shows that this conjecture would imply the Hirsh conjecture on embedding π-manifolds. See [6] for a partial converse when $X = S^i$. The Corollary (0.3) implies this conjecture is true when we localize at odd primes.

2. *Compute the Hopf invariants of stably trivial elements.* If $a \in \pi_n(\Sigma X)$ is stably trivial, then in the metastable range $a = \partial(w)$ for some element $w \in \pi_{n+1}(S^\infty \ltimes_{Z_2} \Sigma X \wedge \Sigma X)$.

Conjecture. $H(a) = t(q(w))$ in $\pi_n^s(\Sigma X \wedge \Sigma X)$, when t is the transfer of the double cover $S^\infty \ltimes \Sigma X \wedge \Sigma X \to S^\infty \times_{Z_2} \Sigma X \wedge \Sigma X$, and q comes from the stable equivalence

$$S^\infty \times_{Z_2} \Sigma X \wedge \Sigma X \sim (S^\infty \times_{Z_2} {}^*) \vee S^\infty \ltimes_{Z_2} \Sigma X \wedge \Sigma X .$$

The conjecture is equivalent to stably computing the map t_1 in the cofibre sequence

$$\Sigma X \wedge X \longrightarrow \Sigma(S^\infty \ltimes_{Z_2} X \wedge X) \longrightarrow S^\infty \ltimes_{Z_2} \Sigma X \wedge \Sigma X \xrightarrow{t_1} \Sigma X \wedge \Sigma X .$$

3. *Conjecture.* (Localize at odd primes.) If $m \leqq 3$ (connectivity X), then

$$\pi_i(X) \xrightarrow{\Sigma^\infty} \pi_i^s(X) \xrightarrow{\bar{\Delta}} \pi_i^s(X \wedge X)$$

is exact, where $\bar{\Delta}$ is the reduced diagonal map.

Since $\pi_i^s(S^\infty \ltimes_{Z_2} X \wedge X) \simeq \pi_i^s(X \wedge X)^{Z_2}$, there exists some map $k\colon \pi_i^s(X) \to \pi_i^s(X \wedge X)$ such that image Σ^∞ = kernel k. Furthermore, an easy Postnikov decomposition argument shows the conjecture is true when localized at 0.

REMARK. Even if we do not localize, there is a close connection between the Hopf invariant and the reduced diagonal.

If $X \cong \Sigma \bar{X}$, then the pinch map $X \to X \vee X$ yields a trivialization Γ_X: cone $X \to X \wedge X$ of $\bar{\Delta}_X\colon X \to X \wedge X$.

PROPOSITION. *If $f \in [X, Y]$, where $X = \Sigma X$ and $Y = \Sigma \bar{Y}$, then $\Sigma H(f) \in [\Sigma X, Y \wedge Y]$ is represented by*

$$\Sigma X \cong \text{cone } X \cup_X \text{cone } X \xrightarrow{(f \wedge f)\cdot \Gamma_X \cup \Gamma_Y \cdot c(f)} Y \wedge Y$$

where $c(f)$: cone $X \to$ cone Y is the extension of f.

Proof. This is just a reinterpretation of the proof of Theorem 5.14 in [3].

II.

LEMMA 2.1. *Let Z^n be a simply connected finite CW complex of dimension n, and let Φ be a $S^N_{(\text{odd})}$-fibration over $Z^n (N > n + 1)$. If $n \leqq 2q$, then there exists a $S^{q-1}_{(\text{odd})}$-fibration θ^q over Z^n such that θ^q has a cross section, and such that θ^q is stably equivalent to Φ.*

Proof. Recall that for simply connected spaces stable $S^N_{(\text{odd})}$-fibrations are classified by $BSG_{(\text{odd})}$ and $S^{q-1}_{(\text{odd})}$-fibrations with cross section are classified by $BSF_{q-1(\text{odd})}$. (See [20] § 4.)

Thus we are done if we can show that the map which classifies Φ lifts to $BSF_{q-1(\text{odd})}$. If q is odd we shall show the map in fact lifts to $BSF_{q-2(\mathrm{odd})}$. It suffices to show $\pi_i(SG/SF_{k-1})_{(\mathrm{odd})} = 0$ when k is even and $i \leqq 2k+1$. Consider the exact sequence:

$$\pi_{i+k-1}(S^{k-1})_{(\mathrm{odd})} \xrightarrow{\Sigma_1^\infty} \pi^S_{i\,(\mathrm{odd})} \longrightarrow \pi_i(SG/SF_{k-1})_{(\mathrm{odd})} \longrightarrow \pi_{i+k+2}(S^{k-1})_{(\mathrm{odd})} \xrightarrow{\Sigma^\infty} \pi^S_{i-1(\mathrm{odd})} .$$

By studying the double suspension (see [7] Appendix 2) one gets that Σ_1^∞ is an epimorphism, Σ^∞ is an isomorphism, and $\pi_i(SG/SF_{k-1})_{(\mathrm{odd})}=0$ when $i \leqq 2k+1$.

The following result was proved in [10].

THEOREM 2.2. *Let $(W, A)^m$ be an oriented, finite Poincare pair of formal dimension m. Assume $\pi_1 A \overset{\sim}{\rightarrow} \pi_1 W$, $m \geqq 6$, and $2m \geqq 3(n+1)$, where n = homotopy dimension of W. Then (W, A) Poincare embeds in S^m if and only if $\pi_m(W/A)$ contains an element of degree* 1.

Although this is a purely homotopy theoretic result, the proof in [10] consists of converting (W, A) to a manifold and then using smooth embedding theory. In § III progress is made towards a homotopy theoretic proof.

Proof of 0.1. Assume Q Poincare embeds in S^{n+k}. Let $f\colon P_{(2)} \rightarrow Q_{(2)}$ be a homotopy equivalence. Let η^k be the normal fibration for the Poincare embedding of Q in S^{n+k}, and let $d \in \pi_{n+k}(T(\eta))$ be the associated normal invariant. $\eta^k_{(2)}$ is the $S^k_{(2)}$-fibration associated to η (see Sullivan [20] for definition and properties). Let $\xi^k_t = f^*\eta^k_{(2)}$. f^{-1} lifts to a map of $S^{k-1}_{(2)}$-fibrations $b(f^{-1})\colon S(\eta^k_{(2)}) \rightarrow S(\xi^k_t)$ which induces a map of Thom complexes $T(f^{-1})\colon T(\eta_{(2)}) \rightarrow T(\xi_t)$. Notice that $c_t = T(f^{-1})(d_{(2)})$ is a unit in $\pi_{n+k}(T(\xi_t))$, i.e. $\deg c_t \in z_{(2)}$ is a unit.

Suppose that we could construct a S^{k-1}-fibration ξ over P such that $\xi_{(2)} = \xi_t$ and a degree 1 map $c\colon S^{n+k} \rightarrow T(\xi)$. Then $(D(\xi), S(\xi))$ is an oriented, finite Poincare pair of formal dimension $n+k$, and Theorem 2.2 implies there exists a Poincare embedding of $(D(\xi), S(\xi))$ in S^{n+k} which determines a Poincare embedding of X in S^{n+k}.

Lemma 2.1 implies there exists a $S^{k-1}_{(\mathrm{odd})}$-fibration ξ_0 such that ξ_0 is stably equivalent to $\gamma_{P(\mathrm{odd})}$ (where γ_P = Spivak fibration of P) and such that $T(\xi_0)$ is a suspension. If k is even, $BG_{k(0)} \overset{\sim}{\rightarrow} K(Q, k)$ is a homotopy equivalence where the map is given by the Euler class; and if k is odd, $BG_{k_{(0)}} \cong K(Q, 2(k-1))$ (see [20] 4.12). Since η^k is the normal fibration of an embedding in a sphere, the Euler class of η and ξ_t are trivial. Since ξ_0 has a cross section, it has trivial Euler class. Thus ξ_t and ξ_0 fit together to yield a S^k-fibration ξ^k

when k is even. If k is odd, $BG_{k(0)}^{2k-3} \simeq *$, and ξ_t and ξ_0 fit together to yield a S^k-fibration ξ^k.

Theorem 0.2 implies that $\pi_{n+k}(T(\xi^k)_{(\mathrm{odd})})$ contains a unit. Furthermore, $\pi_{n+k}(T(\xi^k)_{(2)}) \cong \pi_{n+k}(T(\xi_{(2)}))$ contains c_t which is a unit. Thus $\pi_{n+k}(T(\xi^k))$ contains an element of degree 1.

III. A Poincare embedding of $(W, A)^m$ in S^m consists of a finite complex C (the complement) and a map $a: A \to C$ such that the double mapping cylinder $M(c, a)$ is homotopy equivalent to S^m, where c is the inclusion of A in W. A Poincare embedding determines a deg 1 element α in $\pi_m(W/A)$ which is represented by the composition

$$S^m \simeq M(c, a) \longrightarrow M(c, a)/C \xrightarrow{\text{excision}} W/A .$$

Notice that $\Sigma C \simeq (W/A) \bigcup_\alpha e^{m+1}$.

In this section we give homotopy theoretic proofs that the hypothesis of Theorem 2.2 imply that

(1) $(W/A) \bigcup_\alpha e^{m+1}$ is a suspension

(2) There exists a map $a': \Sigma A \to (W/A) \bigcup_\alpha e^{m+1}$ such that $M(\Sigma c, a') \simeq S^{m+1}$.

If one could prove that the Hopf invariant $H(a')$ were trivial, then one would have a homotopy theoretic proof of Theorem 2.2.

Browder ([4]) has observed that the composition

$$b: W \times 0 \cup A \times I \cup W \times 1 \longrightarrow W \times 0 \cup A \times I \cup W \times 1/W \times 0 \simeq W/A \longrightarrow W/A \bigcup_\alpha e^{m+1}$$

determines an embedding of $(W, A) \times I$ in S^{m+1}. In result (2) we are showing Browder's map b factors through

$$W \times 0 \cup A \times I \cup W \times 1/W \times 0 \cup W \times 1 \simeq \Sigma A .$$

PROPOSITION 3.1. *Let $(W, A)^m$ be an oriented, finite Poincare pair of formal dimension m. If $\pi_m(W/A)$ contains an element α of degree 1, then the map $j: W \to W/A$ which pinches A to a point is stably homotopic to a trivial map.*

Proof. Let $W^+ = W \cup \{+\}$ with $+$ as base point. Let $j^+ = W^+ \to W/A$ be the map which sends $+$ to the collapse point and which equals j on W. Suppose $e: S^n \to D_n(W^+) \wedge W^+$ is an n-duality pairing. Then the map : $\{W^+, W/A\} \to \{S^n, D_n W^+ \wedge W/A\}$ which sends f to $(I_{D_n W^+} \wedge f) \circ e$ is an isomorphism, and we are done if we can show $(I_{D_n W^+} \wedge j^+) \circ e$ is trivial.

Let $\bar{\Delta}\colon (W, A) \to (W, A) \times W$ be the relative diagonal map. $\bar{\Delta}$ induces a map $\widetilde{\Delta}\colon W/A \to W \times W/A \times W \cong W/A \wedge W^{+}$. Since (W, A) satisfies Poincare duality, $e = \widetilde{\Delta} \circ \alpha$ is an n-duality map. Notice that the following diagram commutes:

$$\begin{array}{ccccc} S^n & \xrightarrow{\alpha} & W/A & \xrightarrow{\tilde{\Delta}} & W/A \wedge W^{+} \\ \downarrow{\scriptstyle \bar{\Delta}_{S^n}} & & & \searrow{\scriptstyle \bar{\Delta}_{W/A}} & \downarrow{\scriptstyle I_{W/A} \wedge j^{+}} \\ S^n \wedge S^n & & \xrightarrow{\alpha \wedge \alpha} & & W/A \wedge W/A \end{array} \tag{3.1.1}$$

where $\bar{\Delta}_{S^n}$ and $\bar{\Delta}_{W/A}$ are reduced diagonal maps. Since S^n is a suspension, $\bar{\Delta}_{S^n} \cong *$ and j^{+} is stably homotopy trivial.

Lemma 3.2. (*Jurca* [9] *Prop.* 3.2.) *If* $3 \leqq q$, *Z is a* $(q-1)$-*connected CW complex, and* $\dim Z \leqq 3q - 3$, *then Z desuspends if and only if* $\bar{\Delta}_Z \cong *$.

Proof of (1). Poincare duality implies W/A is $(m - n - 1)$-connected. $\bar{\Delta}_{W/A} = (I_{W/A} \wedge j^{+}) \circ \widetilde{\Delta}$ which is stably trivial by Proposition 3.1. Since $m = \dim W/A \leqq 2$ (connectivity $W/A \wedge W/A) = 2(2(m-n) -1)$, $\bar{\Delta}_{W/A}$ is in fact unstably trivial and Lemma 3.3 implies W/A is a suspension. Then $W/A \bigcup_{\alpha} e^{m+1} \cong (W/A)^{m-1}$ is also a suspension.

Proof of (3). Consider the cofibration sequence $A \xrightarrow{c} W \xrightarrow{j} W/A \xrightarrow{l} \Sigma A$. Since j is homotopy trivial, l has a left inverse l'. Let a' be the composition $\Sigma A \xrightarrow{l'} W/A \to W/A \bigcup_{\alpha} e^{n+1}$. An easy homology and van Kampen's argument shows, $M(\Sigma c, a') \cong S^{m+1}$.

References

1. W. D. Barcus, *The stable suspension of an Eilenberg-MacLane space*, Trans. Amer. Math. Soc., **96** (1960), 101–114.
2. J. C. Becker and D. H. Gottlieb, *The transfer map and fiber bundles*, Topology, **14** (1975), 1–12.
3. J. M. Boardman and B. Steer, *On Hopf invariants*, Comment. Math. Helv., **42** (1967), 180–221.
4. W. Browder, *Embedding smooth manifolds*, in Proc. I.C.M. (Moscow, 1966), Mir, (1968), 712–719.
5. F. R. Cohen, J. P. May, and L. R. Taylor, *Splittings of certain spaces CX*, to appear, Proc. Cambridge Phil. Soc.
6. H. Glover and G. Mislin, *Metastable annihilation in the homotopy groups of spheres*, Notices of the Amer. Math. Soc., **18** (1971), 385.
7. D. Husemoller, *Fibre Bundles*, 2nd ed., Springer-Verlag, Graduate Texts in Mathematics 20.
8. L. Jones, *Path spaces: A geometric representation for Poincare spaces*, Ann. of Math., **97** (1973), 306–343.
9. D. R. Jurca, *Obstructions to desuspending a complex*, Northwestern Thesis, 1973.
10. L. L. Larmore and B. Williams, *Single obstructions to embedding and Boardman-*

Vogt little cubes, to appear.
11. J. Levine, *A classification of differentiable knots*, Ann. of Math., **82** (1965), 15–50.
12. J. Levine, *On differentiable embeddings of simply connected manifolds*, Bull. Amer. Math. Soc., **69** (1963), 806–809.
13. I. Madsen, L. Taylor and B. Williams, *Tangential homotopy equivalence*, to appear.
14. M. Mahowald, *A new infinite family in* ${}_2\pi^s_*$, Topology, **16** (1977), 249–256.
15. J. Milgram, *Unstable Homotopy From the Stable Point of View*, Lecture Notes No. 368, Springer-Verlag, 1974.
16. J. Moore, *The double suspension and p-primary components of the homotopy groups of spheres*, Bol. Soc. Mat. Mexicana (2), **1** (1956), 28–37.
17. F. Quinn, *Surgery on Poincaré and normal spaces*, Bull. Amer. Math. Soc., **78** (1972), 262–267.
18. R. Rigdon, *P-equivalences and embeddings of manifolds*, Proc. London Math. Soc., **11** (1975), 233–244.
19. F. Rouch, *Transfer in Generalized Cohomology Theories*, Princeton thesis, 1971.
20 D. Sullivan, *Geometric Topology. Part I. Localization, Periodicity, and Galois Symmetry*, Notes, M.I.I., 1970.
21. S. Thomeier, *Über eine Beziehurg zwischen unstabilen und stabilen Homotopiegruppen von Sphären*, Arch. Math., **15** (1964), 351–353.
22. H. Toda, *On the double suspension* E^2, J. Inst. Polytech. Osaka City University Ser. A., **7** (1956), 103–145.
23. C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press, 1971.

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